SCALING OF THE GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

WITH NEGATIVE VALUE OF PARAMETER α

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BALANCED DENSITIES

Definition:

Function $f(x)$: $\mathbb{R} \rightarrow \mathbb{R}$ is called balanced density, when following axioms are fullfilled:

1. $Ran(f) \subset \mathbb{R}_0^+$;
2. $Dom(f) = \mathbb{R}$;
3. $f(x) \in \mathcal{PC}(\mathbb{R})$;
4. $f(x) \in \mathcal{L}(\mathbb{R})$;

DENSITY

+

$$\int_{-\infty}^{+\infty} \boldsymbol{f(x)dx = 1}$$

PROBABILITY DENSITY

5. $supp(f) \subset (0, +\infty)$;
6. $\exists \varkappa \in \mathbb{R}_0^+$:

$$\alpha > \varkappa \Rightarrow \lim_{x \to +\infty} f(x)e^{\alpha x} = +\infty\,,$$
$$\alpha < \varkappa \Rightarrow \lim_{x \to +\infty} f(x)e^{\alpha x} = 0\,.$$

BALANCED DENSITIES

EXPONENTIAL DISTRIBUTION	$f(x) = Ae^{-\lambda x}$	$A > 0, \lambda > 0$
ERLANG DISTRIBUTION	$f(x) = Ax^n e^{-\lambda x}$	$A > 0, \lambda > 0, n \in \mathbb{N}_0$
GAMMA DISTRIBUTION	$f(x) = Ax^\alpha e^{-\lambda x}$	$A > 0, \lambda > 0, \alpha \geq 0$

GENERALIZED INVERSE GAUSSIAN DISTRIBUTION (GIG)

$$g(x) = Ax^\alpha e^{-\lambda x} e^{-\frac{\beta}{x}}$$

$A > 0, \alpha \in \mathbb{R}, \beta > 0, \lambda > 0$

BALANCED DENSITIES

Proof: $(\forall\alpha < 0)(g(x) \in \mathcal{L}(\mathbb{R}))$ $(\Leftrightarrow g(x) \in \mathcal{L}(0,+\infty))$

$$|g(x)| = \left|\underbrace{x^{\alpha}e^{-\frac{b}{x}}}_{h_1(x)}e^{-\lambda x}\right| \leq \left|e^{-\lambda x}h_1(x_{max})\right| \in \mathcal{L}(0,+\infty)$$

$$x_{max} = -\frac{\beta}{\alpha}$$

$$|g(x)| = \left|x^{\alpha}e^{-\lambda x}e^{-\frac{\beta}{x}}\right| \leq \left|\left(-\frac{\beta}{\lambda}\right)e^{\alpha}e^{-\lambda x}\right| \leq Ke^{-\lambda x} \in \mathcal{L}(0,+\infty)$$

CONTENT

GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

Normalization and asymptotic approximation of scaling constant for α ≥ 0:

$$g(x) = Ax^{\alpha} e^{-\lambda x} e^{-\frac{\beta}{x}}$$

Normalization constant: $$A = \frac{\left(\sqrt{\frac{\lambda}{\beta}}\right)^{\alpha+1}}{2K_{\alpha+1}\left(2\sqrt{\beta\lambda}\right)}$$

$$K_{\alpha}(x) = K_{-\alpha}(x)$$

Asymptotic approximation: $$\lambda = \alpha + \beta + \frac{3}{2}$$

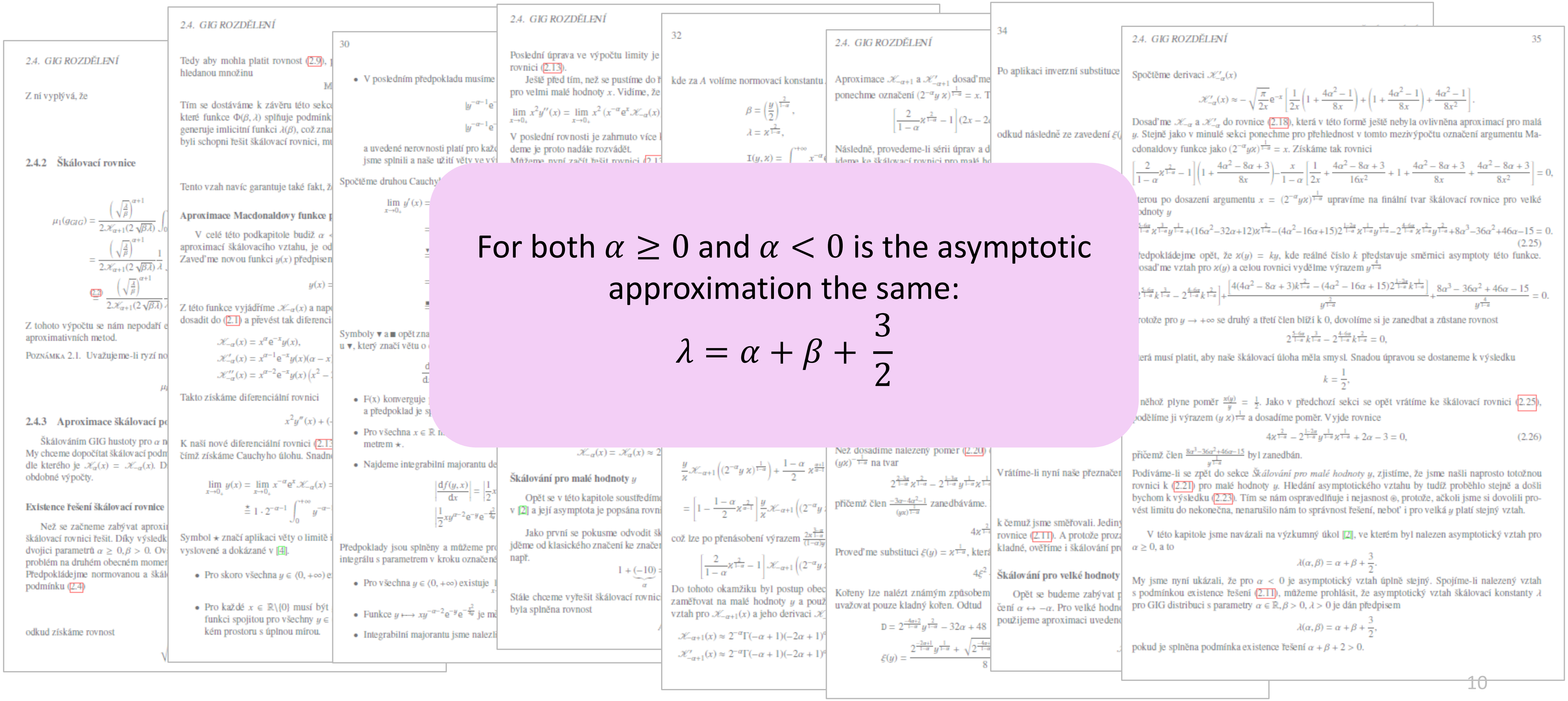
VACKOVÁ, Jana: Multi-headway statistika systémů s kombinovanými potenciály, Výzkumný úkol FJFI ČVUT, 2016

GENERALIZED INVERSE GAUSSIAN DISTRIBUTION

1. scaling equation: $\int_{-\infty}^{+\infty} xg(x) = 1$;
2. derivation of integral depending on parameter;
3. differential equation for $K_\alpha(\sqrt{\beta\lambda})$;
4. solution for $\beta \ll 1$ and $\beta \to +\infty$;
5. asymptotic approximation

$$K_\alpha(x) = K_{-\alpha}(\mathrm{x})$$

GENERALIZED INVERSE GAUSSIAN DISTRIBUTION



For both $\alpha \geq 0$ and $\alpha < 0$ is the asymptotic approximation the same:

$$\lambda = \alpha + \beta + \frac{3}{2}$$

CONTENT

1) BALANCED DENSITIES
2) GENERALIZED INVERSE GAUSSIAN DISTRIBUTION
3) PROBLEM IN APPLICATION
4) SCALING CONDITION

PROBLEM IN APPLICATION

Let X ~ GIG(x;α,β,λ), $\mu_0 = \int_{-\infty}^{+\infty} g(x)dx = 1$ and $\mu_1 = \int_{-\infty}^{+\infty} xg(x)dx = 1.$

Then

$$\mu_2 = \left\{ \mu_k(g_{GIG}) = \left(\frac{\beta}{\lambda}\right)^{\frac{k}{2}} \frac{\mathscr{K}_{\alpha+1+k}(2\sqrt{\beta\lambda})}{\mathscr{K}_{\alpha+1}(2\sqrt{\beta\lambda})} \right\} = \frac{\alpha+\beta+2}{\lambda}.$$

$$\mu_2 = \int_{-\infty}^{+\infty} x^2 f(x)dx \ngeq 0 \qquad \forall \alpha \in \mathbb{R} \text{ !!!}$$

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SCALING CONDITION

Examination of the existence of the scaling equations‘ solution:

$$\mu_1 = \sqrt{\frac{\beta}{\lambda}} \frac{\mathcal{K}_{\alpha+2}(2\sqrt{\beta\lambda})}{\mathcal{K}_{\alpha+1}(2\sqrt{\beta\lambda})} \overset{!}{=} 1 \qquad \Rightarrow \qquad \Phi(\beta, \lambda) = \sqrt{\frac{\beta}{\lambda}} \frac{\mathcal{K}_{\alpha+2}(2\sqrt{\beta\lambda})}{\mathcal{K}_{\alpha+1}(2\sqrt{\beta\lambda})} - 1$$

Idea: If we find $M = \{\beta > 0 \colon \Phi(\beta, \lambda) = 0\}$, then $\Phi(\beta, \lambda)$ generates unique function $\lambda(\beta)$.

We found scaling condition for $\alpha < -2$ in the following form: $\alpha + \beta + 2 > 0.$

THANK YOU FOR YOUR ATTENTION!

References:

[1] *VACKOVÁ, Jana: Multi-headway statistika systémů s kombinovanými potenciály, Výzkumný úkol FJFI ČVUT, 2016*