Miniworkshop on Diffraction and Ultraperipheral Collisions

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Contents

1 The deep inelastic scattering

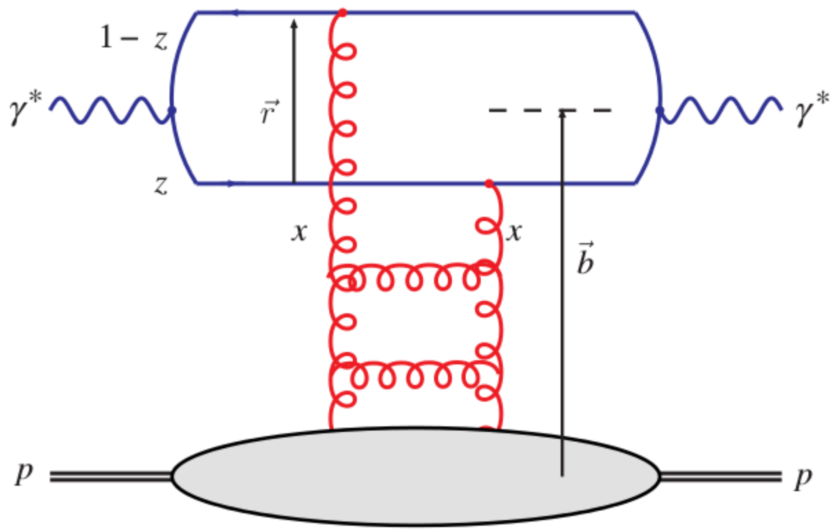

Figure 1: Interaction scheme of the deep inelastic scattering

The main ingredient of the total cross section of the scattering of an electron on a proton with large transfer of the four-momenta Q^2 is the structure function of a proton $F_2(x, Q^2)$, where x is the Bjorken-x

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}(\sigma_T^{\gamma^* p}(x, Q^2) + \sigma_L^{\gamma^* p}(x, Q^2)), \tag{1}$$

where α_{em} is the structure constant, $\sigma_{T,L}^{\gamma^* p}(x, Q^2)$ are transversal and longitudinal cross sections of the scattering of transversally and longitudinally polarized virtual photon with target proton. They can be calculated in the light-cone color dipole model in a frame, where proton is at rest, as

$$\sigma_{T,L}^{\gamma^* p}(x, Q^2) = \int \mathrm{d}\vec{r} \int_0^1 \mathrm{d}z |\psi_{\gamma^* \to q\bar{q}}^{T,L}(\vec{r}, z, Q^2)|^2 \sigma_{q\bar{q}}(\vec{r}, x), \tag{2}$$

where $\sigma_{q\bar{q}}(\vec{r}, x)$ is a cross section of the strong interaction of a quark-antiquark dipole with the target proton, $\psi_{\gamma^* \to q\bar{q}}^{T,L}(\vec{r}, z, Q^2)$ is a wave function (probability amplitude) of a situation where you split photon into a quark-antiquark pair (or better probability amplitude that the $q\bar{q}$ Fock state of the infinite Fock decomposition of a photon joins the interaction), $\vec{r}$ is the transverse dipole size, $\vec{b}$ is the impact parameter of the dipole(transverse distance from the center of the proton to the center of mass of the dipole) and z is a part of photon momenta carried by one of the quarks from the dipole, Q is the scale of the incoming photon and Bjorken-x of the scattering is

$$x = \frac{Q^2}{W^2 + Q^2}, \tag{3}$$

where W is the total energy of the $\gamma - p$ system. The dipole cross section is model dependent and cannot be derived from the first principles. However, the wave function can be derived from QED as a vertex $\gamma \to q\bar{q}$ in a light-cone frame. The square of the wave function is

$$\begin{aligned} |\psi^T_{\gamma^* \to q\bar{q}}(\vec{r}, z, Q^2)|^2 &= \frac{2N_c\alpha_{em}}{(2\pi)^2} \sum_f Z_f^2 \left((z^2 + (1-z)^2)\varepsilon^2 K_1^2(\varepsilon r) + m_f^2 K_0^2(\varepsilon r)\right) \\ |\psi^L_{\gamma^* \to q\bar{q}}(\vec{r}, z, Q^2)|^2 &= \frac{2N_c\alpha_{em}}{(2\pi)^2} \sum_f Z_f^2 4Q^2 z^2 (1-z)^2 K_0^2(\varepsilon r), \end{aligned} \tag{4}$$

where $\varepsilon^2 = z(1-z)Q^2 + m_f^2$, N_c is a number of colors, Z_f is a fractional charge of a flavor f, K_0 and K_1 are modified Bessel functions.

2 The wave function

The wave function can be defined in a mixed representation as

$$\begin{aligned} \psi^{T,L}_{\gamma^* \to q\bar{q}}(\vec{r}, z, Q^2) &= \frac{\sqrt{N_c\alpha_{em}}}{2\pi} Z_f \chi_j^\dagger \hat{\mathcal{O}}^{T,L} \chi_i K_0(\varepsilon r) \\ \hat{\mathcal{O}}^T &= m_f(\vec{\sigma}.\vec{e}) + i(1-2z)(\vec{\sigma}.\vec{n})(\vec{e}.\vec{\nabla}_r) + (\vec{\sigma} \times \vec{e}).\vec{\nabla}_r \\ \hat{\mathcal{O}}^L &= 2Qz(1-z)(\vec{\sigma}.\vec{n}) \end{aligned} \tag{5}$$

where χ_j and χ_i represent spinors of a quark and antiquark, m_f is a mass of a quark with flavor f, $\vec{\sigma}$ is a vector of Pauli matrices, $\vec{e}$ is a polarization vector of incoming photon, $\vec{n}$ is a unit vector in the direction of photon propagation and $\vec{\nabla}_r$ is a two-dimensional vector of derivatives w.r.t. r.
Let's assume, that the frame in which we will be working is such that the photon comes in the positive direction of the z axis. Therefore, the vector $\vec{n}$ points in the direction of $\vec{e}_z$. Also, it is obvious that the vector $\vec{r}$ is perpendicular to the z axis.

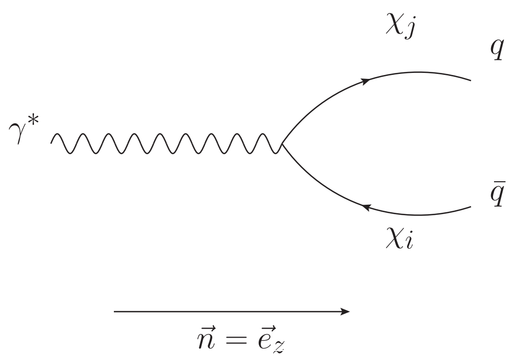

Figure 2: Scheme of the γ^* to $q\bar{q}$ splitting

2.1 Longitudinal polarization

For longitudinally polarized photons the polarization vector goes along the photon propagation and so the system looks like

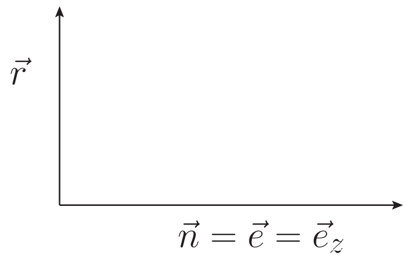

Figure 3: Coordinate system for longitudinally polarized photon

and the term

$$\chi_j^\dagger 2Qz(1-z)(\vec{\sigma}.\vec{n})\chi_i K_0(\varepsilon r) = 2Qz(1-z)\chi_j^\dagger(\vec{\sigma}.\vec{e}_z)\chi_i K_0(\varepsilon r) \tag{6}$$

simplifies by using the fact that $\vec{e}_z = (0,0,1)$ to

$$2Qz(1-z)\chi_j^\dagger \sigma_3 \chi_i K_0(\varepsilon r). \tag{7}$$

Now, the the square of the wave function means that we have to sum over spinors of quark and antiquark and over possible flavours as

$$\begin{aligned}
|\psi^L_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)|^2 &= \sum_{i,j,f} \psi^{L*}_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)\psi^{L}_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2) \\
&= \sum_{i,j,f} \frac{N_c\alpha_{em}}{(2\pi)^2} Z_f^2 4Q^2 z^2(1-z)^2 (\chi_j^\dagger\sigma_3\chi_i)^*(\chi_j^\dagger\sigma_3\chi_i)K_0^2(\varepsilon r) \\
&= \sum_{i,j,f} \frac{N_c\alpha_{em}}{(2\pi)^2} Z_f^2 4Q^2 z^2(1-z)^2 (\chi_i^*\sigma_3^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_3\chi_i)K_0^2(\varepsilon r)
\end{aligned} \tag{8}$$

Now, we use the fact that the spinors are normalized to one and that the square of any Pauli matrix is unit operator

$$\sum_i \chi_i^*\chi_i = \mathbb{1} \qquad \sum_j \chi_j^{\dagger *}\chi_j^\dagger = \mathbb{1} \qquad \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1} \tag{9}$$

to get

$$\begin{aligned}
|\psi^L_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)|^2 &= \sum_{i,j,f} \frac{N_c\alpha_{em}}{(2\pi)^2} Z_f^2 4Q^2 z^2(1-z)^2 (\chi_i^*\sigma_3^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_3\chi_i)K_0^2(\varepsilon r) \\
&= \sum_{f} \frac{N_c\alpha_{em}}{(2\pi)^2} Z_f^2 4Q^2 z^2(1-z)^2 K_0^2(\varepsilon r)
\end{aligned} \tag{10}$$

2.2 Transversal polarization

For transversally polarized photons the polarization vector goes perpendicular to the photon propagation and so the system looks like

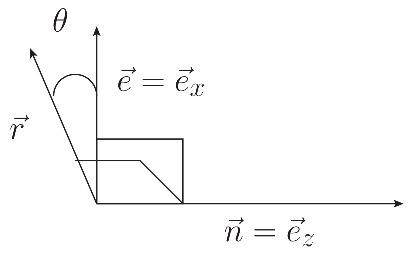

Figure 4: Coordinate system for transversally polarized photon

and the term

$$\chi_j^\dagger m_f(\vec{\sigma}.\vec{e})\chi_i K_0(\varepsilon r) + \chi_j^\dagger i(1-2z)(\vec{\sigma}.\vec{n})(\vec{e}.\vec{\nabla}_r)\chi_i K_0(\varepsilon r) + \chi_j^\dagger(\vec{\sigma}\times\vec{e})\vec{\nabla}_r\chi_i K_0(\varepsilon r) \tag{11}$$

separates into three terms

First term

$$\chi_j^\dagger m_f(\vec{\sigma}.\vec{e})\chi_i K_0(\varepsilon r) = \chi_j^\dagger m_f \sigma_1 \chi_i K_0(\varepsilon r) \tag{12}$$

Second term

$$\begin{aligned} \chi_j^\dagger i(1-2z)(\vec{\sigma}.\vec{n})(\vec{e}.\vec{\nabla}_r)\chi_i K_0(\varepsilon r) &= i(1-2z)\chi_j^\dagger(\vec{\sigma}.\vec{e}_z)(\vec{e}_x.\vec{\nabla}_r)\chi_i K_0(\varepsilon r) \\ &= i(1-2z)\chi_j^\dagger \sigma_3(\vec{e}_x.\vec{\nabla}_r)\chi_i K_0(\varepsilon r) \end{aligned} \tag{13}$$

now we can use the fact that spinors are not dependent on r and K_0 is a scalar function so we can exchange them and get rid of ∇ using

$$\vec{\nabla}_r K_0(\varepsilon r) = (-\varepsilon)\vec{n}_r K_1(\varepsilon r) \tag{14}$$

to get

$$-i\varepsilon(1-2z)\chi_j^\dagger\sigma_3(\vec{e}_x.\vec{n}_r)\chi_i K_1(\varepsilon r) = -i\varepsilon(1-2z)\chi_j^\dagger\sigma_3 cos\theta\chi_i K_1(\varepsilon r) \tag{15}$$

Third term

$$\chi_j^\dagger(\vec{\sigma}\times\vec{e})\vec{\nabla}_r\chi_i K_0(\varepsilon r) \quad = \quad -\varepsilon\chi_j^\dagger(\vec{\sigma}\times\vec{e}_x)\vec{n}_r\chi_i K_1(\varepsilon r) \tag{16}$$

now we use the fact that $\vec{a}.(\vec{b}\times\vec{c}) = \vec{c}.(\vec{a}\times\vec{b}) = \vec{b}.(\vec{c}\times\vec{a})$ to rewrite the term

$$\begin{aligned}
-\varepsilon\chi_j^\dagger(\vec{\sigma}\times\vec{e}_x)\vec{n}_r\chi_i K_1(\varepsilon r) &= -\varepsilon\chi_j^\dagger\vec{\sigma}(\vec{e}_x\times\vec{n}_r)\chi_i K_1(\varepsilon r) \\
&= -\varepsilon\chi_j^\dagger\vec{\sigma}\vec{e}_z\chi_i K_1(\varepsilon r) \\
&= -\varepsilon\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r)
\end{aligned} \tag{17}$$

Now, the the square of the wave function means that we have to sum over spinors of quark and antiquark and over possible flavours as

$$\begin{aligned}
&|\psi^T_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)|^2 = \sum_{i,j,f}\psi^{T*}_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)\psi^{T}_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2) \\
&= \sum_{i,j,f}\frac{N_c\alpha_{em}}{(2\pi)^2}Z_f^2(m_f\chi_j^\dagger\sigma_1\chi_i K_0(\varepsilon r) - \varepsilon\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r) - i\varepsilon\cos\theta(1-2z)\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r))^2 \\
&= \sum_{i,j,f}\frac{N_c\alpha_{em}}{(2\pi)^2}Z_f^2[(m_f\chi_j^\dagger\sigma_1\chi_i K_0(\varepsilon r) - \varepsilon\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r))^*(m_f\chi_j^\dagger\sigma_1\chi_i K_0(\varepsilon r) - \varepsilon\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r)) \\
&+(\varepsilon\cos\theta(1-2z)\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r))^*(\varepsilon\cos\theta(1-2z)\chi_j^\dagger\sigma_3\chi_i K_1(\varepsilon r))] \\
&= \sum_{i,j,f}\frac{N_c\alpha_{em}}{(2\pi)^2}Z_f^2(m_f^2\chi_i^*\sigma_1^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_1\chi_i K_0^2(\varepsilon r) - m_f\varepsilon\chi_i^*\sigma_1^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_3\chi_i K_0(\varepsilon r)K_1(\varepsilon r) \\
&-m_f\varepsilon\chi_i^*\sigma_3^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_1\chi_i K_0(\varepsilon r)K_1(\varepsilon r) + \varepsilon^2\chi_i^*\sigma_3^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_3\chi_i K_1^2(\varepsilon r) \\
&+\varepsilon^2\cos^2\theta(1-2z)^2\chi_i^*\sigma_3^*\chi_j^{\dagger *}\chi_j^\dagger\sigma_3\chi_i K_1^2(\varepsilon r))
\end{aligned} \tag{18}$$

Due to the fact that $\sigma_3^*\sigma_1 = -\sigma_1^*\sigma_3$ the second and the third term cancels

$$|\psi^T_{\gamma^*\to q\bar{q}}(\vec{r},z,Q^2)|^2 = \sum_{i,j,f}\frac{N_c\alpha_{em}}{(2\pi)^2}Z_f^2\left(m_f^2K_0^2(\varepsilon r) + \varepsilon^2K_1^2(\varepsilon r) + \varepsilon^2\cos^2\theta(1-2z)^2K_1^2(\varepsilon r)\right) \tag{19}$$

which resembles the correct formula. One note, the angle between $\vec{n}_r$ and $\vec{e}_x$ is to be specified. However, we have the freedom to choose the orientation of the dipole w.r.t the plane $\vec{e}_x\times\vec{e}_z$. It is usually chosen to be in the plane so that $\theta = 0$.

3 Alternative definition

Instead of formulating the vertex in mixed representation, it is more clear to formulate the vertex in momentum representation (conventional QED) and then use the Fourier transformation

$$\psi_{h\bar{h}}(\vec{r}, z, Q^2) = \int \frac{\mathrm{d}k}{(2\pi)^2} e^{i\vec{k}\vec{r}} \psi_{h,\bar{h}}(\vec{k}, z, Q^2), \tag{20}$$

where the momentum space light-cone wave function $\psi(\vec{,}k, z, Q^2)$ in the lowest order of QED reads [1, 2]

$$\psi^{\lambda}_{h,\bar{h}}(\vec{k}, z, Q^2) = \sqrt{N_c \alpha_{em}} \frac{\bar{u}_h(\vec{k})}{\sqrt{z}} (Z_f \gamma \varepsilon^{\lambda}) \frac{v_h(-\vec{k})}{\sqrt{1-z}} \Phi(k, z), \tag{21}$$

where the scalar part of the photon light-cone wave function $\Phi(k, z)$ is given by

$$\Phi(k, z) = \frac{z(1-z)}{z(1-z)Q^2 + k^2 + m_f^2} \tag{22}$$

Performing the Fourier transformation leads to

$$\begin{aligned}
\Psi^{\gamma^*}_{h\bar{h}\lambda=0}(r, z, Q) &= e_f \delta_{f\bar{f}} e \sqrt{N_c} \delta_{h,-\bar{h}} 2Qz(1-z) \frac{K_0(\epsilon r)}{2\pi} \qquad (23) \\
\Psi^{\gamma^*}_{h\bar{h}\lambda=\pm1}(r, z, Q) &= \pm e_f \delta_{f\bar{f}} e \sqrt{2N_c} \left(i e^{\pm i\theta_r} \left(z\delta_{h,\pm1}\delta_{\bar{h},\mp1} - (1-z)\delta_{h,\mp1}\delta_{\bar{h},\pm1} \right) \partial_r + m_f \delta_{h,\pm1}\delta_{\bar{h},\pm1} \right) \frac{K_0(\epsilon r)}{2\pi},
\end{aligned}$$

where $e = \sqrt{4\pi\alpha_{em}}$, h, θ_r is the azimuthal angle between the vector $\vec{r}$ and the x-axis in the transverse plane, $\epsilon^2 = z(1-z)Q^2 + m_f^2$, $N_c = 3$ is the number of colors, $e_f \delta_{f\bar{f}}$ and m_f are the fractional charge and effective mass of the quark respectively. The partial derivative of the modified Bessel function K_0 with respect to r can be done using the equation $\partial_r K_0(\epsilon r) = -\epsilon K_1(\epsilon r)$.

References

[1] H. G. Dosch, T. Gousset, G. Kulzinger, and H. Pirner, Phys.Rev. **D55**, 2602 (1997), arXiv:hep-ph/9608203.

[2] G. P. Lepage and S. J. Brodsky, Phys.Rev. **D22**, 2157 (1980).