Estimates of Similarity Dimension via Diffusion over Compact Subsets of Regular Grids

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Outline

- Theory of noninteger dimensional spaces and diffusion
- Estimate of dimension
- Discrete fractal model
- Implementation
- Simulation study results

Diffusion in the Space of noninteger dimension

- Adopted noninteger dimensional space $\mathcal{S}_d$ definition from *Stillinger, Frank H., Axiomatic basis for spaces with noninteger dimension*
- Unbounded metric space, dense in itself with defined integration measure
- Redefines Laplace operator
- Allows integrals of radially symmetric functions to be computed:

$$\int_{\mathcal{S}_d} f\big(\varrho(x, x_0)\big)\,\mathrm{d}x = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} f(r)\; r^{d-1}\;\mathrm{d}r$$

Diffusion equation and solution

$$\frac{\partial c(r,t)}{\partial t} = D\;\nabla^2 c(r,t)$$

$$c(r,t) = \big(4\pi\,Dt\big)^{-d/2} \exp\left(-\frac{r^2}{4\,Dt}\right)$$

Moment based invariant

Absolute noninteger moments

$$M_\alpha = \mathrm{E}\, r^\alpha = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} r^{d-1+\alpha} c(r,t)\,\mathrm{d}r, \quad \alpha \in \mathbb{R}^+$$

Time and diffusion coefficient invariant

$$I_{\alpha,\beta} = \frac{M_{\alpha+\beta}}{M_\alpha\, M_\beta} = \frac{\Gamma\left(\frac{d}{2}\right)\ \Gamma\left(\frac{d+\alpha+\beta}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)\ \Gamma\left(\frac{d+\beta}{2}\right)}$$

- Moment estimates can be easily calculated from random sample.
- Invariance on diffusion coef. allows us to fully focus on dimension.
- Invariant is understood as an implicitly defined dimension estimate.

Bias and Variance of Dimension estimate

Invariant based Dimension Estimate

$$d_{\alpha,\beta} = \Phi\big(M_\alpha, M_\beta, M_{\alpha+\beta}\big)$$

- Implicitly defined dimension estimate still allows calculation of derivatives
- Bias and Variance can be approximated using Taylor expansion

Bias Estimates

$$n\,\mathrm{Bias}(d_{2,2}) \approx 3\,(2+d)$$

$$n\,\mathrm{Bias}(d_{4,2}) \approx 23 + 3d + \frac{22}{2+d}$$

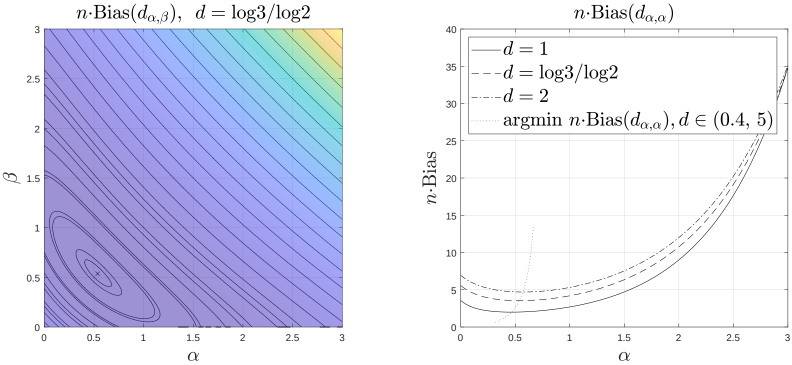
Variance Estimates

$$n\,\mathrm{Var}(d_{2,2}) \approx 2d\,(2+d)$$

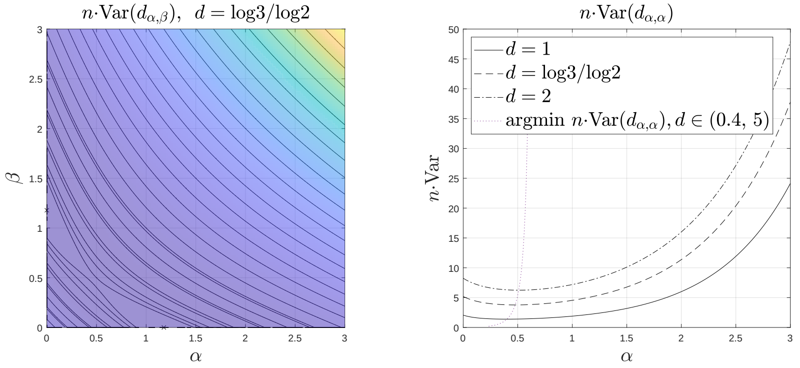
$$n\,\mathrm{Var}(d_{4,2}) \approx \frac{d\,(4+d)(11+2d)}{2+d}$$

- General forms of $\mathrm{Bias}(d_{\alpha,\beta})$ and $\mathrm{Var}(d_{\alpha,\beta})$ are too large to present

Bias Estimate



Variance Estimate



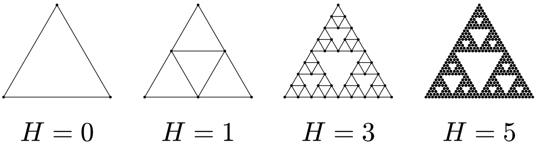
Diffusion over Discrete Fractal Model

Stochastic particle movement

$$\Pr\left(\boldsymbol{x}_{\text{new}} \middle| \boldsymbol{x}\right) = \begin{cases} p, & \{\boldsymbol{x}_{\text{new}}, \boldsymbol{x}\} \in E(\mathcal{F}), \\ 1 - p \deg_{\mathcal{F}} \boldsymbol{x}, & \boldsymbol{x}_{\text{new}} = \boldsymbol{x} \end{cases}$$

- Random starting vertex
- Stochastic particle movement model
- Measured travelled distance after T steps

Sierpinsky gasket as finite compact substrate $\mathcal{F}$



Implementation

Vertexes

- List of coordinates as array `X`.
- Translation vectors `v1`, `v2`.
- New more detailed list: `X = [X, X+v1, X+v2]`
- Remove duplicates.

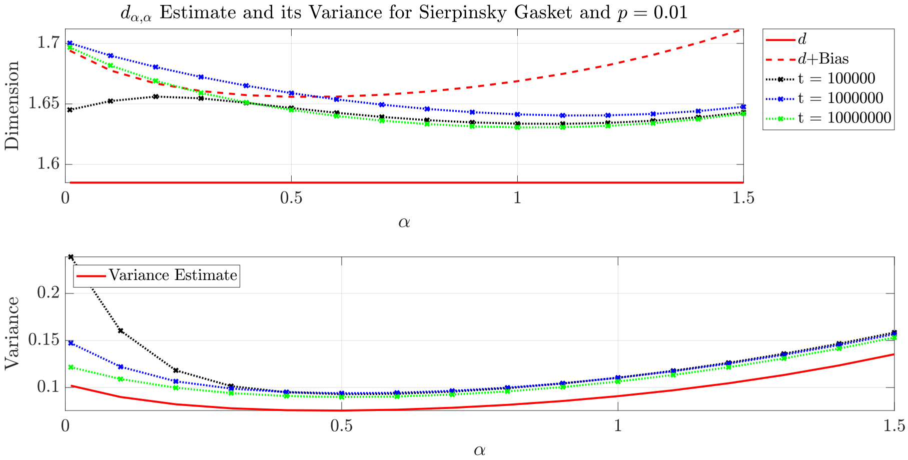
Adjacency

- Raw adjacency matrix size needed: $2^7 \times 2^7$.
- Code using sparsity and low max. vertex degree.
- Saving created structures for more detailed structures.

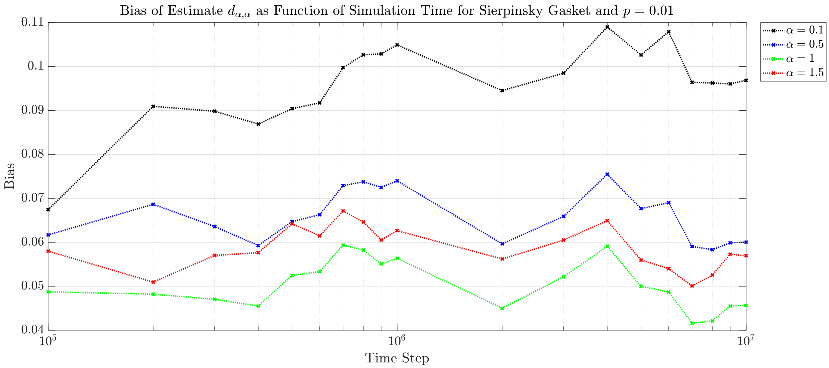
Simulation

- Only using vertex index and adjacency.
- **Warning**: MATLAB switches random generator of `parpool` workers.

Simulation Results



Simulation Results



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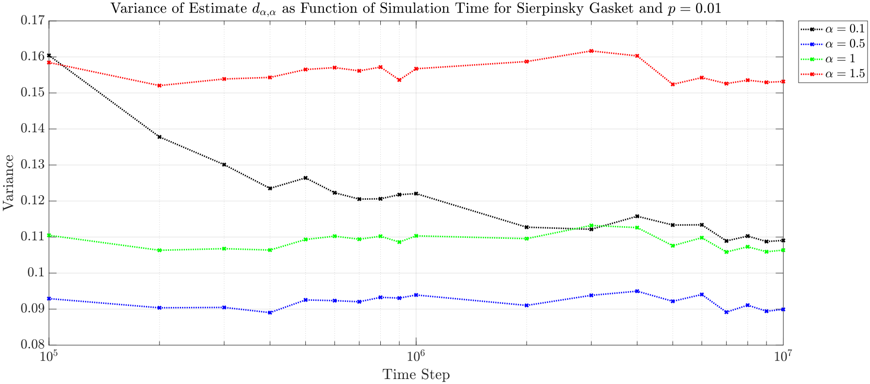

Future Research Focus

- Bias correction of dimension estimate.
- Specify optimal values of dimension estimate order.
- Detailed statistical testing of dimension estimate properties.
- Expand the simulation study to other fractals.
- Explore other diffusion models.

Thank You For Your Attention