Vortex solutions of Liouville equation and quasi spherical surfaces[1]

11$^{\text{th}}$ CSSSC

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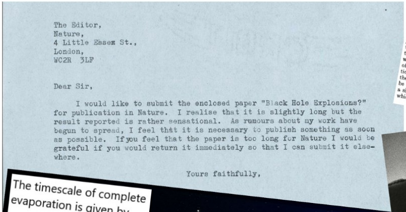
[1] Based on Iorio A., **Kůs P.**, Int. J. Geom. Meth. Mod. Phys. (2020) 2050106

Analog gravity

Black holes are (almost surely) evaporating...

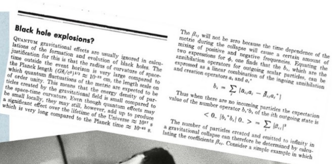


The Editor,
Nature,
4 Little Essex St.,
London,
WC2R 3LF

Dear Sir,

I would like to submit the enclosed paper "Black Hole Explosions?" for publication in Nature. I realise that it is slightly long but the result reported is rather sensational. As rumours about my work have begun to spread, I feel that it is necessary to publish something as soon as possible. If you feel that the paper is too long for Nature I would be grateful if you would return it immediately so that I can submit it elsewhere.

Yours faithfully,



Black hole explosions?

QUANTUM gravitational effects are usually ignored in calculations of the formation and evolution of black holes. The justification for this is that the radius of curvature of space-time outside the event horizon is very large compared to the Planck length $(G\hbar/c^3)^{1/2} \approx 10^{-33}$ cm, the length scale on which quantum fluctuations of the metric are expected to be of order unity. This means that the energy density of particles created by the gravitational field is small compared to the space-time curvature. Even though quantum effects may be small locally, they may still, however, add up to produce a significant effect over the lifetime of the Universe $\approx 10^{17}$ s which is very long compared to the Planck time $\approx 10^{-43}$ s.

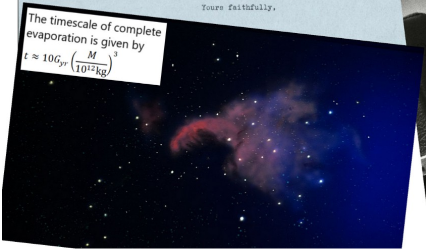
The β_{ij} will not be zero because the time dependence of the metric during the collapse will cause a certain amount of mixing of positive and negative frequencies. Equating the two expressions for ϕ, one finds that the b_i, which are the annihilation operators for outgoing scalar particles, can be expressed as a linear combination of the ingoing annihilation and creation operators a_i and a_i^+

$$b_i = \sum_j \{\bar{\alpha}_{ij} a_j - \bar{\beta}_{ij} a_j^+\}$$

Thus when there are no incoming particles the expectation value of the number operator $b_i^+ b_i$ of the ith outgoing state is

$$< 0_- | b_i^+ b_i | 0_- > = \sum_j |\beta_{ij}|^2$$

The number of particles created and emitted to infinity in a gravitational collapse can therefore be determined by calculating the coefficients β_{ij}. Consider a simple example in which

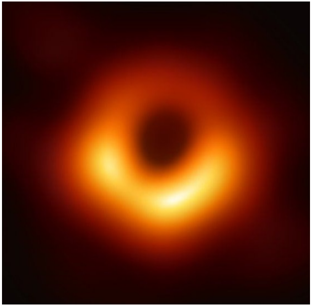


The timescale of complete evaporation is given by

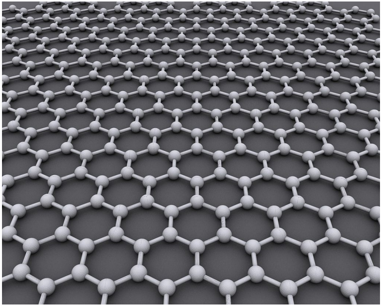
$t \approx 10 G_{yr} \left(\frac{M}{10^{12}\mathrm{kg}}\right)^3$

An example

Q: „*Is it possible to mimic key features of black holes in lab?*“
A: *Yes, in graphene!* [1] [2]



(a) Black hole



(b) Graphene

Note: Graphene is not the only analog. Hawking radiation has been observed in Bose-Einstein condensates, see e.g. [3]

Graphene membrane as a 2+1 dim spacetime

- Table top relativistic-like system described by $g^{(3)}_{\mu\nu}$
- Assumptions about $g^{(3)}_{\mu\nu}$:

  i) flat in time, $g^{(3)}_{ij}$ belongs to a surface

$$g^{(3)}_{\mu\nu}(t,\tilde{x},\tilde{y}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\phi^2(\tilde{x},\tilde{y}) & 0 \\ 0 & 0 & -\phi^2(\tilde{x},\tilde{y}) \end{pmatrix} \tag{1}$$

  ii) conformally flat $\leftrightarrow$ Cotton tensor vanishes $C_{\mu\nu} = 0$

$\rightarrow$ Constraints on the shape of the graphene membrane!

$$\Delta_{(\tilde{x},\tilde{y})} \ln \phi(\tilde{x},\tilde{y}) = -K\phi^2(\tilde{x},\tilde{y}) \tag{2}$$

with Gaussian curvature K to be a constant!

General solution of the Liouville equation

- Liouville equation

$$\Delta_{(\tilde{x},\tilde{y})} \ln \phi(\tilde{x},\tilde{y}) = -K\phi^2(\tilde{x},\tilde{y}) \tag{3}$$

- Mapping to the complex plane $z = \tilde{x} + i\tilde{y}$, $\bar{z} = \tilde{x} - i\tilde{y}$

$$\Delta_{(z,\bar{z})} \ln \phi(z,\bar{z}) = -K\phi^2(z,\bar{z}) \tag{4}$$

- General solution [4]

$$\phi(z) = \frac{2}{\sqrt{|K|}} \frac{|f'(z)|}{1 \pm |f(z)|^2} \tag{5}$$

$f(z)$ arbitrary meromorphic function: $df/dz \neq 0$, $f(z)$ has at most simple poles
$\rightarrow$ **infinite family of solutions**!

Where does my study begin?

- Vortices become interesting for graphene [1]
- Specific vortex solutions proposed in [1]:

$$f(z) = z^{-N}, \quad \phi_+(\tilde{r}) = \frac{2N}{\sqrt{K}} \frac{\tilde{r}^{N-1}}{\tilde{r}^{2N}+1} \tag{6}$$

$\rightarrow$ local geometry:

$$dl^2 = \phi_+^2(\tilde{r})(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) \tag{7}$$

where $\tilde{r} = |z| = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, $\tilde{\theta} \in [0, 2\pi]$, N **natural number**

- named "non-topological vortex", or "H-Y vortex" ($K > 0$; Horváthy, Yéra [5])

Our tasks

- **Tasks to solve** (proposed in [1])
  1. **Find surfaces corresponding to H-Y vortices**

$$(\tilde{r}, \tilde{\theta}) \to (x, y, z(x, y))$$

  2. **Describe** associated **spacetime**
  3. **Generalization to pseudospheres** $K < 0$ (if possible)

Problem to solve...

- Coordinates transformation:

$$(\tilde{r}, \tilde{\theta}) \to (x, y, z(x, y))$$

$\to$ **a set of non-linear PDEs**:

$$\phi_+^2(\tilde{r}) = \left(\frac{\partial x}{\partial \tilde{r}}\right)^2 + \left(\frac{\partial y}{\partial \tilde{r}}\right)^2 + \left(\frac{\partial z}{\partial \tilde{r}}\right)^2 \quad (8)$$

$$\phi_+^2(\tilde{r})\tilde{r}^2 = \left(\frac{\partial x}{\partial \tilde{\theta}}\right)^2 + \left(\frac{\partial y}{\partial \tilde{\theta}}\right)^2 + \left(\frac{\partial z}{\partial \tilde{\theta}}\right)^2 \quad (9)$$

$$0 = \frac{\partial x}{\partial \tilde{r}}\frac{\partial x}{\partial \tilde{\theta}} + \frac{\partial y}{\partial \tilde{r}}\frac{\partial y}{\partial \tilde{\theta}} + \frac{\partial z}{\partial \tilde{r}}\frac{\partial z}{\partial \tilde{\theta}} \quad (10)$$

...and its solution

- Ansatz:

$$x = R_+(\tilde{r})\cos\tilde{\theta},\ \ y = R_+(\tilde{r})\sin\tilde{\theta},\ \ z = z_+(\tilde{r}) \qquad (11)$$

→ Radial function

$$R_+(\tilde{r}) = \tilde{r}\phi_+(\tilde{r}) = \frac{2N}{\sqrt{K}}\frac{\tilde{r}^N}{\tilde{r}^{2N}+1} \qquad (12)$$

→ z-coordinate

$$z_+ = \int\sqrt{\phi_+^2(\tilde{r}) - [R_+'(\tilde{r})]^2}\,d\tilde{r} \qquad (13)$$

→ analytic expression exists, but is complicated

...and its solution

- $\tilde{r}$-range

$$\frac{4N^2}{K}\frac{\tilde{r}^{2N-2}}{(\tilde{r}^{2N}+1)^2}-\frac{4N^4}{K}\left[\frac{\tilde{r}^{N-1}(1-\tilde{r}^{2N})}{(\tilde{r}^{2N}+1)^2}\right]^2\geq 0 \qquad (14)$$

$\rightarrow$ $N=1$

$$\tilde{r}_{\min}=0,\quad \tilde{r}_{\max}=+\infty \qquad (15)$$

$\rightarrow$ $N\geq 2$

$$\tilde{r}_{\min}=\sqrt[2N]{\frac{N-1}{N+1}},\quad \tilde{r}_{\max}=\sqrt[2N]{\frac{N+1}{N-1}} \qquad (16)$$

Bulge surfaces and N

- Next coordinates transformation

$$dl^2 = \frac{4N^2}{K}\frac{\tilde{r}^{2(N-1)}}{(\tilde{r}^{2N}+1)^2}(d\tilde{r}^2+\tilde{r}^2 d\tilde{\theta}^2) = du^2 + a^2N^2\cos^2\frac{u}{a}d\tilde{\theta} \tag{17}$$

where

$$K = 1/a^2$$

$$u = 2a\arctan e^{\tilde{r}^N} - \pi/2$$

$$u \in [-\arcsin(1/N), \arcsin(1/N)]$$

- **Surfaces of revolution** with $K > 0$

$$\mathrm{d}l^2 = \mathrm{d}u^2 + R^2(u)\mathrm{d}v^2, R(u) = c\cos\frac{u}{a} \tag{18}$$

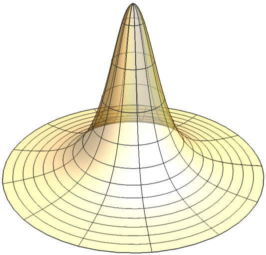
→ Interpretation of N

$$c = aN \tag{19}$$

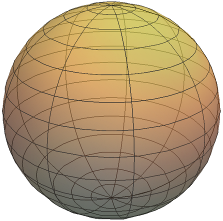
→ **Singularities match with Bulge's!**

Sphere and $\phi_+^2(\tilde{r})$ plots

$N = 1$



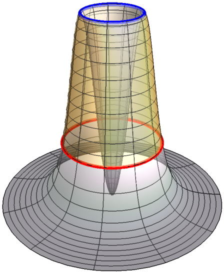
(c) $\phi^2(\tilde{r})$



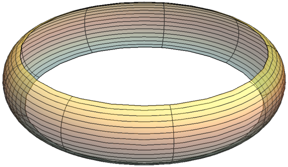
(d) Sphere

Bulge surfaces and $\phi_+^2(\tilde{r})$ plots

$N = 2$



(e) $\phi^2(\tilde{r})$



(f) Bulge

$\rightarrow$ Only part of $\phi_+^2(\tilde{r})$ is realized (holds for $N \geq 2$)!

Surfaces with constant $K < 0$

- Coorinates transformation:

$$dl^2 = \frac{4N^2}{|K|}\frac{\tilde{r}^{2(N-1)}}{(\tilde{r}^{2N}-1)^2}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2) = du^2 + a^2N^2\sinh^2\frac{u}{a}d\tilde{\theta}^2 \tag{20}$$

where $u = 2a\,\mathrm{arctanh}\, e^{\tilde{r}^N}$, $\tilde{\theta} \in [0, 2\pi]$

- local geometry of the **elliptical pseudosphere**:

$$R(u) = c\sinh\frac{u}{a} \tag{21}$$

with $a > c > 0$

$\rightarrow$ local geometry corresponds to elliptic pseudosphere, but a **surface does not exist**!

Spacetime associated to H-Y vortex solution

- Spacetime element

$$ds^2 = dt^2 - \frac{4N^2}{K}\frac{\tilde{r}^{2(N-1)}}{(1+\tilde{r}^{2N})^2}d\tilde{r}^2 - \frac{4N^2}{K}\frac{\tilde{r}^{2N}}{(1+\tilde{r}^{2N})^2}d\tilde{\theta}^2 \quad (22)$$

→ Suitable coordinates transformation leads to:

$$ds^2 = dt^2 - \frac{dR_+^2}{1-\frac{K}{N^2}R_+^2}\frac{1}{N^2} - R_+^2 d\tilde{\theta}^2 \quad (23)$$

→ Compare with the **Einstein static universe** (FLRW metric)

$$ds^2 = dt^2 - \frac{dr^2}{1-\frac{k}{A^2}r^2} - r^2 d\tilde{\Omega}^2\,, \quad (24)$$

Conclusions

1. H-Y vortices $\rightarrow$ Sphere ($N = 1$), Bulge surfaces ($N \geq 2$)
2. N = scale factor
4. No pseudospheres with H-Y geometry exist
5. Graphene with H-Y geometry $\rightarrow$ Einstein static universe

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- [2] Curved spacetimes and curved graphene: A status report of the Weyl symmetry approach, Iorio A., Int. J. Mod. Phys. D 4 (2015) 1530013, arXiv:1412.4554
- [3] Observation of thermal Hawking radiation and its temperature in an analogue black hole, Steinhauer J. et al., Nature 569 (2019) 688–691
- [4] Sur l'équation aux différences partielles $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$, Liouville J., Journal de mathématiques pures et appliquées 1re série, tome 18 (1853), p. 71-72.
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- [6] The electronic properties of graphene, Castro Neto A.H. & col., Reviews of modern physics, volume 81, January–March 2009

Thank you for your attention!

Graphene, relativistic?

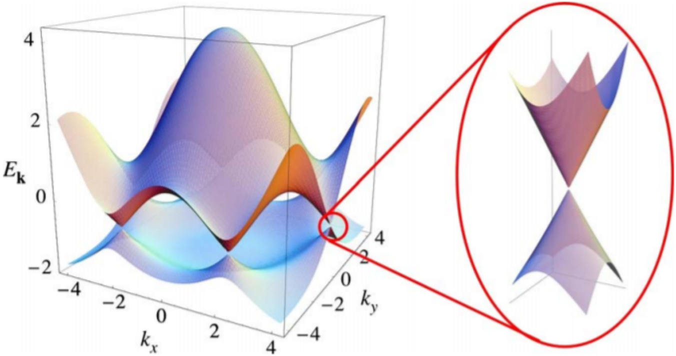

Figure: Band structure of graphene [6]