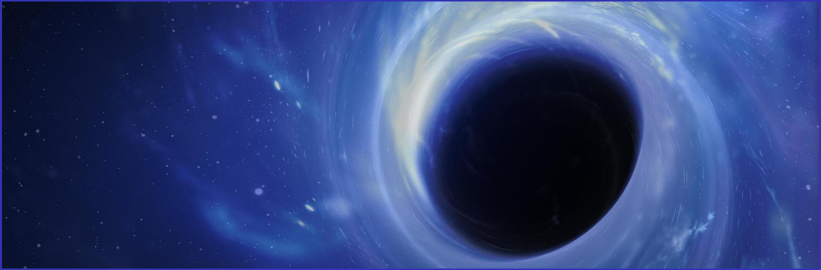

Imaginary magnetic field in Relativistic Quantum Mechanics

Alexandra Ridziková

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Content

Magnetic field in Quantum Mechanics

- The state of the system is described by a vector ψ in a Hilbert space and its time evolution is governed by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

with H being a linear self-adjoint operator representing the total energy of the system.

- Hamiltonian of charged particle in electromagnetic field

$$H = \frac{1}{2m}(\vec{P} - q\vec{A})^2 + q\varphi,$$

where $\vec{A}$ and φ are potentials for which it applies:

$$\vec{B} = \mathrm{rot}\vec{A}, \qquad \vec{E} = -\mathrm{grad}\varphi - \frac{\partial\vec{A}}{\partial t}.$$

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- Why self-adjointness is required?
  The Spectrum of the linear operator assigned to observable must be identical to the set of values that can be measured for that quantity.

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Can Quantum theory be extended by non-self-adjoint operators playing the role of observables?

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Can Quantum theory be extended by non-self-adjoint operators playing the role of observables?

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Quasi-Hermitian Quantum Mechanics

- This unconventional representation of observables is consistent with fundamental axioms of Quantum Mechanics if, and only if, the *non-self-adjoint* observable H is *quasi-self-adjoint*, i.e:

$$H^* = \Theta H \Theta^{-1}, \tag{1}$$

  where Θ is positive, bounded and boundedly invertible operator called *metric*.

- H is self-adjoint with modified inner product $\langle \cdot | \cdot \rangle_\Theta := \langle \cdot | \Theta \, \cdot \rangle$

$$\langle \phi | H\psi \rangle_\Theta = \langle \phi | \Theta \, \Theta^{-1} H^* \Theta \, \psi \rangle = \langle H\phi | \Theta \, \psi \rangle = \langle H\phi | \psi \rangle_\Theta .$$

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Quasi-Hermitian Quantum Mechanics

- H is quasi-self-adjoint if, and only if, it is *similar* to a self-adjoint operator.
- There exists a *self-adjoint* operator h and a bounded and boundedly invertible operator Ω such that

$$h = \Omega H \Omega^{-1}. \tag{2}$$

Indeed if H satisfies (1), then h from (2) is self-adjoint provided that we set $\Omega := \Theta^{1/2}$. Vice versa, an operator H satisfying (2) is quasi-self-adjoint with $\Theta := \Omega \cdot \Omega$.

Content

1. Magnetic field in Quantum Mechanics

2. Imaginary magnetic field?
   - Quasi-Hermitian Quantum Mechanics
   - Black holes

3. Magnetic Dirac operator on circle
   - My research

4. Conclusion

Stability of rotating Black holes

- [Andersson 2005], [Jaramillo 2015]

- The apparent horizon is the boundary between the light that is trapped inside a black hole and the light that is able to escape gravity at a given time. More specifically we will focus on *marginally outer trapped surfaces (MOTS)* that possess the stability notion that guarantees their physical consistency as models of black hole horizons.

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Stability of rotating black holes

- Let us consider $n-$dimensional spacetime (M, g_{ab}) and spacelike, closed (compact and without boundary) and orientable surface $\mathcal{S}$ with codimension-2 embedded in (M, g_{ab}).
- MOTS-stability notion admits a spectral characterization in terms of the so-called principal eigenvalue of the operator

$$L_{\mathcal{S}}\psi = \left[-\Delta + 2\Omega^a D_a - (|\Omega|^2 - D_a\Omega^a - \frac{1}{2}R_{\mathcal{S}} + G_{ab}k^a l^b)\right]\psi$$

defined on the apparent horizon $\mathcal{S}$.

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- By substitution:

$$\Omega_a = \frac{ie}{\hbar c} A_a \qquad R_{\mathcal{S}} = \frac{4me}{\hbar^2}\varphi, \qquad G_{ab} k^a l^b = -\frac{2m}{\hbar^2} V$$

passes the stability operator $L_{\mathcal{S}}$ to Hamiltonian of non-relativistic charged particle.

- Applies

$$\frac{\hbar^2}{2m} L_{\mathcal{S}} = H,$$

where

$$H = \frac{1}{2m}\Big(-i\hbar D - \frac{e}{c} A\Big)^2 + e\varphi + V.$$

- Potential A purely imaginary!

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Dirac operator

- We introduce relativistic Dirac operator D_a on circle with complex-valued magnetic potential $a : (-\pi, \pi) \to \mathbb{C}$ and mass m:

$$(D_a\psi)(x) := \begin{pmatrix} m & -i\partial_x - a \\ -i\partial_x - a & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

with domain

$$\mathrm{Dom}\, D_a := \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in W^{1,2}((-\pi,\pi);\mathbb{C}^2) : \psi(-\pi) = \psi(\pi) \right\}.$$

- Operátor D_a is self-adjoint $\iff$ $\mathrm{Im}\, a = 0$.

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Theorem

- *Operator D_a is quasi-self-adjoint if, and only if*

$$\operatorname{Im}\langle a\rangle = 0,$$

with metric operator

$$(\Theta\psi)(x) := \begin{pmatrix} \exp\left(2\operatorname{Im} A(x)\right) & 0 \\ 0 & \exp\left(2\operatorname{Im} A(x)\right) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

- *Operator D_a satisfies the similarity relation*

$$\Omega_a D_a \Omega_a^{-1} = D_{\langle a\rangle},$$

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Spectrum

- We obtain the spectrum by solving the equation $D_a\psi = \lambda\psi$:

$$\begin{pmatrix} m & -i\partial_x - a \\ -i\partial_x - a & -m \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

$$\sigma(D_a) = \left\{ \pm\sqrt{m^2 + (n - \langle a \rangle)^2} \right\}_{n\in\mathbb{Z}},$$

$$\sigma(D_a^*) = \left\{ \pm\sqrt{m^2 + (n - \langle \overline{a} \rangle)^2} \right\}_{n\in\mathbb{Z}}.$$

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Spectrum

- Eigenfunctions of D_a:

$$\psi_n(x) = \frac{1}{\sqrt{4\pi\lambda_n}} \begin{pmatrix} \sqrt{\lambda_n + m} \\ \sqrt{\lambda_n - m} \end{pmatrix} \exp\Big(i(\sqrt{\lambda_n^2 - m^2}x + A(x))\Big).$$

- The eigenfunctions of the adjoint D_a^* are given by:

$$\phi_n(x) = \frac{1}{\sqrt{4\pi\lambda_n}} \begin{pmatrix} \sqrt{\overline{\lambda_n} + m} \\ \sqrt{\overline{\lambda_n} - m} \end{pmatrix} \exp\Big(i(\sqrt{\overline{\lambda_n^2} - m^2}x + \overline{A(x)})\Big).$$

- The normalisation factors are chosen in such a way that the standard biorthogonal condition $\langle\psi_m|\phi_n\rangle = \delta_{mn}$ holds.

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Basis properties

Definition (Riesz basis)

$\{\psi_n\}_{n=1}^{+\infty}$ *is Riesz basis on Hilbert space* $\mathcal{H}$*, if there exists orthonormal basis* $\{e_n\}_{n=1}^{+\infty}$ *on Hilbert space* $\mathcal{H}$ *and bounded invertible operator* ξ *that satisfies:*

$$\xi e_n = \psi_n \qquad \forall n \in \mathbb{N}.$$

Basis properties

Theorem

The eigenfunctions of D_a form a Riesz basis.

- If $\operatorname{Im}\langle a\rangle = 0 \implies \lambda_n = \overline{\lambda_n}$, we can write eigenvectors using ON basis

$$e_n(x) := \frac{1}{\sqrt{4\pi\lambda_n}} \begin{pmatrix} \sqrt{\lambda_n + m} \\ \sqrt{\lambda_n - m} \end{pmatrix} \exp\left(i\sqrt{\lambda_n^2 - m^2}\, x + i \operatorname{Re} A(x) \right)$$

and bounded positive function $\xi(x)$

$$\xi(x) := \exp(-\operatorname{Im} A(x)),$$

as:

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Conclusion

- Relevance of complex magnetic field in physics.
- My results for Dirac operator with complex magnetic potential on circle:
  - The purely *real spectrum* of D_a under the condition $\operatorname{Im}\langle a\rangle = 0$, which represents more general condition than self-adjointness (ie. $\operatorname{Im} a = 0$)
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