Stochastic Modeling of Fractal Diffusion

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Cantor Set

Diffusion on Fractal Sets

- Let $\mathcal{G} \subset \mathbb{R}^n$, $d_f = \dim(\mathcal{G})$ and
$$d_f < n, \quad d_f \in \mathbb{R}$$
- Let μ_G be well defined measure on $\mathcal{G}$
- Let $\{X_t\}_{t \in \tau} \subset \mathcal{G}$ be (well behaved) diffusion process
- Let $R_t = ||X_t - X_0||_2$ be absolute travelled distance

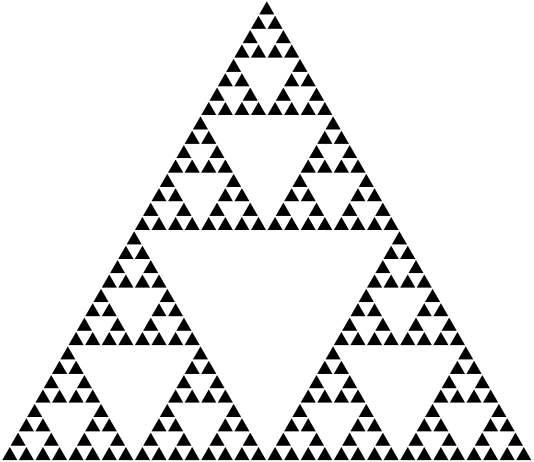
Properties

$$\mu_{\mathcal{G}}(B_n(x, r) \cap \mathcal{G}) \asymp r^{d_f}$$
$$\mathbb{E}\, R_t \asymp t^{\frac{1}{d_w}}$$
$$\Pr(X_t = x | X_0 = x) \asymp t^{-\frac{2\, d_f}{d_w}}$$

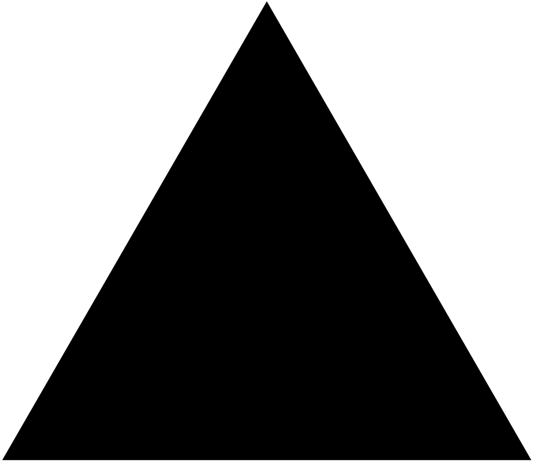
- $f(x) \asymp g(x) \quad \Leftrightarrow \quad \exists a, b \in \mathbb{R}, \quad a\, g(x) \le f(x) \le b\, g(x)$

How to obtain samples $r_i \sim R_t$ to test models and quality of dimension estimates?

Sierpinski Gasket Model



Finite Model Construction



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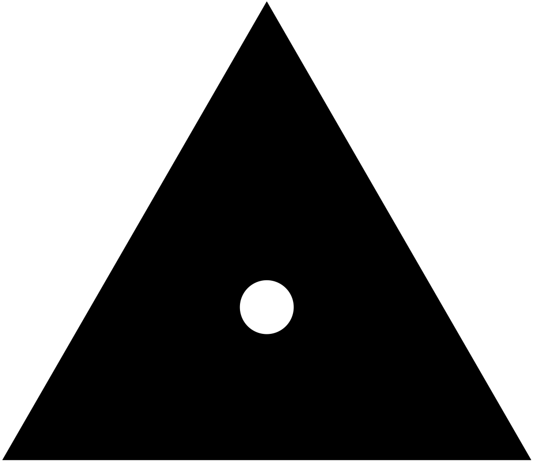

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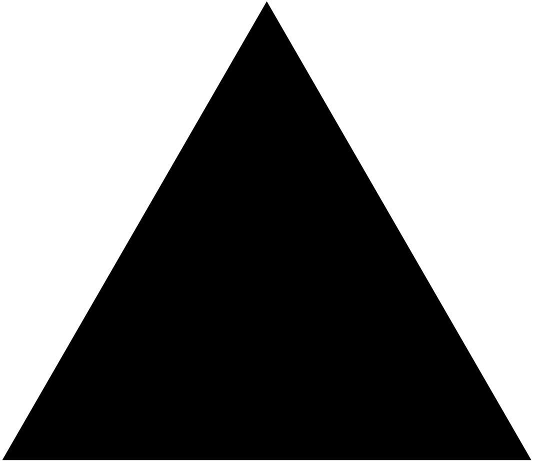

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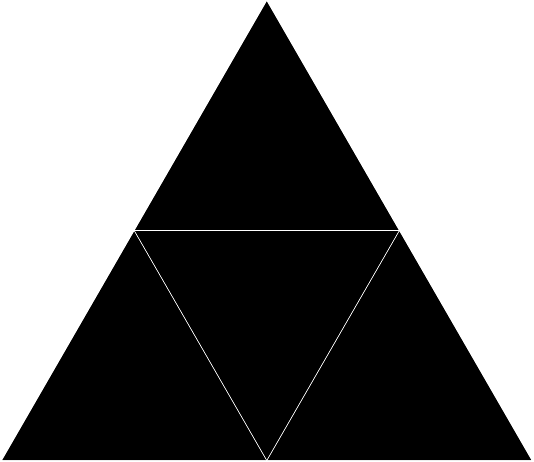

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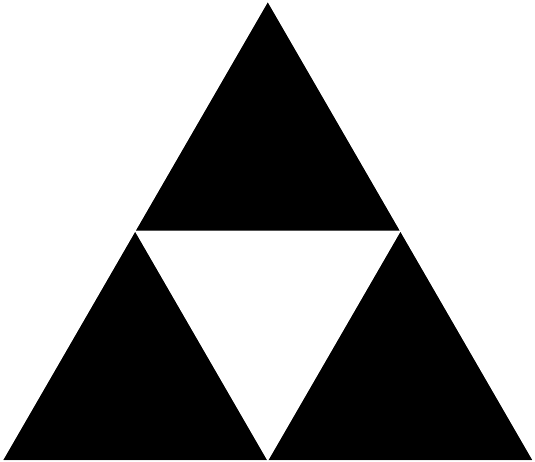

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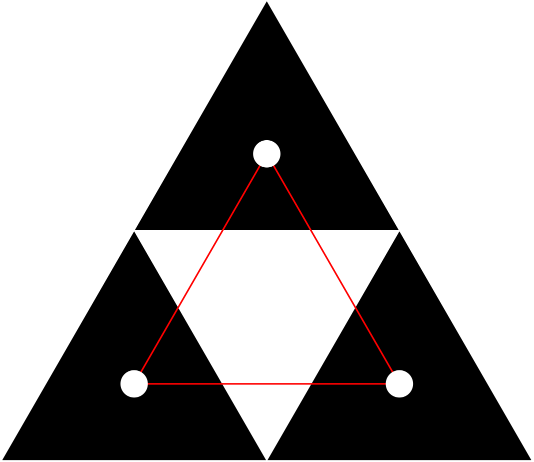

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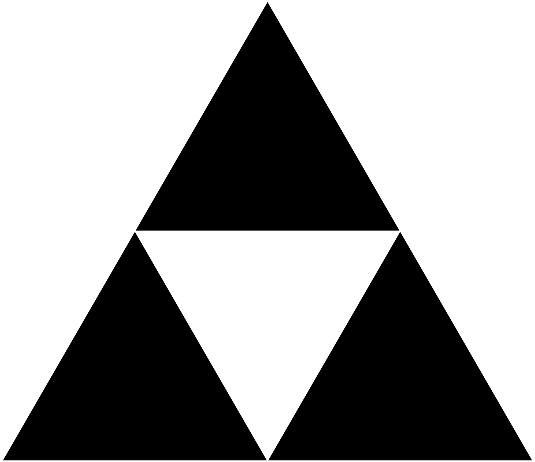

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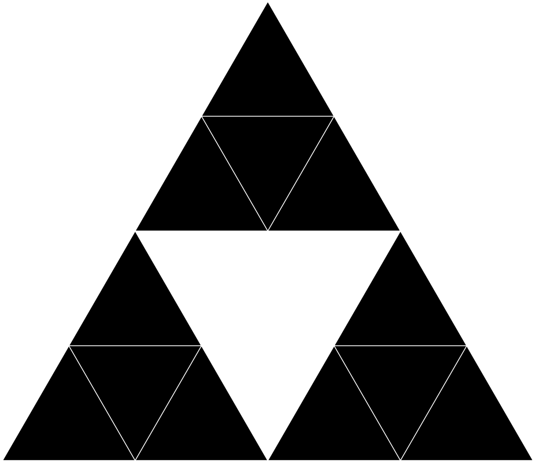

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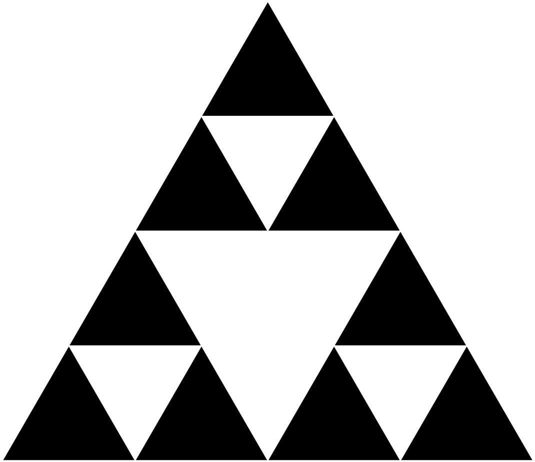

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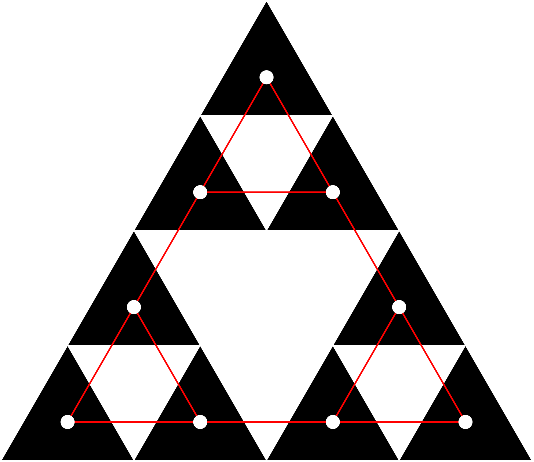

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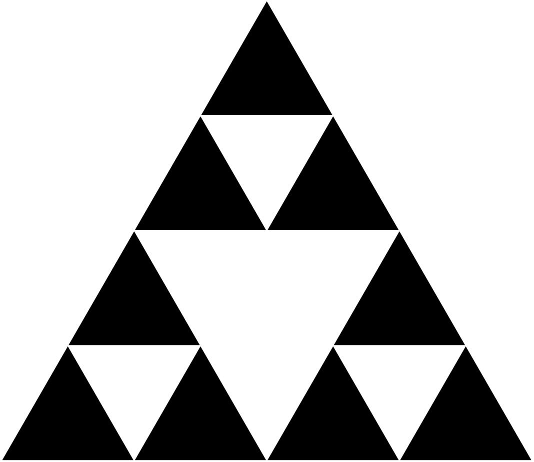

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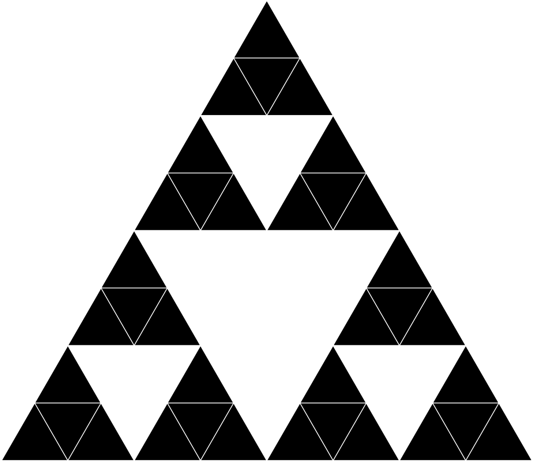

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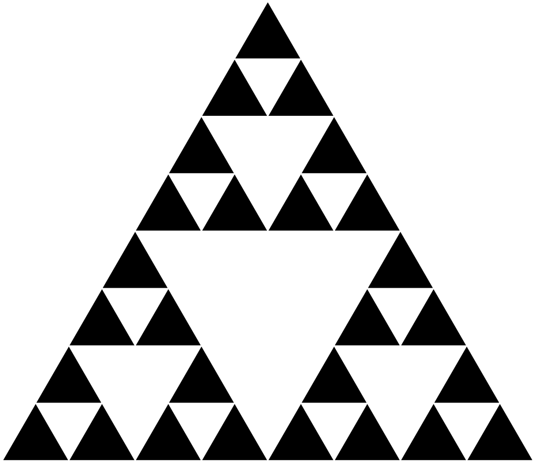

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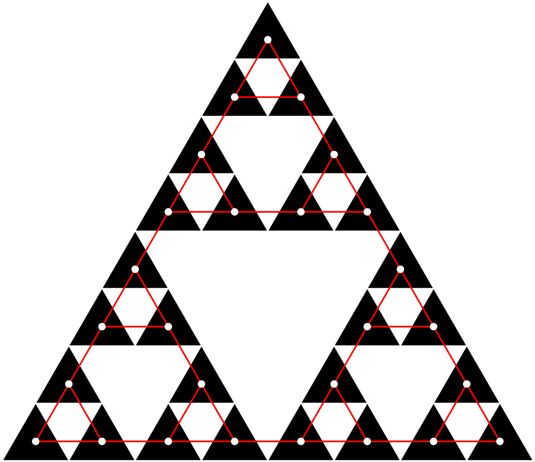

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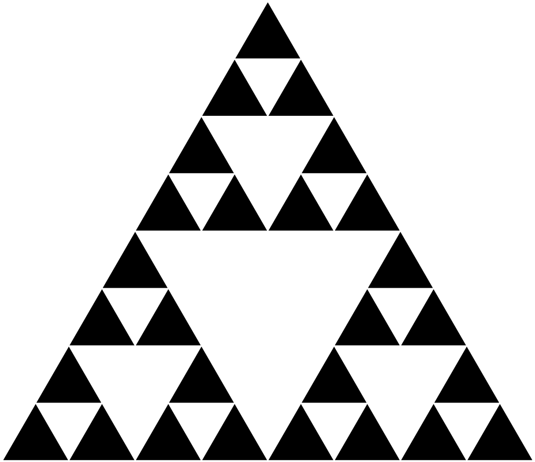

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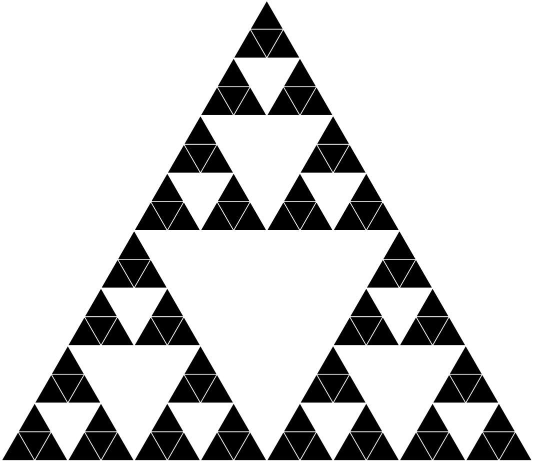

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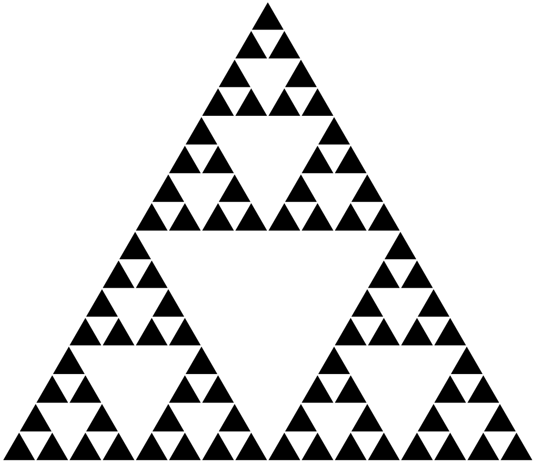

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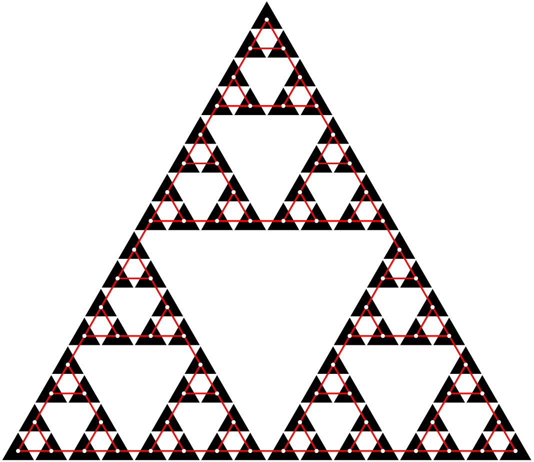

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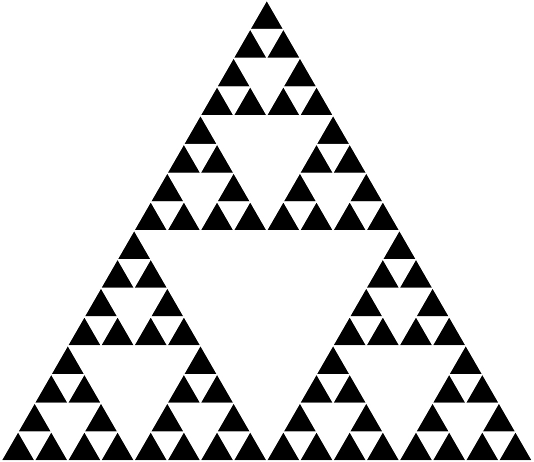

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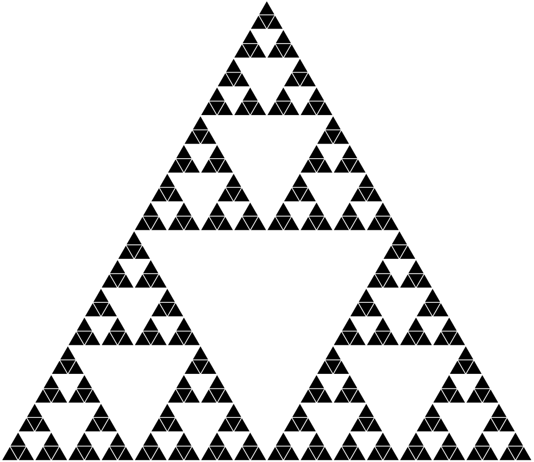

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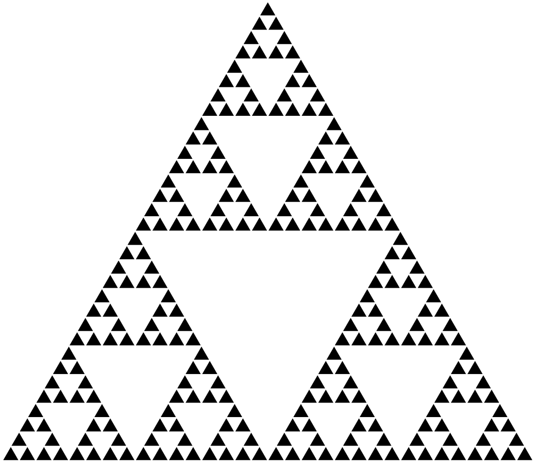

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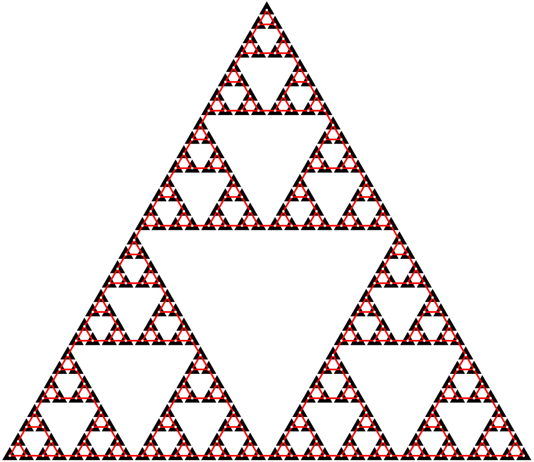

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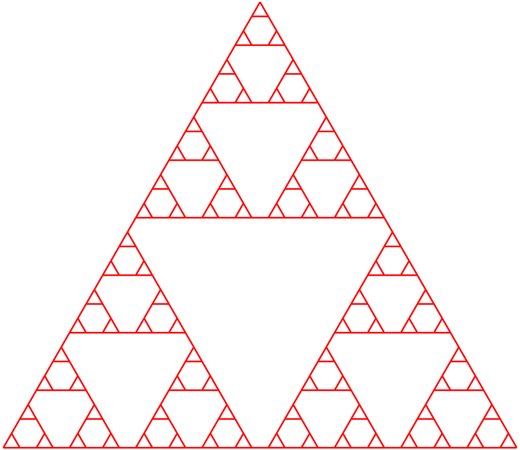

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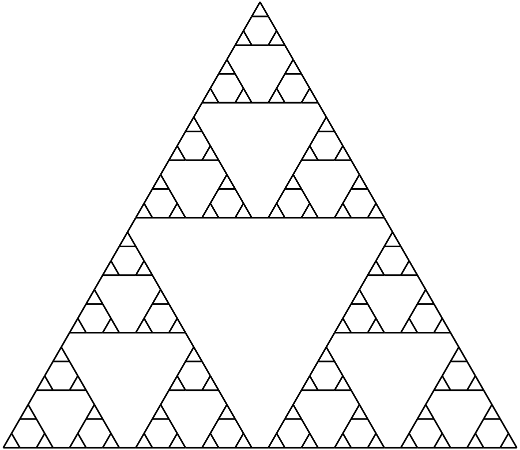

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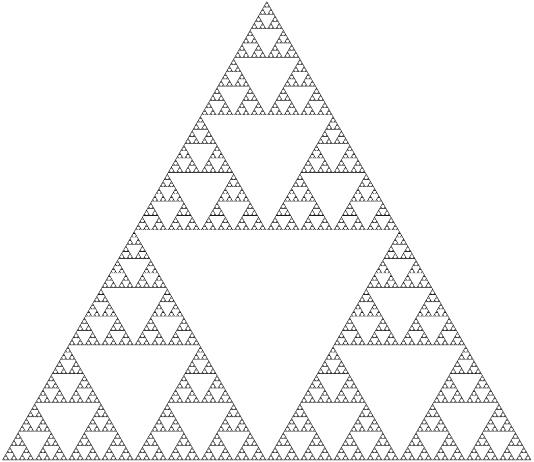


Simulation Outline

Process Dynamics

- Fixed *jump* probability p
- Varying *stay* probability $1 - \deg(x)\, p$
- Uniformly randomly selected starting vertex
- Given number of time steps t

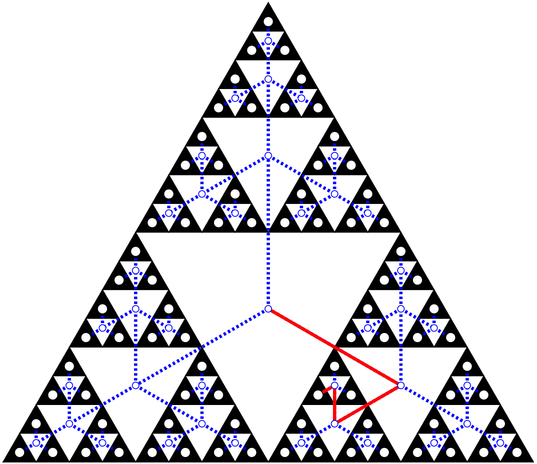
Graph Specification

- Depth level H
- Number of vertexes 3^H
- Number of edges $3^{H+1} - 3$
- Step distance between border vertexes $2^H - 1$

Implementation Result

```
>> X = zeros(3^20, 2);
Error using zeros
Requested 3486784401x2 (52.0GB)
array exceeds maximum array size
preference (31.9GB). This might
cause MATLAB to become
unresponsive.
```

Recursive Coordinates (2,1,3,1)



Recursive Representation

Self-Similar Fractal Property

- Recursive generation of vertexes of new level

$$\boldsymbol{x}^{(H)} \to \left\{\boldsymbol{x}_1^{(H+1)}, \dots, \boldsymbol{x}_k^{(H+1)}\right\}$$

- Cartesian coordinates of new vertex

$$\boldsymbol{x}_i^{(H+1)} = T_{i,H}(\boldsymbol{x}^{(H)})$$

Coding

- Recursive "coordinates" of vertex

$$(i_1, i_2, \dots, i_H) \in \{1, 2, \dots, k\}^H$$

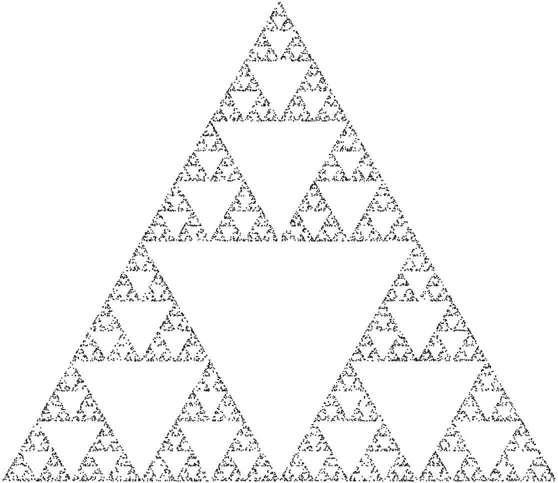
- Cartesian Coordinates of Coded Vertex

$$\boldsymbol{x} = \left(T_{i_1,1} \circ T_{i_2,2} \circ \cdots \circ T_{i_H,H}\right)(\boldsymbol{x}_0)$$

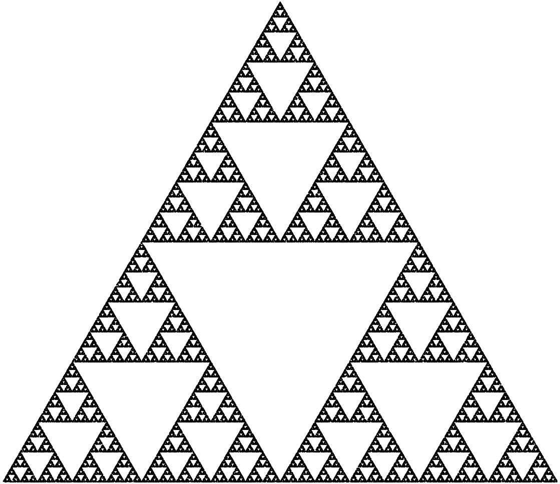
Uniform Sampling, $H = 50$, $N = 100$

Uniform Sampling, $H = 50$, $N = 1000$

Uniform Sampling, $H = 50$, $N = 10000$



Uniform Sampling, $H = 50$, $N = 100000$



Adjacency Representation

Movement using Recursive Coordinates

- Algorithmically represent movement between vertexes
- Moves within the deepest level $\rightarrow$ change of the last coordinate
- Moves outside the deepest level $\rightarrow$ multiple changing coordinates
- Requirement of generality and low computational time

Adjacent Vertexes of Sierpinski Gasket

$$(3,2,3,1) \rightarrow \begin{cases} (3,2,3,2) \\ (3,2,3,3) \\ ? \end{cases}$$

Adjacency Schema for Sierpinsky Gasket

Adjacency Schema

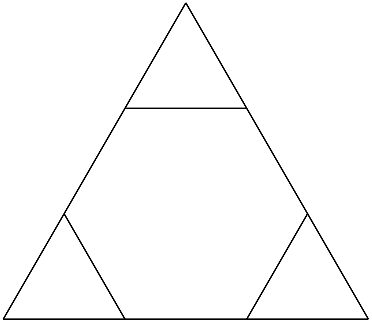
	$\longrightarrow$	$\nearrow$	$\nwarrow$	$\longleftarrow$	$\swarrow$	$\searrow$
1	**2**	**3**	$\times$	2	3	$\times$
2	1	$\times$	**3**	**1**	$\times$	3
3	$\times$	1	2	$\times$	**1**	**2**

Change coords. from right to left until **inner** jump or forbidden step.

Neighbour in Direction

$$\begin{array}{llllllllll}(3, & 2, & 3, & 3) & + & \longleftarrow & = (\sim, & \sim, & \sim, & \times) \\ (3, & 2, & 3, & 3) & + & \nearrow & = (\sim, & \times, & 1, & 1) \\ (3, & 2, & 3, & 3) & + & \nwarrow & = (\sim, & \mathbf{3}, & 2, & 2)\end{array}$$

$H = 2$



Simulation Outline

Process Dynamics

- Fixed *jump* probability p
- Varying *stay* probability $1 - \deg(x)\, p$
- Uniformly random starting vertex
- Given number of time steps t

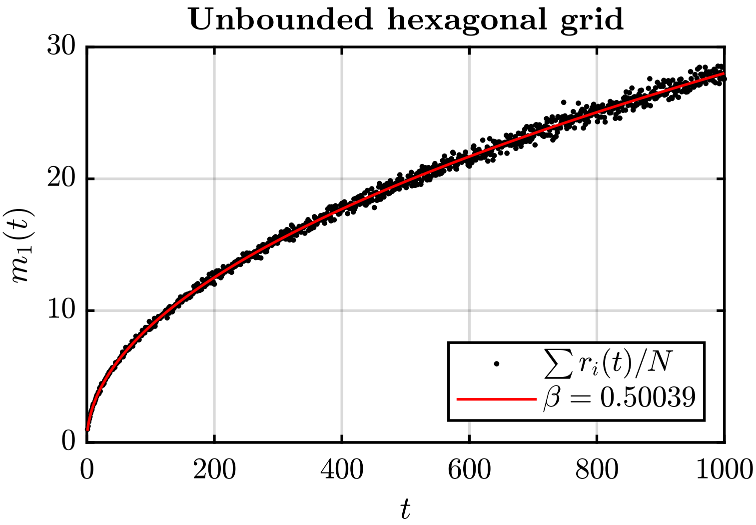
Graph Specification

- Depth level H
- Transformation list
- Scale parameter
- Adjacency schema

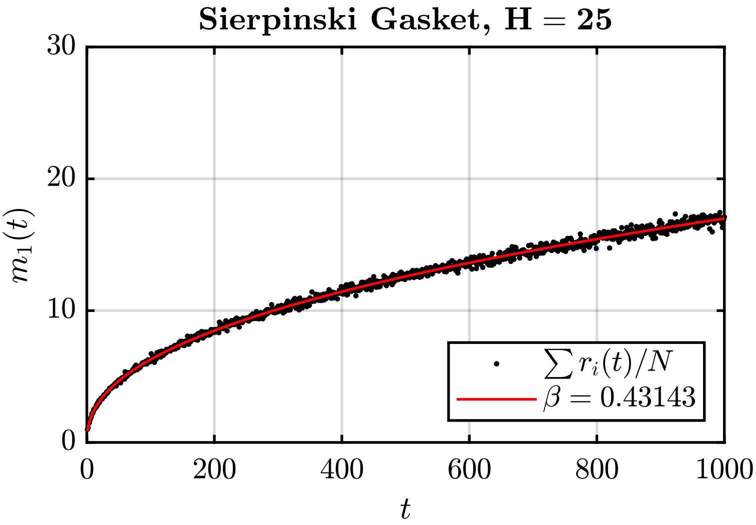
Algorithm

- Generate random recursive coordinates
- Transform to Cartesian coordinates $\boldsymbol{x}_0$
- Movement using adjacency schema
- Transform to Cartesian coordinates $\boldsymbol{x}_t$
- $r_t = ||\boldsymbol{x}_0 - \boldsymbol{x}_t||_2$

Walk dimension test



Walk dimension test



Having proper simulation data we can...

Let $r_1, \ldots, r_N \sim R_t$ i.i.d. and let us present dimension estimate...

Thank You For Your Attention