Dispersion curves and their measurement

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Outline

Dispersion

Lamb waves and frequency equations

Determination of dispersion curves

Wavenumber and dispersion

Wave number is spatial frequency.

period	T	wavelength	λ
frequency	$f = 1/T$	wavenumber	$\nu = 1/\lambda$
angular frequency	$\omega = 2\pi f$	(angular) **wavenumber**	$k = 2\pi\nu$

Dispersion

Propagation characteristics depends on frequency, $k = k(\omega)$. Wave components traveling at different speeds.

Phase velocity $c_{\mathrm{p}} = \dfrac{\omega}{k}$

Group velocity $c_{\mathrm{g}} = \dfrac{\partial\omega}{\partial k}$

Non-dispersive propagation $k(\omega) = \frac{1}{c}k$, $c_{\mathrm{p}} = c_{\mathrm{g}} = c$

Dispersion

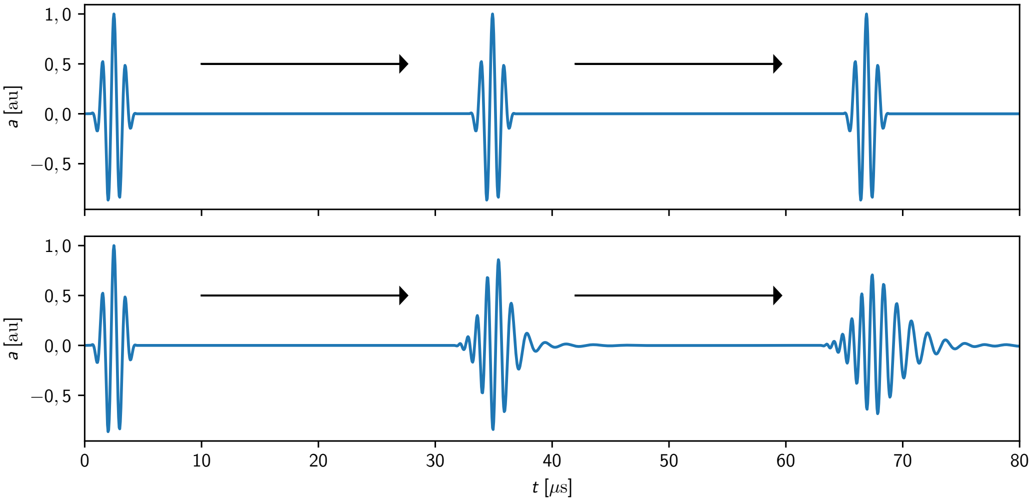

Figure 1: Non-dispersive propagation (top), dispersive propagation (bottom).

Lamb waves

Lamb waves propagate in solid plates or spheres. Superposition of longitudinal and transversal waves refelected by two paralllel surfaces.
Two types of modes – symmetric and antisymmetric. [1]

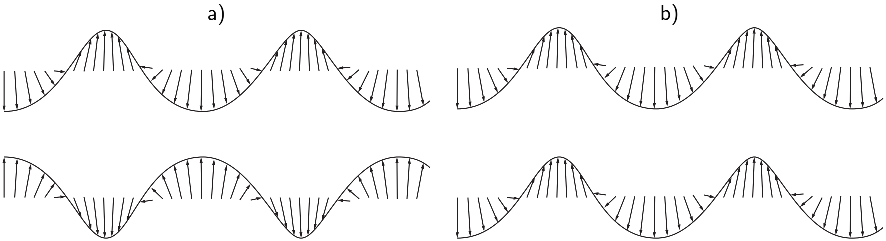

Figure 2: Symmetric (a) and antisymmetric mode (b) .

[1] ROSE, Joseph L. *Ultrasonic guided waves in solid media.* New York: Cambridge University Press, 2014. ISBN 978-1-107-04895-9.

Rayleig–Lamb frequency equations

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{\left(q^2-k^2\right)^2} \quad \text{(symmetric modes)} \tag{1}$$

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{\left(q^2-k^2\right)^2}{4k^2pq} \quad \text{(antisymmetric modes)} \tag{2}$$

$$\text{where} \quad p^2 = \frac{\omega^2}{c_L^2} - k^2 \quad \text{and} \quad q^2 = \frac{\omega^2}{c_T^2} - k^2, \tag{3}$$

h half thickness, c_L longitudinal wave velocity, c_T transversal wave velocity

Rayleig–Lamb frequency equations

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{\left(q^2-k^2\right)^2} \quad \text{(symmetric modes)} \tag{4}$$

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{\left(q^2-k^2\right)^2}{4k^2pq} \quad \text{(antisymmetric modes)} \tag{5}$$

$$\text{where} \quad p^2 = \frac{\omega^2}{c_\mathrm{L}^2} - k^2 \quad \text{and} \quad q^2 = \frac{\omega^2}{c_\mathrm{T}^2} - k^2, \tag{6}$$

Complex solutions

Finite number of real and pure imaginary solutions and infinite number of complex inhomogeneous solutions.

- $\Im k < 0$, wave amplitude increases exponentially with distance (not observed);
- $\Im k > 0$, wave amplitude decreases exponentially with distance (evanescent wave, disappearing quickly);
- $\Im k = 0$ ($k \in \mathbb{R}$), wave propagates without attenuation.

Rayleig–Lamb frequency equations

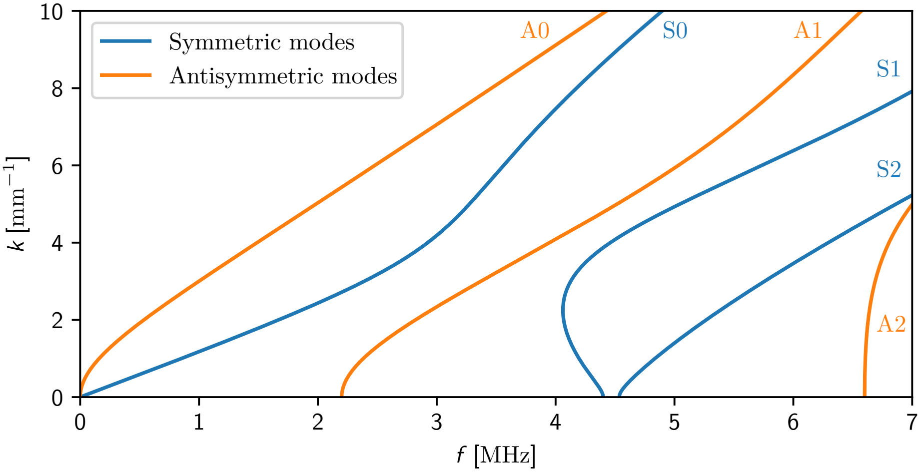

Figure 3: Dispersion curves for Lamb wave modes in aluminium plate, thickeness 0,7 mm. $c_L = 6{,}35\,\mathrm{mm\,\mu s^{-1}}$, $c_T = 3{,}08\,\mathrm{mm\,\mu s^{-1}}$.

Determination of dispersion curves

Temporal Fourier transform converts from the time to frequency domain. Spatial Fourier transform converts from the distance to wavenumber domain. Two-dimensional Fourier tranform is applied on array of wave propagation signals,

$$H(k,\omega) = \int_{\mathbb{R}} \int_{\mathbb{R}} u\left(x,t\right) \exp\left[-i\left(kx+\omega t\right)\right] \mathrm{d}x\mathrm{d}t,$$

the result is two-dimensional array of amplitudes in frequency-wavenumber domain, where the dispersion curves may be distinguished.[2]

[2] ALLEYNE, D. a P. CAWLEY. A two-dimensional Fourier transform method for the measurement of propagating multimode signals. *The Journal of the Acoustical Society of America* [online]. 1991, 89(3), 1159–1168 [cit. 2020-01-06]. DOI: 10.1121/1.400530. ISSN 0001-4966.

Determination of dispersion curves

Signals acquired in equidistantly spaced points along the wave propagation path (through acoustic source) – from experiment or numerical simulation,

$$u(x,t),\ x = x_0 + n\Delta x,\ t \in \langle 0, T\rangle\,.$$

Avoid aliasing using high enough sampling frequency and fine spacing.

Input signal

Wide band needed, e.g. linear chirp (frequencies represented equally in range from f_0 to f_1),

$$s\left(t\right) = \sin\left[2\pi\left(f_0 t + \alpha t^2/2\right)\right], \quad t \in \langle 0, T_0\rangle\,, \quad \text{kde } \alpha = \frac{f_1 - f_0}{T}.$$

Experimental measurement

Linear chirp from 100 kHz to 1 MHz, 2500 points, step 0,1 mm, 50000 samples at 20 MHz.

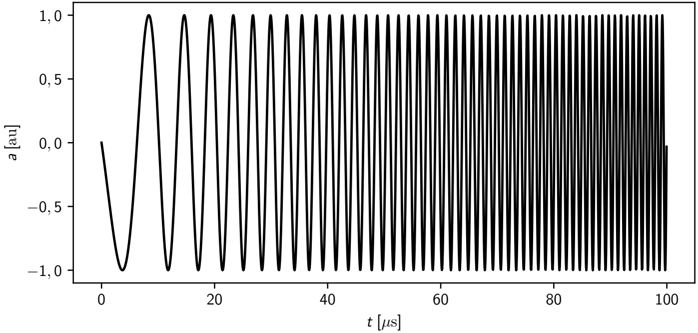

Figure 4: Input signal – linear chirp.

Experimental measurement

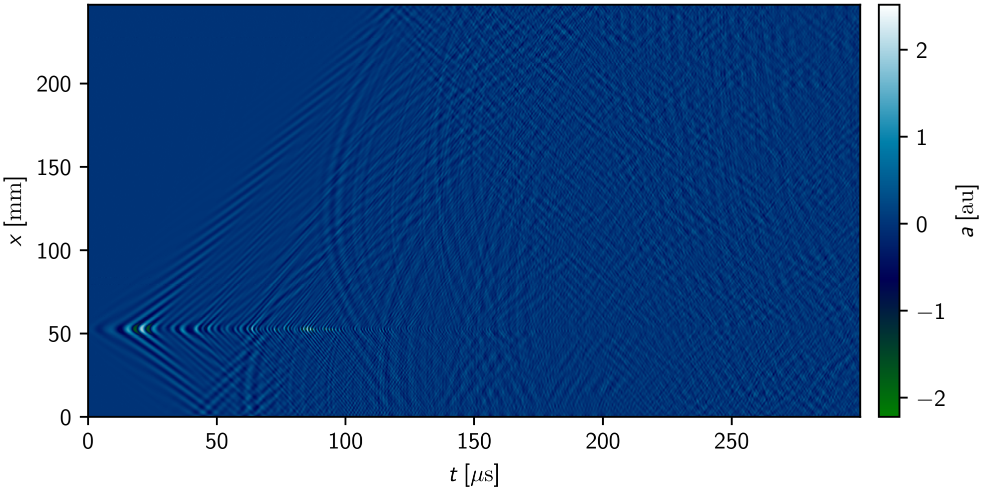

Figure 5: Experimentally acquired signals (cropped).

Dispersion curves

```
H = np.fft.fft2(u)                             # u (2500, 5000)
k = np.fft.fftfreq(H.shape[0], dx)             # dx = 1e-4
f = np.fft.fftfreq(H.shape[1], 1/freq)         # freq = 20e6
H = np.abs(H)                                  # get amplitudes
fi = (f >= 0) & (f < 1.5e6)
ki = (k >= 0) & (k < 700)
H = H[ki, :][:, fi]                            # crop
f = f[fi]
k = k[ki] * 2 * np.pi                          # it's angular wavenumber
```

Dispersion curves

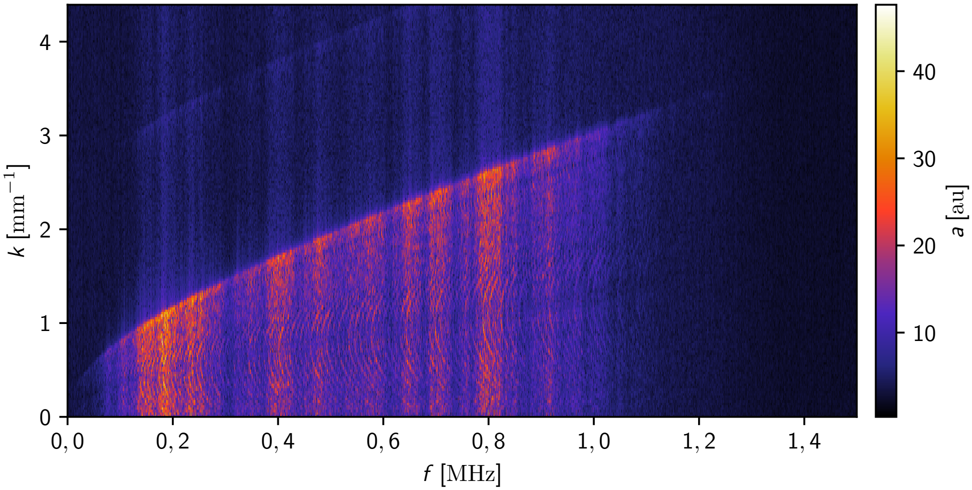

Figure 6: Amplitude of two-dimensional Fourier transform of acquired signals (cropped).

Dispersion curve extraction

Frequency equalization – $\widetilde{H}(k, f) = H(k, f) \,/\, \max_f H(k, f)$

```
H = H / np.max(H, axis=0)[np.newaxis, :]
```

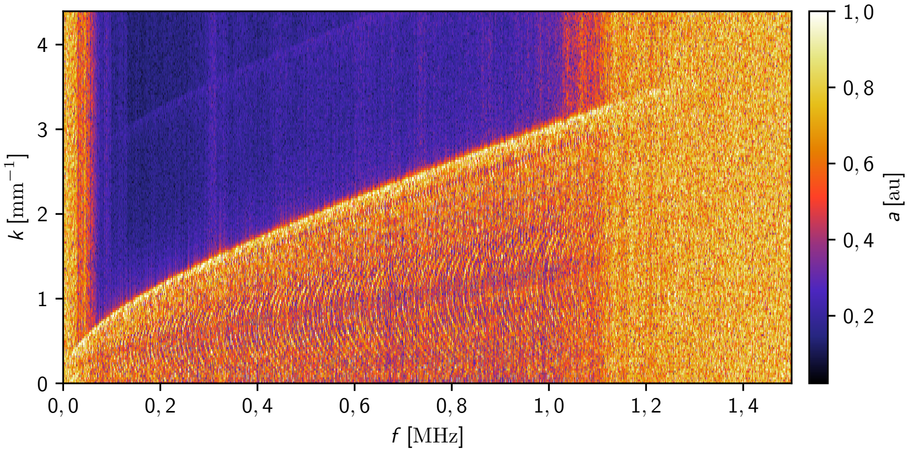

Figure 7: Amplitude of two-dimensional Fourier transform of acquired signals, equalized.

Dispersion curve extraction

Thresholding

```
h = 0.9         # threshold level
H = H > h
```

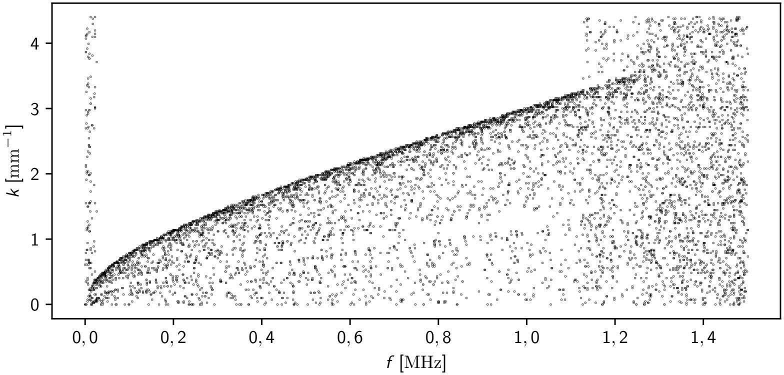

Figure 8: Amplitude of two-dimensional Fourier transform of acquired signals, thresholded.

Dispersion curve extraction

Curve fitting – $\widehat{f}(k) \approx \min_f \left\{ \widetilde{H}(k, f) > t \right\}$, "lay" curve on top of thresholded points

```
X = np.where(H)                       # get indices
X = np.split(X[1], np.cumsum(np.unique(X[0], return_counts=True)[1])[:-1])
Y = np.zeros(H.shape[0])              # index of point for each wavenumber
for i in range(len(X)):
     if len(X[i]) > 0: Y[i] = np.min(X[i])
plt.plot(f[Y.astype('int')], k)    # curve given by (f, k)
```

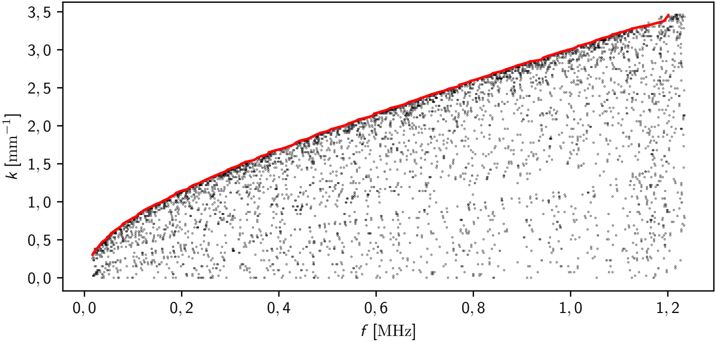

Figure 9: Thresholded points, fitted curve.

Dispersion curve extraction

A0 mode, compare with computed curve.

Parameters estimation – $\min_{c_L, c_T, h} \sum_k \left[\widehat{f}(k) - f(k, c_L, c_T, h)\right]^2$

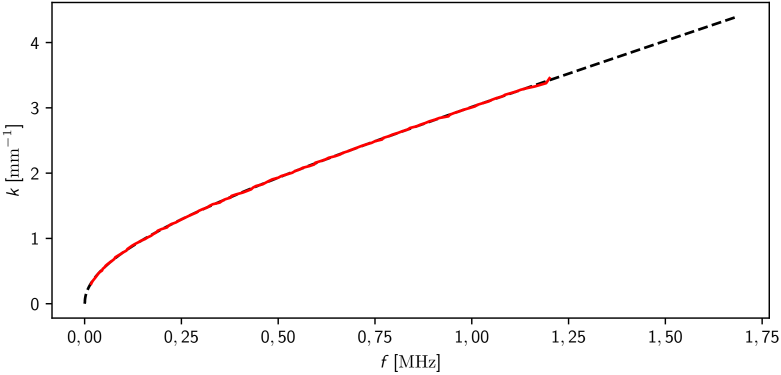

Figure 10: Extracted curve and computed curve.

Summary

- Time-map of Lamb wave propagation acquired using experiment of numerical simulation
- Dispersion curve extracted from timemap by means of Fourier transform
- Two-dimensional Fourier transform is carried to go to frequency-wavenumber domain where dispersion curve is distinguishable
- Material parameters (propagation velocities and thickness) estimated from dispersions curve and theoretical frequency equations