Collective measurement and system disturbance

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Theoretical frame

- Theoretical background for our work mainly theory of quantum measurements and information theory
- We start with von Neumann (projective) measurement, then generalize to Positive operator-valued measure (POVM)
- From information theory we use mostly the concept of informational entropy

Theoretical frame

Projectors, measurement operators of the von Neumann measurement, are defined as follows:

$$\hat{P}_n = |\lambda\rangle_n \langle\lambda|_n = |n\rangle \langle n| \,, \tag{1}$$

where $\{|\lambda\rangle_n\}$ form an orthonormal basis on our $\mathcal{H}$. Any projector $\hat{P}_n$ need to satisfy the following:

- → completeness relation: $\sum_n \hat{P}_n = \hat{I}$
- → $\hat{P}_m \hat{P}_n = \delta_{mn} \hat{P}_n$
- → hermiticity: $\hat{P}_n^\dagger = \hat{P}_n$

Theoretical frame

We switch to a more convenient representation of a state, to density matrix:

$$\hat{\rho} = \sum_i P(i) \left|\lambda\right\rangle_i \left\langle\lambda\right|_i . \quad (2)$$

The probability of measuring the n-th outcome can then be written as:

$$P(\lambda_n) = P(n) = \langle\lambda_n|\hat{\rho}|\lambda_n\rangle = \mathrm{Tr}(\hat{\rho}\hat{P}_n). \quad (3)$$

Acting with $\hat{P}_n$ on $\hat{\rho}$ creates a post-measurement state:

$$\hat{\rho} \to \hat{\rho}'_n = \frac{\hat{P}_n\hat{\rho}}{\mathrm{Tr}(\hat{P}_n\hat{\rho})} = \frac{\hat{P}_n\hat{\rho}}{P(n)} \quad (4)$$

Theoretical frame

Generalizing a quantum measurement $\Leftrightarrow$ generalizing the measurement operators:

$$\hat{P}_n \to \hat{M}_n, \tag{5}$$

where $\hat{M}_n$ must satisfy the completeness relation. The equations for calculating probability and post-measurement state remain in the same form.

POVM, or positive-operator-valued measure is then defined by following operators:

$$\hat{\Pi}_n \equiv \hat{M}_n^{\dagger}\hat{M}_n \tag{6}$$

Theoretical frame

The main differences are the following:

- $\rightarrow$ number of elements in a POVM set $\neq$ dimension of our $\mathcal{H}$
- $\rightarrow$ expectational value of $\hat{\Pi}_n$ gives probability of measuring n-th outcome
- $\rightarrow$ any POVM set is utilisable for description of any generalized measurement

Introducing POVM allows for non-idealized measurement with noise distortion. If m are real outcomes and n outcomes of idealized measurement, then $P(m|n)$ represents the source of distortion.

Theoretical frame

In order to correctly define the concepts of noise and disturbance, we need the information entropy $H(X)$ which is, for a random discrete variable X, defined as:

$$H(X) = -\sum_{x \in \mathfrak{X}} P(X = x) \log_2 P(X = x), \quad (7)$$

where x are realizations of X.

We also need the concept of discreet channel, which Shannon[1] defined as any stochastic process producing discrete sequence of symbols $X \in \mathfrak{X}$.

Defining noise and disturbance

One usually tends to use the following definition when addressing measurement disturbance:

$$\Delta x \Delta p \geq \frac{\hbar}{2}, \tag{8}$$

which represent the limit of knowledge precision of two non-commuting variables. To include what we call the observer effect, we need the mathematically correct reformulation by Ozawa[2]:

$$\epsilon_A \eta_B \geq \frac{|\langle\psi|[\hat{A}, \hat{B}]|\psi\rangle|}{2} \tag{9}$$

Both ϵ and η are root-mean-square deviations of our system observables.

Defining noise and disturbance

We mainly used definitions and notations from Buscemi, Francesco, et al.[3] for our calculation. They defined both noise and disturbance using information entropy as follows:

- we have a quantum system S with observables $\hat{A}$ and $\hat{B}$, whose set of eigenvalues are denoted as $\{|\psi\rangle_a\}$ and $\{|\phi\rangle_b\}$, respectively
- we subject S to measurement M, which yields outcomes m
- m is then compared to eigenvalues of measured observables, we denote them as $\{a\}$ and $\{b\}$
- correlations between $\{a\}$ and m can be expressed using conditional probability distribution $P(a|m)$ for noise, meaning we can write:

$$N(M, \hat{A}) \equiv H(\hat{A}|M) \tag{10}$$

Defining noise and disturbance

- situation is more complicated for disturbance, since we can ”correct” it by post-processing
- such operation yields ”guesses” denoted as $\{b'\}$
- quantifying disturbance then means finding $P(b|b')$
- after that, we can write:

$$D(M, \hat{B}) \equiv H(B|B') \tag{11}$$

- using previous notation and results, we can write a noise-disturbance relation defined for any measurement M and any $\hat{A}, \hat{B}$ as follows:

$$N(M, \hat{A}) + D(M, \hat{B}) \geq -\log_2 \max_{a,b} |\langle\psi_a|\phi_b\rangle|^2 \tag{12}$$

Reducing disturbance without addressing specific particles

In our work, we first showed how to construct a measurement from $\{\hat{P}_0, \hat{P}_1\}$ on a system of three-identical spin-1/2 particles, described by a following state:

$$|\phi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \,, \tag{13}$$

where $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,\ \alpha, \beta \in \mathbb{C}$. We required particle indistinguishability, which gave us three options for measurement construction:

1. tensor product of respective projectors only
2. tensor product of respective projectors with identity operator
3. projector weakening

Reducing disturbance without addressing specific particles

Since we chose to measure the total spin of our system, we made the following choice for the observables $\hat{A}$ and $\hat{B}$:

$$\hat{A} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{B} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

with respective projectors:

$$\begin{aligned} \hat{P}_+ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad & \hat{P}_- &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \hat{P}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad & \hat{P}_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{15}$$

We then chose the input state:

$$|\phi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \, . \tag{16}$$

Reducing disturbance without addressing specific particles

- the first type of measurement, tensor product of respective projectors only, requires the post-processing we mentioned earlier
- we use a discrete channel, depicted right
- this then yield us $P(b|b') \implies D_1(M, \hat{B}) = 1\,\text{bit}$

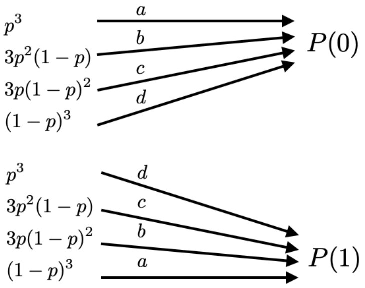

Figure: Depiction of the channel that maps spin outcomes $m \in M$ to either $P(0)$ or $P(1)$.

Reducing disturbance without addressing specific particles

The second type of measurement is more convenient – it does not require post-processing and reduces disturbance:

$$D_2(M, \hat{B}) = 0.65\,\mathrm{bit}. \tag{17}$$

The most interesting case is the last one – projector weakening:

$$\begin{aligned} \hat{A}_0 &= \mu\hat{I} + \nu\hat{P}_0 \\ \hat{A}_1 &= \mu\hat{I} + \nu\hat{P}_1, \end{aligned} \tag{18}$$

where $2\mu^2 + 2\mu\nu + \nu^2 = 1$, which we know from the completeness relation for POVM set.

Reducing disturbance without addressing specific particles

- because of the measurement construction, we again need a post-processing channel
- sadly, no real channel, which yields outcomes with the same probability as the tensor product of projectors only, exists
- this fact forced us to make several adjustments

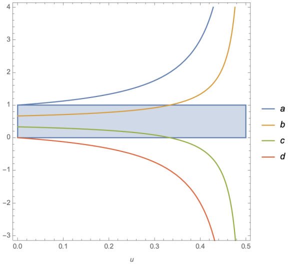

Figure: The blue frame depicts the values of $a, b, c, d \in \langle 0, 1 \rangle$ where they can be considered as a description of an information channel.

Reducing disturbance without addressing specific particles

We made the following substitutions to reduce the length of our calculations: $u = \mu^2$, $v = \nu^2 + 2\mu\nu$, and plotted $D_3 = D_3(u)$:

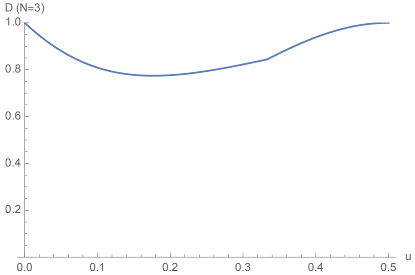

Figure: Disturbance for the three-particle case as a function of u.

N-particle system disturbance

We then made the previous calculation for N-particle system and calculated disturbance for $N \to \infty$:

- $\to$ the first case remained unchanged in terms of disturbance
- $\to$ the second case:

$$D(M, \hat{B}_j^{(1,2,\dots,N)}) = H\left(\frac{2N-1}{2N}, \frac{1}{2N}\right) \tag{19}$$

which means for $N \to \infty \implies D \to 0$, which is exactly what we expected

N-particle system disturbance

Disturbance for the third-case of projector-weakening was again plotted against u:

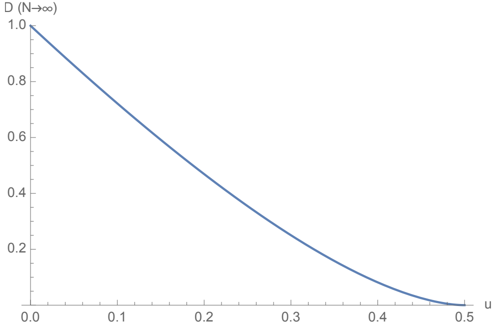

Figure: The N-particle system disturbance as a function of u.

Sources

- Shannon, Claude Elwood. A mathematical theory of communication. Bell system technical journal 27.3 (1948): 379-423.
- Ozawa, M. ”Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement.” Physical Review A, 67(4), 042105.
- JBuscemi, Francesco, et al. ”Noise and disturbance in quantum measurements: an information-theoretic approach.” Physical review letters 112.5 (2014): 050401.