Balitsky–Kovchegov equation at next-to-leading order

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Big questions in QCD

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- Does the number of gluons grow infinitely?
- Or does it saturate?

Deep Inelastic Scattering (DIS)

- Excellent tool to probe inner structure of hadrons.
- DIS cross section

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[y^2 F_1(x,Q^2) + (1-y)\frac{F_2(x,Q^2)}{x}\right]$$

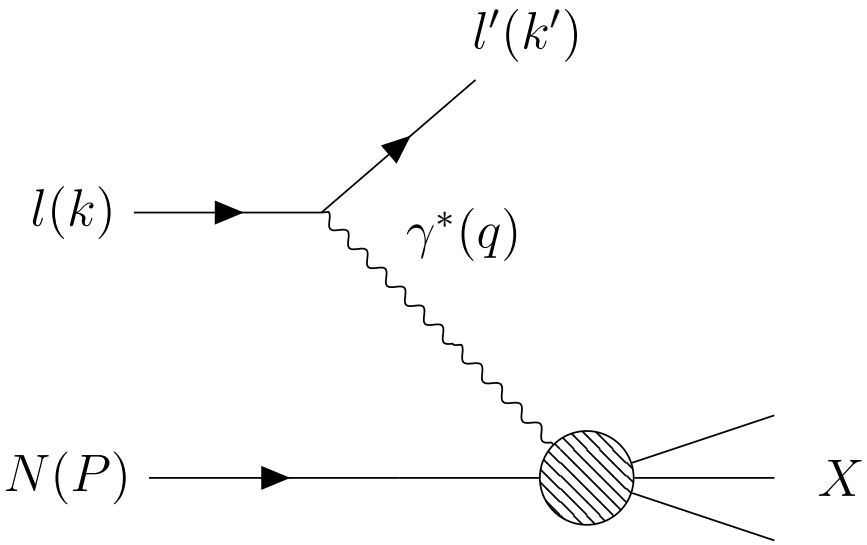
- $F_1,\ F_2$ – structure functions, include photon-proton interaction.

$$W^2_{\gamma^* p} = (P+q)^2$$

$$Q^2 = -q^2 = -(k-k')^2$$

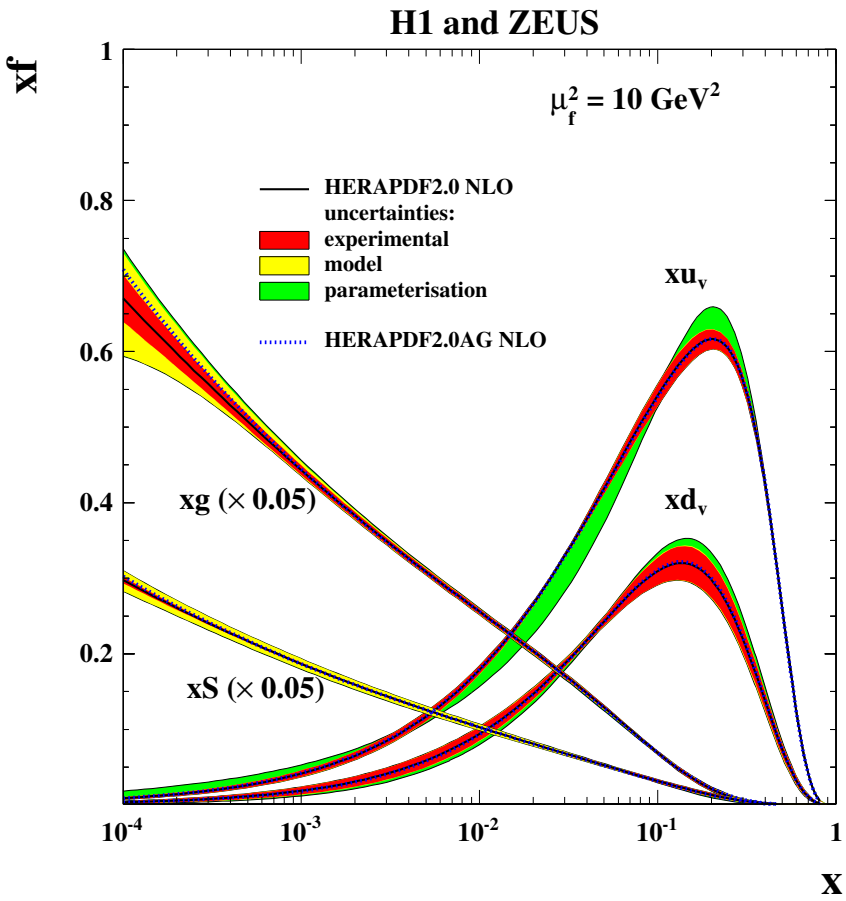
$$x = \frac{Q^2}{2P\cdot q}$$

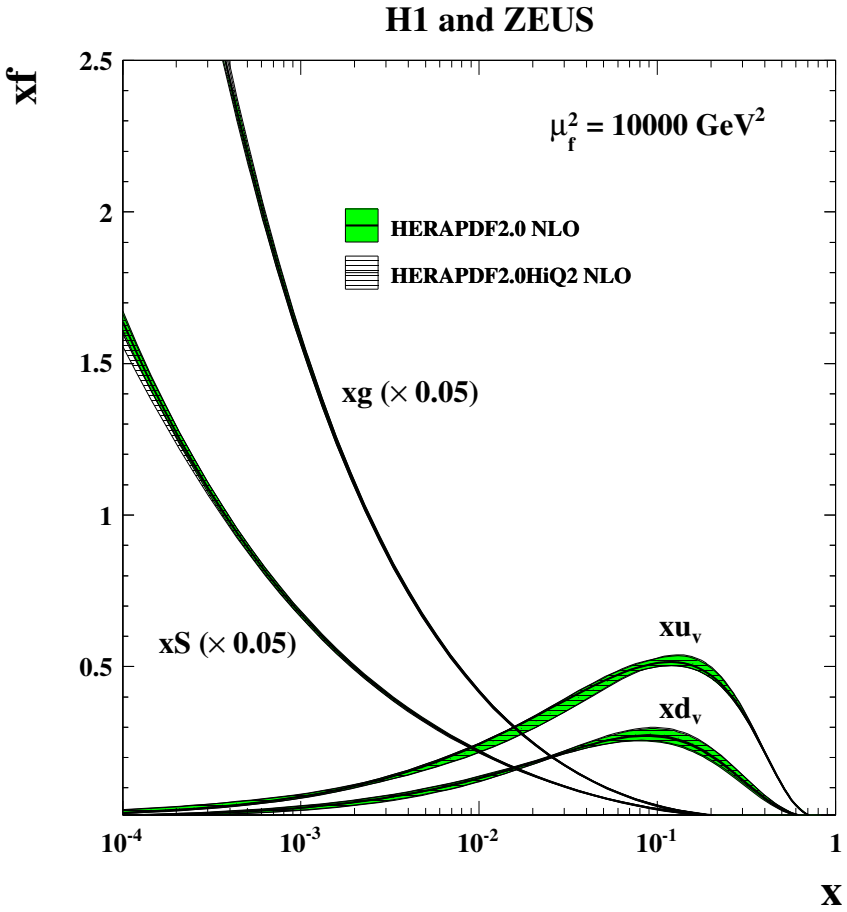
$$y = \frac{P\cdot Q}{P\cdot k}$$



Evolution of proton structure

- Composition of the proton changes with x and Q^2.
- At low energies proton dominated by valence quarks.
- At large energies gluons dominate.





Are gluon densities growing to infinity?

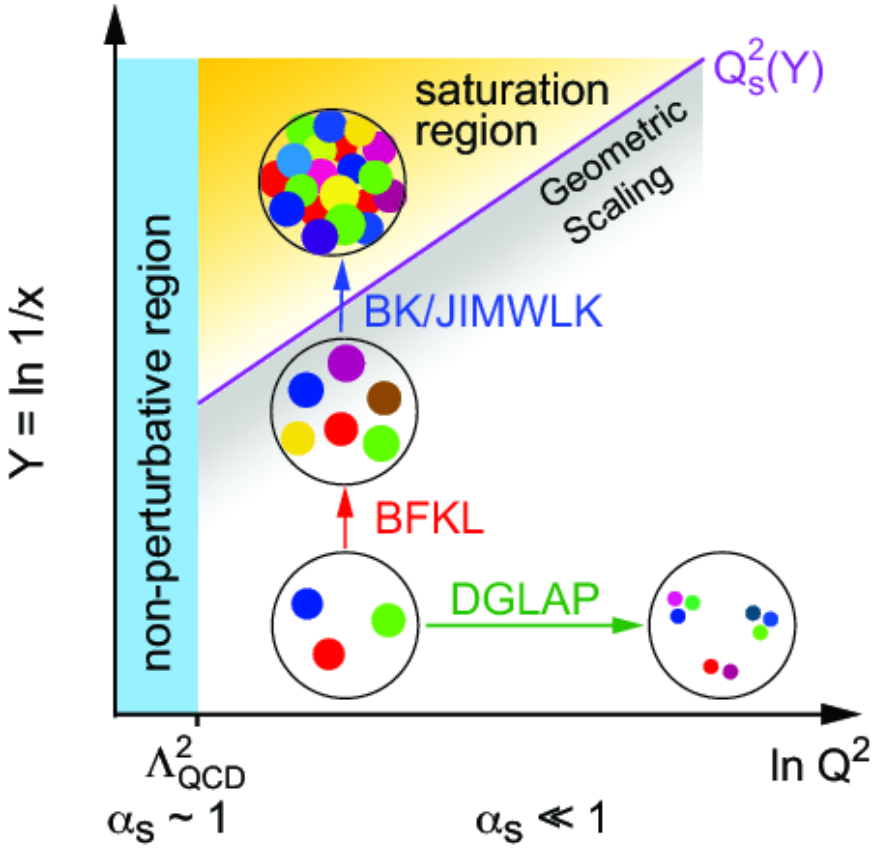


Figure: Diagram picturing the QCD evolution of the partonic structure of the proton.

C. Marquet, Nucl.Phys. A904-905 (2013) 294c-301c.

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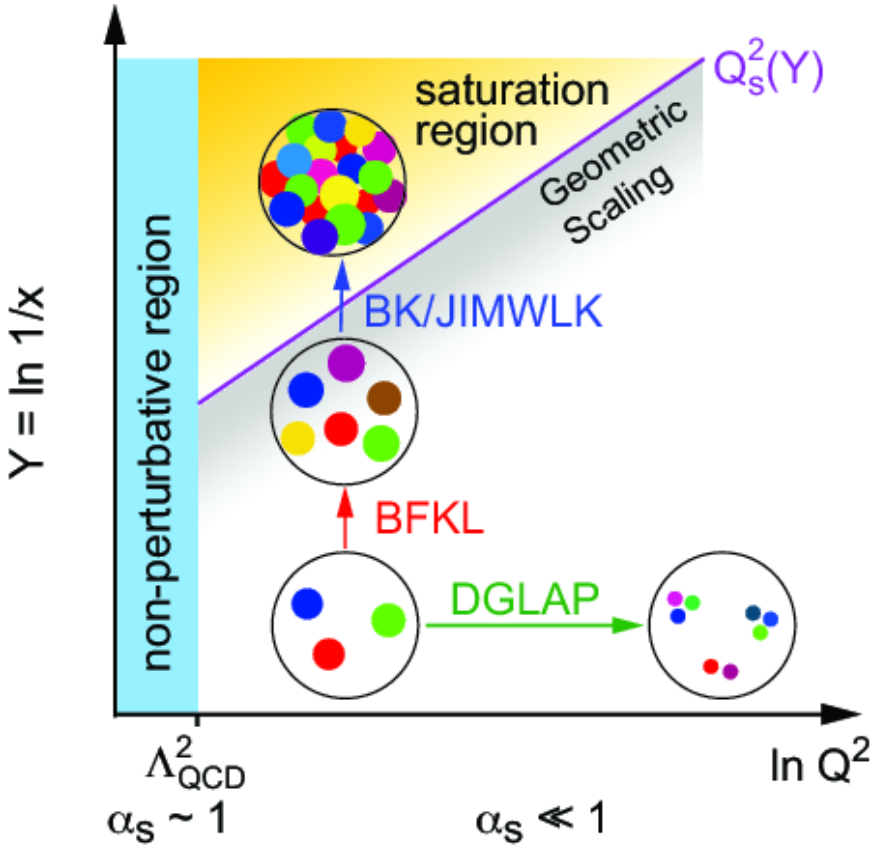

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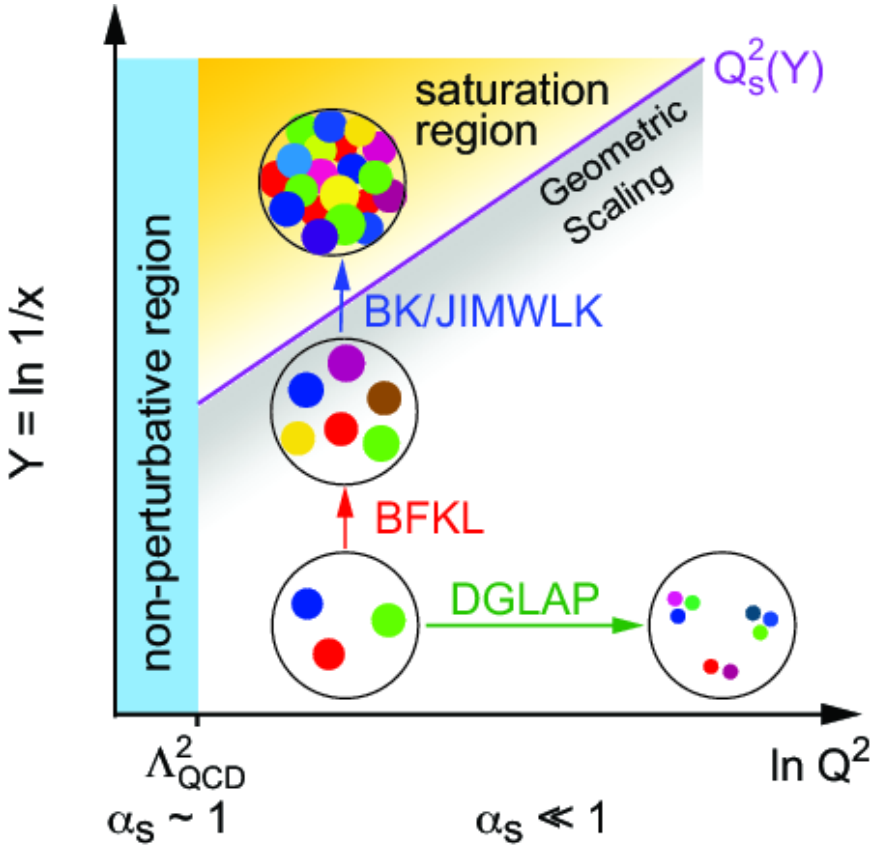

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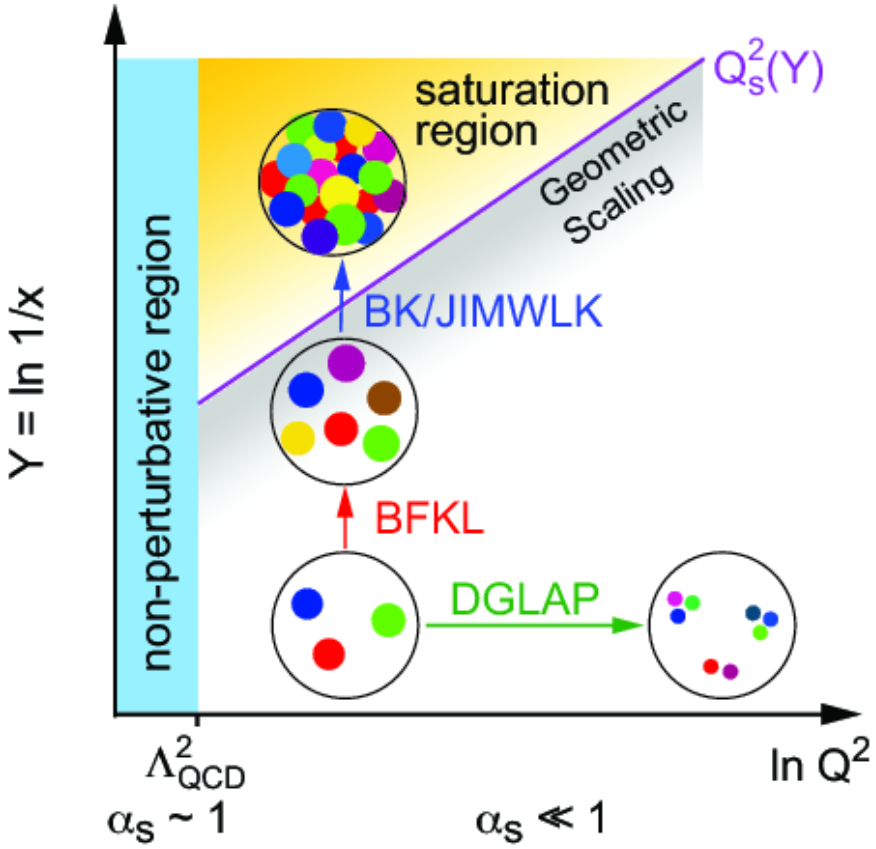

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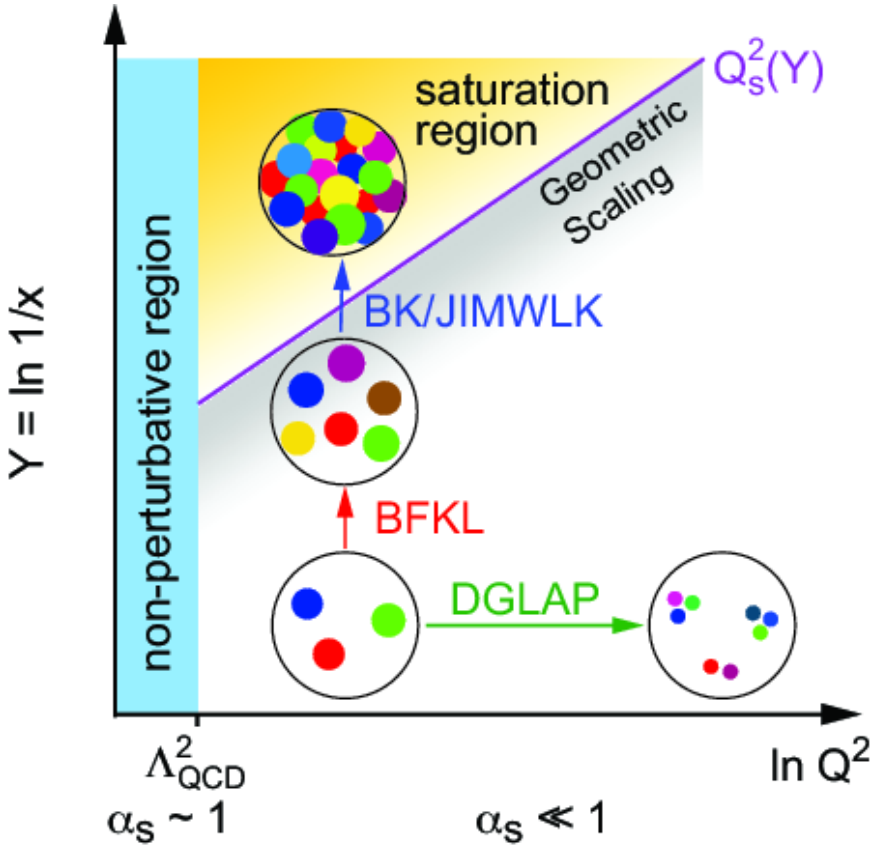

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  - Above $Q_S^2(x) \rightarrow$ dense regime, non-linear evolution of the gluon density (JIMWLK, BK).

What other process is sensitive to the proton structure?

- Exclusive and dissociative production of vector mesons (VM).
- Advantage: Easily observable final state.

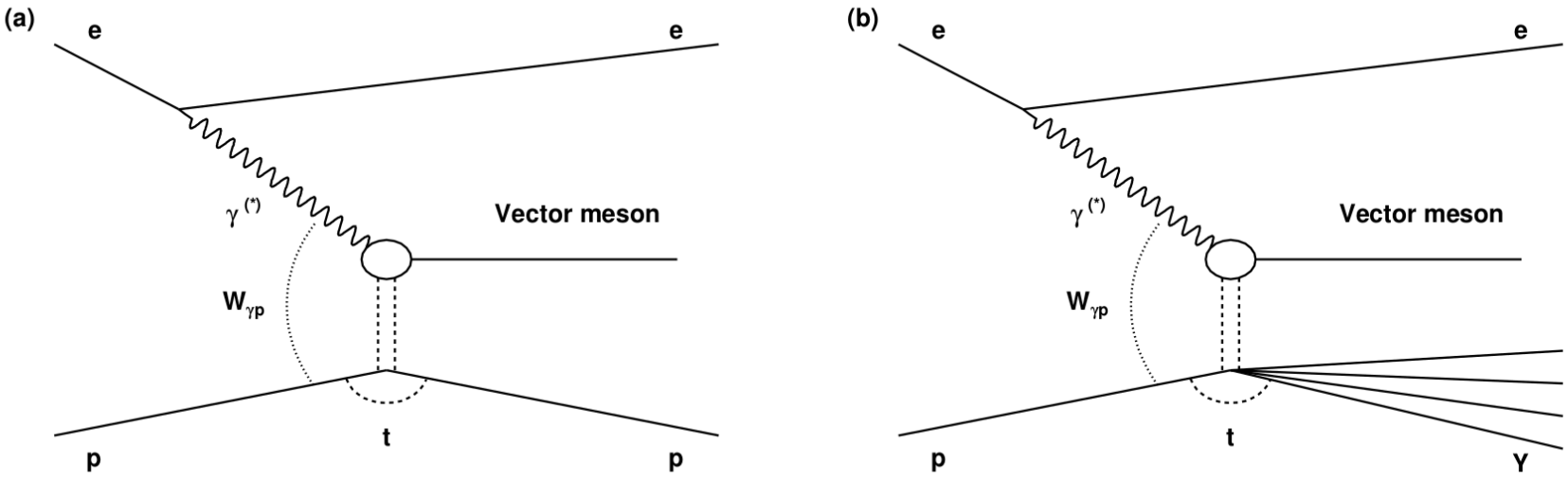

Figure: Diagrams for exclusive (a) and dissociative (b) production of vector mesons.

The color dipole picture

- The photon interacts via its $q\bar{q}$ Fock state with the proton in the target rest frame.
- At low x, lifetime of the fluctuation is larger than dipole-proton interaction time.

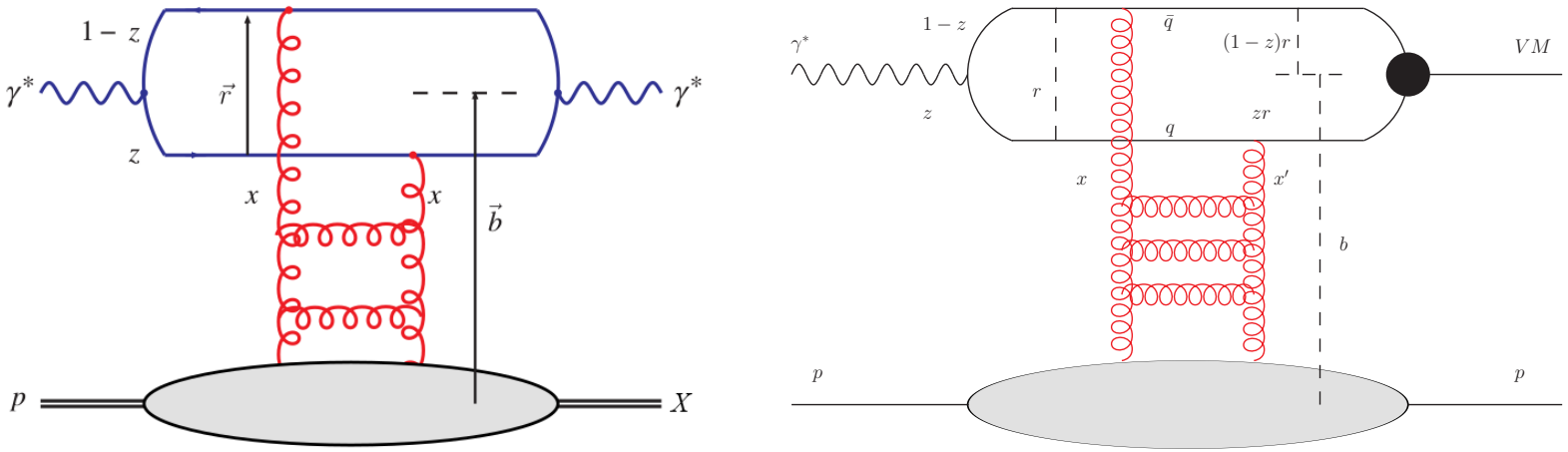

Figure: Schematic pictures of DIS and VM production within the color dipole approach.

Dipole-proton cross section

- From optical theorem

$$\frac{\mathrm{d}\sigma_{q\bar{q}}}{\mathrm{d}\vec{b}} = 2N\left(x, \vec{r}, \vec{b}\right)$$

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- Dipole scattering amplitude $N\left(x, \vec{r}, \vec{b}\right)$
  - From a dipole model like GBW, IP-Sat, b-CGC
  - From an evolution equation $\rightarrow$ Balitsky–Kovchegov equation
  - Note: impact-parameter dependence can be factorized out

$$N\left(x, \vec{r}, \vec{b}\right) \rightarrow \sigma_0 N\left(x, \vec{r}\right) T_p\left(\vec{b}\right)$$

Balitsky–Kovchegov equation at leading-order

$$\frac{\partial N(r_{xy}, b_{xy}, Y)}{\partial Y} = \int \mathrm{d}\vec{r}_{xz} K(r_{xy}, r_{xz}, r_{zy})[N(r_{xz}, b_{xz}, Y) + N(r_{zy}, b_{zy}, Y) - N(r_{xy}, b_{xy}, Y) - N(r_{xz}, b_{xz}, Y)N(r_{zy}, b_{zy}, Y)]$$

$$Y = \ln\left(\frac{x_0}{x}\right)$$

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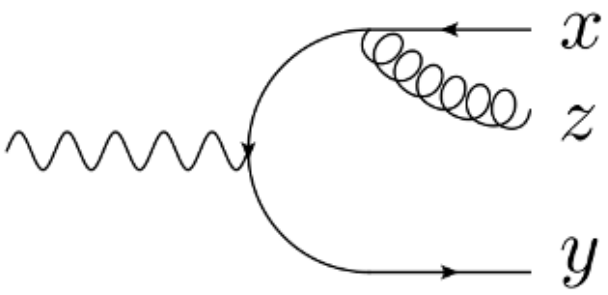
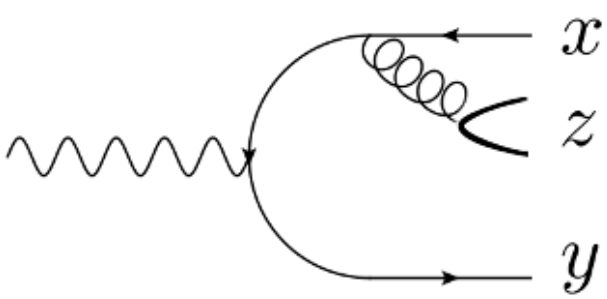
Simplest kernel at LO:

$$K_{\mathrm{BFKL}}(r_{xy}, r_{xz}, r_{zy}) = \bar{\alpha}_S \frac{r_{xy}}{r_{xz} r_{zy}}; \qquad \bar{\alpha}_S = \alpha_S \frac{N_C}{\pi}; \qquad \alpha_S \sim 1$$





What steps can be taken to include higher order contributions?

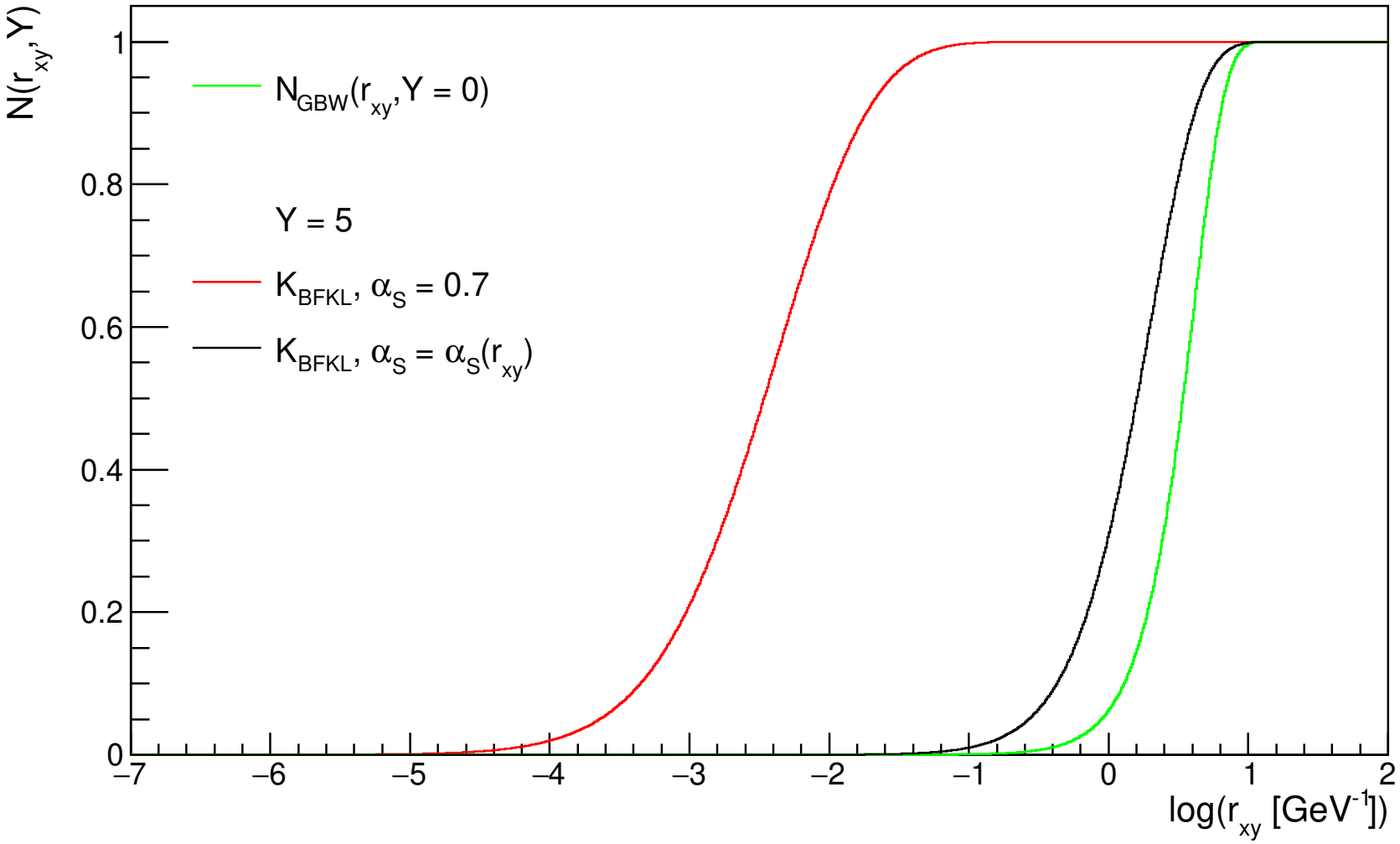
- Include running of the coupling constant α_S

$$\alpha_S \to \alpha_S(r) = \frac{4\pi}{\beta_{n_f} \ln\left(\frac{4C^2}{r^2\Lambda_{n_f}^2}\right)}; \qquad \beta_{n_f} = \frac{11}{3}N_C - \frac{2}{3}n_f$$

- Various choices of the argument of $\alpha_S(r)$, most common:
  - Parent dipole size $\alpha_S = \alpha_S(r_{xy})$
  - Smallest dipole prescription

$$\alpha_S = \alpha_S(r_{min}); \qquad r_{min} = \min\{r_{xy}, r_{xz}, r_{zy}\},$$

Solution to BK equation at LO - fixed vs running coupling



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  $\rightarrow$ *running coupling (rc) kernel* by Balitsky

$$K_{\mathrm{Bal}}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi}\left[\frac{r_{\mathrm{xy}}^2}{r_{\mathrm{xz}}^2 r_{\mathrm{zy}}^2} + \frac{1}{r_{\mathrm{xz}}^2}\left(\frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1\right) + \frac{1}{r_{\mathrm{zy}}^2}\left(\frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1\right)\right]$$

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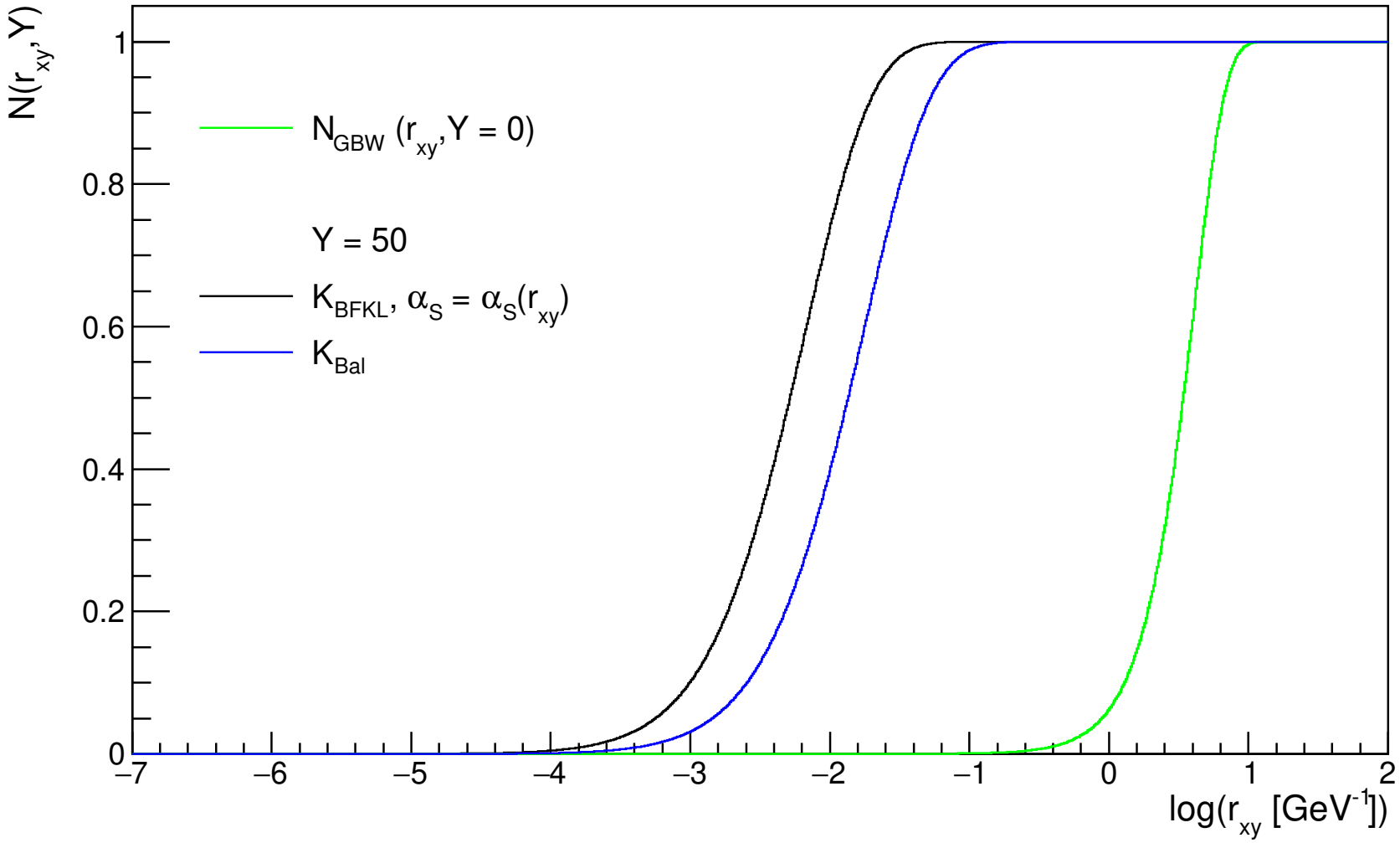
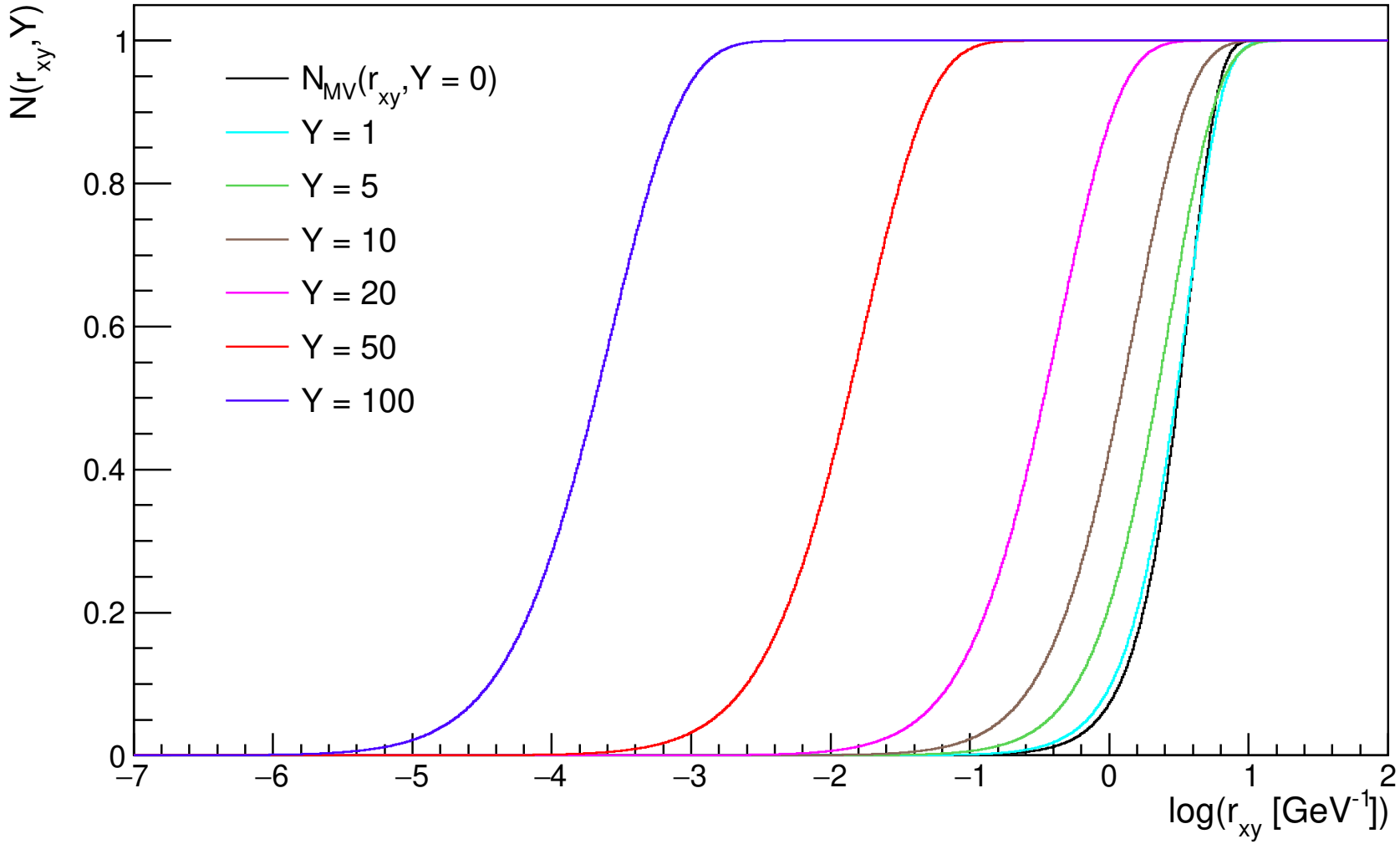
  - Resums higher-order corrections associated with the running coupling.
  - Slows down the evolution significantly.
  - Very popular in phenomenological applications of LO BK.

Solution to LO BK equation - BFKL vs Balitsky kernel



Solution to rc-BK equation at LO - large rapidities



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  - Corrections where each power of α_S is accompanied by a double logarithm
  - Corrections with a collinear single-logarithmic term from pure NLO integrals
  - Results in so called *collinearly-improved* BK equation

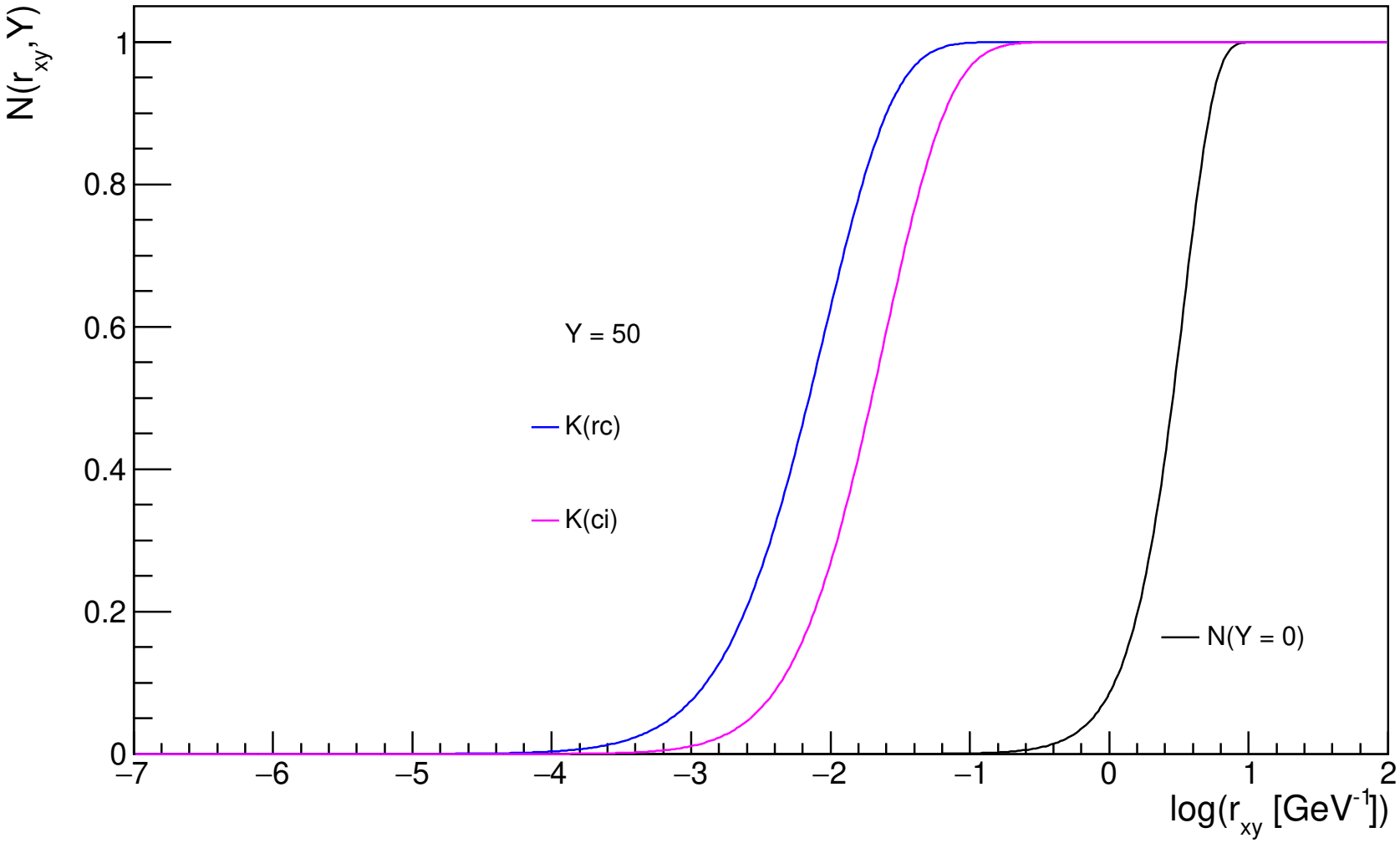
$$K_{ci} = \frac{\bar{\alpha}_S}{2\pi} \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \left[\frac{r_{xy}^2}{\min\left\{r_{xz}^2, r_{zy}^2\right\}} \right]^{\pm\bar{\alpha}_S A_1} K_{\mathrm{DLA}}(\rho),$$

$$K_{\mathrm{DLA}}(\rho) = \frac{J_1\left(2\sqrt{\bar{\alpha}_S \rho^2}\right)}{\sqrt{\bar{\alpha}_S \rho}} = 1 - \frac{\bar{\alpha}_S \rho^2}{2} + \frac{(\bar{\alpha}_S \rho^2)^2}{12} + \dots$$

$$\rho = \sqrt{L_{r_{xz} r_{xy}} L_{r_{zy} r_{xy}}}; \qquad L_{r_i r_{xy}} = \ln\left(\frac{r_i^2}{r_{xy}^2}\right).$$

  - $+$ in the exponent $\pm\bar{\alpha}_S A_1$ is taken when $r_{xy} < \min\left\{r_{xz}, r_{zy}\right\}$, the negative sign otherwise.
  - Term $A_1 = \frac{11}{12}$ gives an extra power-law suppresion of the kernel, treats single-transverse logarithms.

Solution to collinearly-improved BK equation



BK equation at next-to-leading order in its original form

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}Y}\,\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\} = \\
&= \frac{\alpha_S}{2\pi^2}\int \mathrm{d}^2 z\,\frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{xz}^{\,2}\vec{r}_{zy}^{\,2}}\left\{1+\frac{\alpha_S}{4\pi}\left[\beta_{n_f}\ln\left(\vec{r}_{xy}^{\,2}\,\mu^2\right)-\beta_{n_f}\frac{\vec{r}_{xz}^{\,2}-\vec{r}_{zy}^{\,2}}{\vec{r}_{xy}^{\,2}}\ln\frac{\vec{r}_{xz}^{\,2}}{\vec{r}_{zy}^{\,2}}+\left(\frac{67}{9}-\frac{\pi^2}{3}\right)N_{\mathrm{C}}-\frac{10}{9}n_f\right.\right. \\
&\qquad\left.\left.-\,2N_{\mathrm{C}}\ln\frac{\vec{r}_{xz}^{\,2}}{\vec{r}_{xy}^{\,2}}\ln\frac{\vec{r}_{zy}^{\,2}}{\vec{r}_{xy}^{\,2}}\right]\right\}\times\left[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\}-N_{\mathrm{C}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\right] \\
&+\frac{\alpha_S^2}{16\pi^4}\int \mathrm{d}^2 z\,\mathrm{d}^2 w\left[\left(-\frac{4}{\vec{r}_{zw}^{\,4}}+\left\{2\,\frac{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}+\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}-4\,\vec{r}_{xy}^{\,2}\,\vec{r}_{zw}^{\,2}}{\vec{r}_{zw}^{\,4}\left[\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}-\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}\right]}+\frac{\vec{r}_{xy}^{\,4}}{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}-\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\left[\frac{1}{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}+\frac{1}{\vec{r}_{zy}^{\,2}\,\vec{r}_{xw}^{\,2}}\right]+\right.\right. \\
&\qquad\left.\left.+\frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{zw}^{\,2}}\left[\frac{1}{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}-\frac{1}{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\right]\right\}\ln\frac{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\right)\times \\
&\qquad\times\left[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_w^\dagger\}\mathrm{Tr}\{\hat{U}_w\hat{U}_y^\dagger\}-\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\hat{U}_w U_y^\dagger\hat{U}_z\hat{U}_w^\dagger\}-(w\to z)\right]+ \\
&\qquad+\left\{\frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{zw}^{\,2}}\left[\frac{1}{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}+\frac{1}{\vec{r}_{zy}^{\,2}\,\vec{r}_{xw}^{\,2}}\right]-\frac{\vec{r}_{xy}^{\,4}}{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}\,\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\right\}\ln\frac{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\times \\
&\qquad\times\left[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_w^\dagger\}\mathrm{Tr}\{\hat{U}_w\hat{U}_y^\dagger\}\right]+ \\
&\qquad+4n_f\left\{\frac{4}{\vec{r}_{zw}^{\,4}}-2\,\frac{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}+\vec{r}_{wy}^{\,2}\,\vec{r}_{xz}^{\,2}-\vec{r}_{xy}^{\,2}\,\vec{r}_{zw}^{\,2}}{\vec{r}_{zw}^{\,4}\left[\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}-\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}\right]}\ln\frac{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}}\right\}\times \\
&\qquad\times\left[\mathrm{Tr}\{t^a\hat{U}_x t^b\hat{U}_y^\dagger\}\left(\mathrm{Tr}\{t^a\hat{U}_z t^b\hat{U}_w^\dagger\}-(w\to z)\right)\right]\Bigg]
\end{aligned}
$$

Looks scary, so let's break it into individual parts

- First term corresponds to LO BK equation $\rightarrow$ additional corrections $\sim \alpha_S^2$.

$$\frac{\alpha_S}{2\pi^2}\int \mathrm{d}^2 z\ \frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{xz}^{\,2}\vec{r}_{zy}^{\,2}}\left\{1+\frac{\alpha_S}{4\pi}\left[\beta_{n_f}\ln\left(\vec{r}_{xy}^{\,2}\,\mu^2\right)-\beta_{n_f}\frac{\vec{r}_{xz}^{\,2}-\vec{r}_{zy}^{\,2}}{\vec{r}_{xy}^{\,2}}\ln\frac{\vec{r}_{xz}^{\,2}}{\vec{r}_{zy}^{\,2}}+\right.\right.$$
$$\left.\left.+\left(\frac{67}{9}-\frac{\pi^2}{3}\right)N_{\mathrm{C}}-\frac{10}{9}n_f-2N_{\mathrm{C}}\ln\frac{\vec{r}_{xz}^{\,2}}{\vec{r}_{xy}^{\,2}}\ln\frac{\vec{r}_{zy}^{\,2}}{\vec{r}_{xy}^{\,2}}\right]\right\}$$
$$\times\left[\mathrm{Tr}\{\hat{U}_x\hat{U}_z^\dagger\}\mathrm{Tr}\{\hat{U}_z\hat{U}_y^\dagger\}-N_{\mathrm{C}}\mathrm{Tr}\{\hat{U}_x\hat{U}_y^\dagger\}\right]$$

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  - Parent dipole $\vec{r}_{xy}$ emits a gluon $\rightarrow$ two daughter dipoles $\vec{r}_{xz}$ and $\vec{r}_{zy}$

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  - Parent dipole $\vec{r}_{xy}$ emits a gluon $\rightarrow$ two daughter dipoles $\vec{r}_{xz}$ and $\vec{r}_{zy}$
  - Dipole $\vec{r}_{zy}$ emits a gluon at $\vec{w}$ $\rightarrow$ fluctuates into $q\bar{q}$ $\rightarrow$ daughter dipoles $\vec{r}_{xz}$, $\vec{r}_{zw}$, $\vec{r}_{wy}$

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  - Dipole $\vec{r}_{zy}$ emits a gluon at $\vec{w}$ $\rightarrow$ fluctuates into $q\bar{q}$ $\rightarrow$ daughter dipoles $\vec{r}_{xz}$, $\vec{r}_{zw}$, $\vec{r}_{wy}$
  - Cubic terms — real contribution, all daughter dipoles interact with the target
  - Quadratic term — virtual contribution, dipole at $\vec{w}$ emitted and reabsorbed before or after interaction, also substracts term with gluons emitted at $\vec{z} = \vec{w}$

$$
\begin{aligned}
&+ \frac{\alpha_S^2}{16\pi^4} \int \mathrm{d}^2 z \, \mathrm{d}^2 w \Bigg[\Bigg(- \frac{4}{\vec{r}_{zw}^{\,4}} + \Bigg\{ 2 \, \frac{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2} + \vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2} - 4 \, \vec{r}_{xy}^{\,2} \, \vec{r}_{zw}^{\,2}}{\vec{r}_{zw}^{\,4} \left[\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2} - \vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2} \right]} + \frac{\vec{r}_{xy}^{\,4}}{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2} - \vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2}} \left[\frac{1}{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2}} + \frac{1}{\vec{r}_{zy}^{\,2} \, \vec{r}_{xw}^{\,2}} \right] + \\
&\qquad + \frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{zw}^{\,2}} \left[\frac{1}{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2}} - \frac{1}{\vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2}} \right] \Bigg\} \ln \frac{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2}} \Bigg) \times \\
&\qquad \times \left[\mathrm{Tr}\{ \hat{U}_x \hat{U}_z^\dagger \} \mathrm{Tr}\{ \hat{U}_z \hat{U}_w^\dagger \} \mathrm{Tr}\{ \hat{U}_w \hat{U}_y^\dagger \} - \mathrm{Tr}\{ \hat{U}_x \hat{U}_z^\dagger \hat{U}_w U_y^\dagger \hat{U}_z \hat{U}_w^\dagger \} - (w \rightarrow z) \right] + \\
&\qquad + \left\{ \frac{\vec{r}_{xy}^{\,2}}{\vec{r}_{zw}^{\,2}} \left[\frac{1}{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2}} + \frac{1}{\vec{r}_{zy}^{\,2} \, \vec{r}_{xw}^{\,2}} \right] - \frac{\vec{r}_{xy}^{\,4}}{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2} \, \vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2}} \right\} \ln \frac{\vec{r}_{xz}^{\,2} \, \vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2} \, \vec{r}_{zy}^{\,2}} \times \\
&\qquad \times \left[\mathrm{Tr}\{ \hat{U}_x \hat{U}_z^\dagger \} \mathrm{Tr}\{ \hat{U}_z \hat{U}_w^\dagger \} \mathrm{Tr}\{ \hat{U}_w \hat{U}_y^\dagger \} \right] + \text{quark part}
\end{aligned}
$$

Looks scary, so let's break it into individual parts

- Terms of order $\sim \alpha_S^2$ in double integrals (DI) $\rightarrow$ fluctuations involving 2 additional partons at the time of the scattering beside the parent dipole.
- Quark part of NLO corrections $\sim n_f$ $\rightarrow$ similar situation as with gluons, daughter partons at the time of the scattering are a quark or an anti-quark

$$\frac{\alpha_S^2}{16\pi^4}\int \mathrm{d}^2 z\, \mathrm{d}^2 w \left[\text{gluon part} + \; 4n_f \left\{ \frac{4}{\vec{r}_{zw}^{\,4}} - 2\,\frac{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2} + \vec{r}_{wy}^{\,2}\,\vec{r}_{xz}^{\,2} - \vec{r}_{xy}^{\,2}\,\vec{r}_{zw}^{\,2}}{\vec{r}_{zw}^{\,4}\left[\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2} - \vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}\right]}\,\ln\frac{\vec{r}_{xz}^{\,2}\,\vec{r}_{wy}^{\,2}}{\vec{r}_{xw}^{\,2}\,\vec{r}_{zy}^{\,2}} \right\} \times \right.$$
$$\left. \times \left[\mathrm{Tr}\{t^a \hat{U}_x t^b \hat{U}_y^\dagger\}\left(\mathrm{Tr}\{t^a \hat{U}_z t^b \hat{U}_w^\dagger\} - (w \rightarrow z)\right)\right]\right]$$

BK equation at NLO in the mean-field approximation

- Suppose a large N_C (3 is large enough :))

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\begin{aligned}
\partial_Y N(r_{xy}) &= \int d^2z K_a \Big[N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \Big] \\
&+ \int d^2z d^2w K_b \Big[N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \\
&\qquad - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \Big] \\
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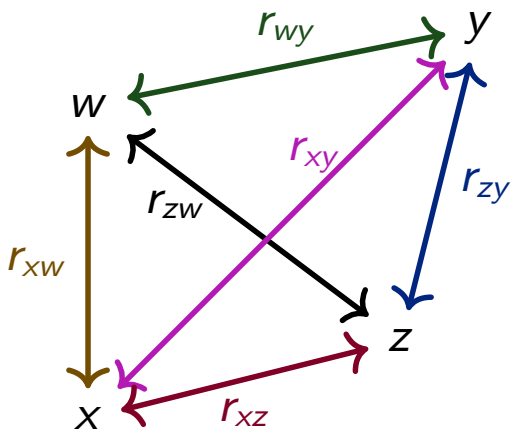
\end{aligned}
$$

- Definitely looks less scary!
- Single-integration term — corrections to LO BK associated with α_S^2
- Double integration term — pure NLO contributions



NLO kernels

$$K_a = K_{\mathrm{Bal}} + \frac{\alpha_S^2(r_{xy})N_C^2}{8\pi^3}\frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2}\left[\frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{N_C} - 2\ln\frac{r_{xz}^2}{r_{xy}^2}\ln\frac{r_{zy}^2}{r_{xy}^2}\right]$$

- Belongs to term with single integration over transverse coordinate z
- Similar structure of the term as in LO BK equation + some NLO corrections
- NLO corrections $\sim \alpha_S^2$
- Note the double logarithm!

Purely NLO kernels

- Gluon part:

$$K_b = \frac{\alpha_S^2 N_C^2}{8\pi^4}\Bigg(- \frac{2}{r_{zw}^4} + \Bigg[\frac{r_{xz}^2 r_{wy}^2 + r_{xw}^2 r_{zy}^2 - 4r^2 r_{zw}^2}{r_{zw}^4\left(r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2\right)} + \frac{r_{xy}^4}{r_{xz}^2 r_{wy}^2\left(r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2\right)} + \frac{r_{xy}^2}{r_{xz}^2 r_{wy}^2 r_{zw}^2}\Bigg] \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2}\Bigg)$$

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- Notice the single-logarithms and their collinear behavior in both kernels!

Numerical solutions to NLO BK equation

- NLO BK equation is numerically unstable
  - Dipole amplitude can even turn negative – negative NLO corrections larger than LO contribution
  - Stable solution obtained only for unphysically large values of some parameters

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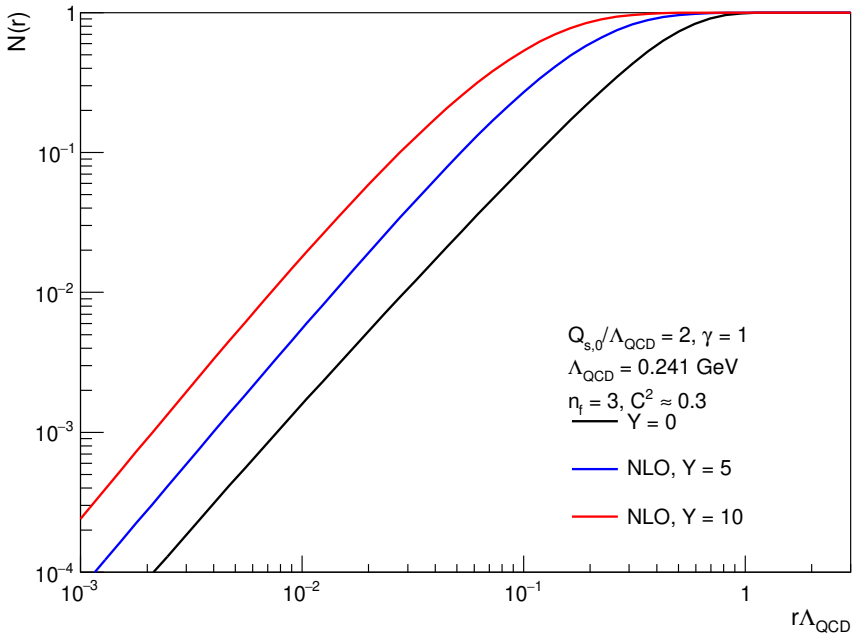
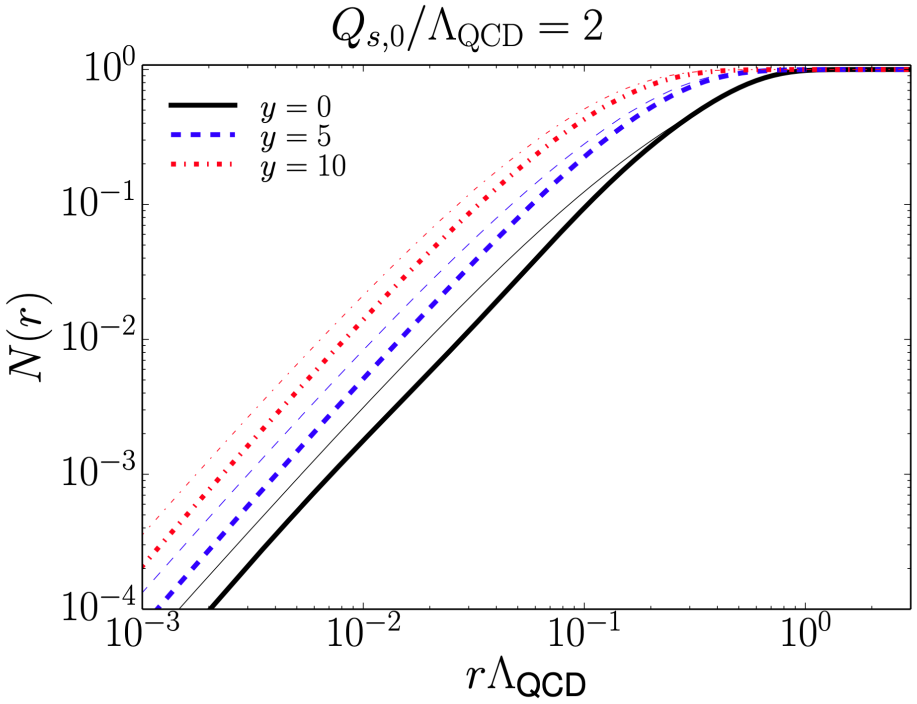
  - Dipole amplitude can even turn negative – negative NLO corrections larger than LO contribution
  - Stable solution obtained only for unphysically large values of some parameters
- Additional resumations of single and double logarithms needed → results in a numerically stable equation





TL, HM – Phys. Rev., D93(9):094004, 2016

Choice of the proper evolution variable

- LC momenta of photon and proton:

$$q^\mu \equiv (q^+, q^-, \vec{q}_\perp) = (q^+, -\frac{Q^2}{2q^+}, \vec{0}_{perp}); \qquad P^\mu = \delta^{\mu^-} P^-$$

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- **Target frame:**
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- **Dipole frame:**
  - Evolution seen as successive gluon emissions within the dipole WF $\rightarrow$ BK
  - For $Q^2 >> Q_0^2$, emissions are strongly ordered in both long. and trans. momenta $\rightarrow$ soft and collinear emissions $\rightarrow$ DL contribution $\sim \alpha_S Y \rho$
  - DL enhancement holds only when gluon lifetimes are also ordered

$$\frac{2q^+}{Q^2} >> \frac{2k_1^+}{k_{1,\perp}^2} >> ... >> \frac{2q_0^+}{Q_0^2}$$

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  - Evolution in $\eta \rightarrow$ time-ordering preserved, no anti-collinear logs

Collinearly-improved BK in target rapidity

- Time-ordering is automatically satisfied, no large anti-collinear logs
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Collinearly-improved BK in target rapidity

- Time-ordering is automatically satisfied, no large anti-collinear logs
- Double-log corrections in NLO still present, but suppressed
- Further resummation leads to an equation non-local in η

$$\frac{\partial S_{xy}(\eta)}{\partial \eta} = \int \mathrm{d}^2 z \frac{\bar{\alpha}_S(r_{min})}{2\pi} \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \left(\frac{r_{xy}^2}{\min[r_{xz}, r_{zy}]} \right)^{\pm A_1} [S_{xz}(\eta - \delta_{xz;r}) S(\eta - \delta_{zy;r}) - S(\eta)]$$

- Rapidity shifts:

$$\delta_{xz;r} \equiv \max \left[0, \ln \frac{r_{xy}^2}{r_{xz}^2} \right]$$

  $\rightarrow$ Non-zero for emissions where one of daughters $<<$ parent

- Iancu et al. $\rightarrow$ proposal of the equation, successful fits to HERA data

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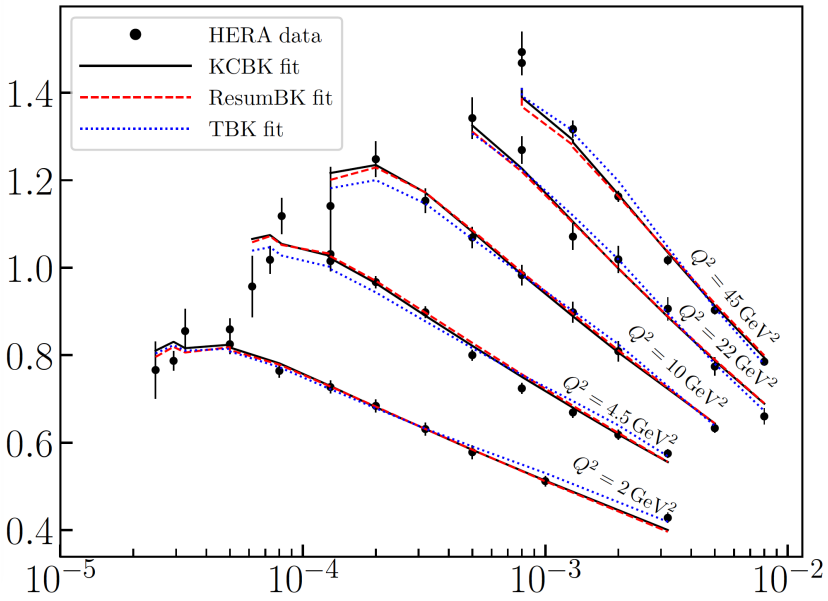
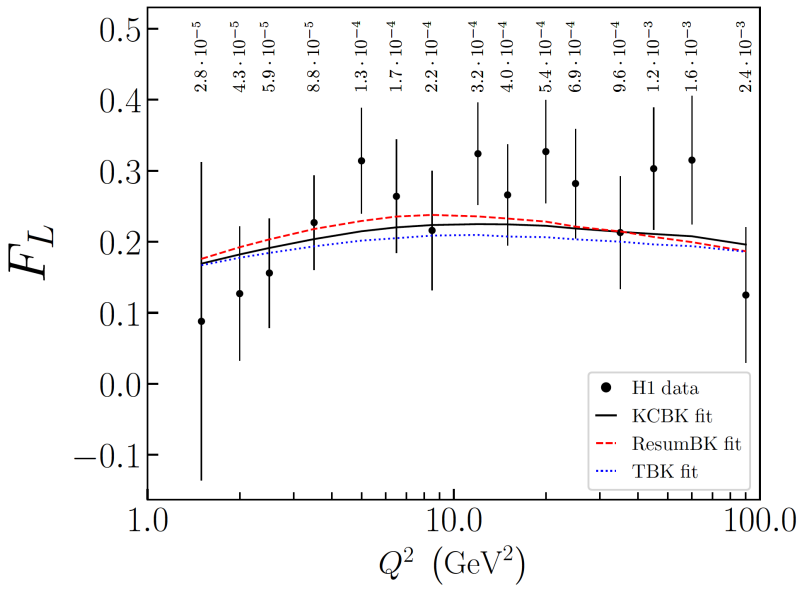
- **Two approches to the fit:**
  - Light-quark contribution to structure functions calculated and compared to inclusive data in appropriate region
  - Interpolated dataset with only light-quark contribution is constructed and fitted

First NLO BK fits to DIS data

- All equations provide equally reasonable description of both F_2 data and pseudodata
- Q^2 dependence of structure functions is weaker than in LO case
- All three setups predict almost the same F_L when compared to H1 data
- In EIC kinematics (very low x_{Bj}, the equations start to differ at very large Q^2





GB, HH, TL, HM – Phys. Rev., D102: 074028, 2020

First pheno results with NLO BK fits

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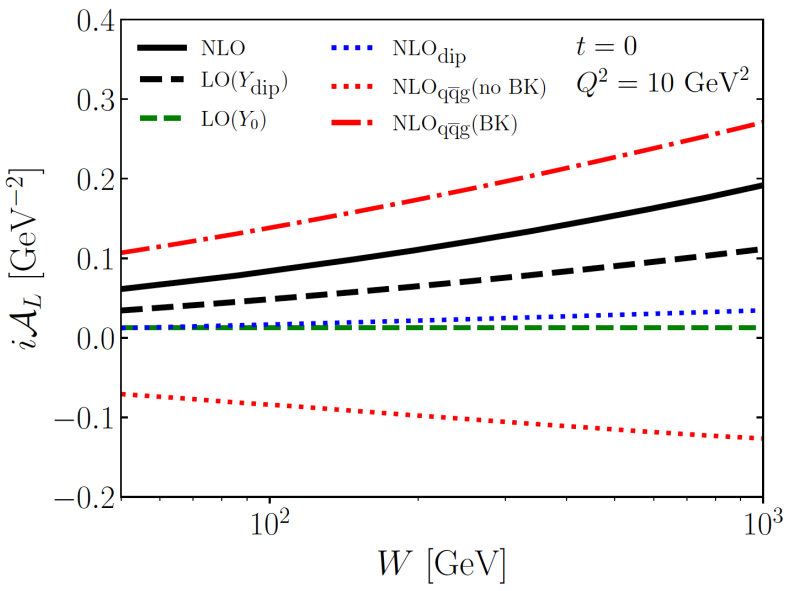
- Exclusive heavy VM production at NLO
- Two problems at NLO:
  - Dipole cross section at NLO - calculated via NLO BK equation (or its approximations)
  - Extension of the wave function to NLO accuracy
- In previous works by Beuf, Lappi, et al., – massless case derived (NLO DIS fits)
- Since 2021, WF with massive quarks are available for the longitudinal polarization



HM, JP – arXiv:2104.02349

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  - N from NLO BK has to be matched with a wave function at NLO accuracy

    $\rightarrow$ available for massless quarks, for massive quarks only in some cases

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