Heavy Tail Distribution Derived with Maximum Renyi Entropy Principle for Returns of Financial Assets

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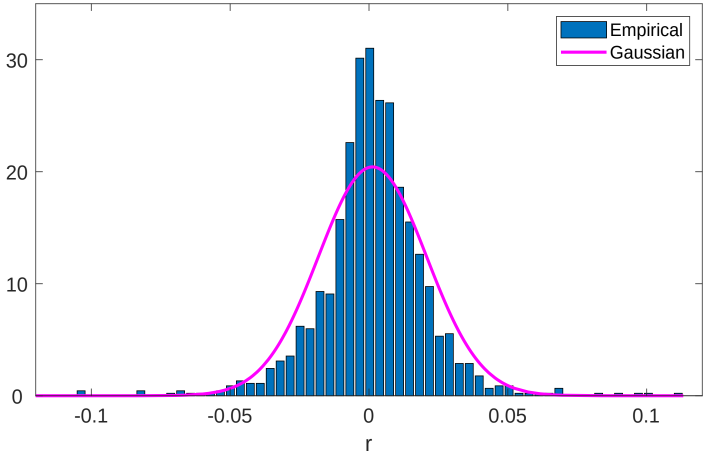
The 13th International Conference on Stochastic and Physical Monitoring Systems
Rumburk, Czech Republic, June 23 - 27, 2022

Motivation

Figure 1: The density of returns of stock price of Apple Inc.



Motivation and Research Objective

- Returns of financial assets exhibit leptokurtic property and normal distribution is not a good candidate for modeling returns of financial assets
- The light tails or normal distribution leads to underestimation of risk exposure of investments which induces unfulfillment of capital adequacy requirement of firms
- Several alternative distributions have been proposed to to deal with this problems: t-distribution, generalized normal distribution, Normal Inverse Gaussian distribution (NIG), Alpha stable distribution
- We want to contribute to this debate and propose a new distribution which is derived from maximum Renyi entropy principle
- We verify its usability and compare its applicability with existing alternatives

Entropy maximization approach

- Entropy maximization is a common approach to derive the probability distribution of a random variable with insufficient information
- The probability distributionhe is derived as a solution to the problem of Renyi entropy maximization subject to a constraint on generalized moments
- Renyi entropy maximization problem for $\alpha > 0, \alpha \neq 1$ is defined as

$$\mathrm{H}_\alpha(X) = \frac{\ln \int_{-\infty}^{+\infty} \mathrm{f}^\alpha(x)\mathrm{d}x}{1-\alpha} = \max \tag{1}$$

s.t.

$$\int_{-\infty}^{+\infty} \mathrm{f}(x)\mathrm{d}x = 1.$$

Properties of Renyi entropy

There are three basic properties of Renyi entropy:

- $\lim_{\alpha \to 1} \mathrm{H}_\alpha(X) = \mathrm{H}_1(X) = \mathrm{H}_{\text{Shannon}}(X)$,
- translation invariancy $\mathrm{H}_\alpha(X + \xi) = \mathrm{H}_\alpha(X)$ for any $\xi \in \mathbb{R}$,
- scaling formula $\mathrm{H}_\alpha(X/s) = \mathrm{H}_\alpha(X) - \ln s$ for any $s > 0$.

It is clear that Renyi entropy is the generalization of Shannon entropy.

Deriving a new distribution

Supposing $\mathrm{E}X^4 < \infty$, μ, σ are location and scale parameters

$$Y = (X - \mu)/\sigma, \tag{2}$$

hence, $\mathrm{E}Y = 0, \mathrm{E}Y^2 = 1$, $Y \sim \mathrm{g}$ having a continuous density g and finite kurtosis $\mathrm{E}Y^4$. The density is transformed

$$\mathrm{f}(x) = \mathrm{g}((x - \mu)/\sigma), \tag{3}$$

$$\mathrm{H}_\alpha(Y) = \mathrm{H}_\alpha(X) - \ln \sigma. \tag{4}$$

so this distribution has only two parameters $\alpha > 0, \beta \in (0, 4]$ that are the shape parameters of the new distribution

$$\mathrm{H}_\alpha(Y) = \max \tag{5}$$

with respect to

$$\mathrm{E}|Y|^\beta = Q, \tag{6}$$

$Q > 0$ only guarantees the final standardization requirement $\mathrm{E}Y^2 = 1$. The condition $\mathrm{E}Y = 0$ is also satisfied as the result of (5), (6).

Case $\alpha = 1$

We maximize Shannon entropy

$$\mathrm{H}_{\mathrm{Shannon}}(Y) = -\int_{-\infty}^{+\infty} \mathrm{g}(y) \ln \mathrm{g}(y) \mathrm{d}y, \tag{7}$$

with respect to the constrains

$$\int_{-\infty}^{+\infty} \mathrm{g}(y) \mathrm{d}y = 1, \tag{8}$$

$$\int_{-\infty}^{+\infty} |y|^{\beta} \mathrm{g}(y) \mathrm{d}y = Q. \tag{9}$$

Using Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$, we design the functional

$$\mathscr{L} = \int_{-\infty}^{+\infty} \Big(-\mathrm{g}(y) \ln \mathrm{g}(y) + \lambda_1 \mathrm{g}(y) + \lambda_2 |y|^{\beta} \mathrm{g}(y) \Big) \mathrm{d}y, \tag{10}$$

which has to be maximized via g and minimized via λ_1, λ_2. The saddle point conditions are

$$\frac{\partial \mathscr{L}}{\partial g} = \frac{\partial \mathscr{L}}{\partial \lambda_1} = \frac{\partial \mathscr{L}}{\partial \lambda_2} = 0, \tag{11}$$

which implies

$$-\ln g(y) - 1 + \lambda_1 + \lambda_2 |y|^\beta = 0 \tag{12}$$

together with the constrains (8), (9). The optimal density is

$$g(y) = \exp(\lambda_1 - 1 + \lambda_2 |y|^\beta). \tag{13}$$

The left hand side integral in (8) is finite only when $\lambda_2 < 0$. Resulting density formula is

$$g(y) = \frac{C}{2} \exp\left(- \left(\frac{|y|}{s} \right)^\beta \right), \tag{14}$$

where $C, s > 0$ are a normalizing factor and the scale. Let $Z = |Y|$, we can easily express its density as

$$h(z, 1, \beta) = C \exp\left(- \left(\frac{z}{s} \right)^\beta \right) \tag{15}$$

for $z \geq 0$.

Case $\alpha = 1$

The normalizing factor C, the CDF $\mathrm{H}(z, 1, \beta)$, and the central absolute moment M_γ^* of Y can be evaluated using the integral

$$\mathrm{P}(z, a, b, s) = \int_0^z x^a \exp\left(-\left(\frac{x}{s}\right)^b\right) \mathrm{d}x. \tag{16}$$

We obtain explicit formulas

$$C = \frac{1}{\mathrm{P}(+\infty, 0, \beta, s)} = \frac{\beta}{s\Gamma(1/\beta)}, \tag{17}$$

$$\mathrm{H}(z, 1, \beta) = C \cdot \mathrm{P}(z, 0, \beta, s) = \Gamma\left(\left(\frac{z}{s}\right)^\beta, \frac{1}{\beta}\right), \tag{18}$$

$$M_\gamma^* = C \cdot \mathrm{P}(+\infty, \gamma, \beta, s) = \frac{\Gamma((\gamma+1)/\beta)s^\gamma}{\Gamma(1/\beta)}. \tag{19}$$

Using $M_2^* = 1$, we obtain the scaling factor

$$s = \left(\frac{\Gamma(1/\beta)}{\Gamma(3/\beta)}\right)^{1/2}. \tag{20}$$

When $\alpha > 1$, the Renyi entropy maximization is equivalent to

$$-\int_{-\infty}^{+\infty} g^{\alpha}(y)dy = \max \tag{21}$$

wrt constrains (8),(9).
Using Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$, we design another functional

$$\mathscr{L} = \int_{-\infty}^{+\infty} \left(-g^{\alpha}(y) + \lambda_1 g(y) + \lambda_2 |y|^{\beta} g(y)\right) dy, \tag{22}$$

which has to be maximized via g and minimized via λ_1, λ_2. The saddle point conditions (11) produce

$$-\alpha g^{\alpha-1}(y) + \lambda_1 + \lambda_2 |y|^{\beta} = 0. \tag{23}$$

With the constrains (8), (9). Let $(\xi)_+ = \max(\xi, 0)$, the optimal density is

$$g(y) = \left(\frac{\lambda_1}{\alpha} + \frac{\lambda_2}{\alpha}|y|^{\beta}\right)_+^{\frac{1}{\alpha-1}}. \tag{24}$$

Case $\alpha > 1$ with Finite Support

The value $\lambda_1 \geq 0$ guarantees $\mathrm{g}(0) \geq 0$. The left hand side integral in (8) is finite only when $\lambda_2 < 0$. Moreover, $\mathrm{g}(y)$ exists only for $\lambda_1 > 0$. The resulting density formula is

$$\mathrm{g}(y) = \frac{C}{2}\left(1 - \left(\frac{|y|}{s}\right)^{\beta}\right)_{+}^{\frac{1}{\alpha-1}}, \tag{25}$$

where $C, s > 0$ are a normalizing factor and the scale. The random variable $Z = |Y|$ has the density

$$\mathrm{h}(z, \alpha, \beta) = C\left(1 - \left(\frac{z}{s}\right)^{\beta}\right)^{\frac{1}{\alpha-1}} \tag{26}$$

for $z \in [0, s]$.

The normalizing factor C, the cumulative distribution function $\mathrm{H}(z,\alpha,\beta)$, and the central absolute moment M_γ^* of Y can be evaluated using the integral

$$\mathrm{Q}(z,a,b,c,s) = \int_0^z x^a \left(1 - \left(\frac{x}{s}\right)^b\right)^c \mathrm{d}x, \tag{27}$$

We obtain explicit formulas

$$\begin{aligned} C &= \frac{1}{\mathrm{Q}(+\infty,0,\beta,1/(\alpha-1),s)} \\ &= \frac{\beta}{s\mathrm{B}(1/\beta,\alpha/(\alpha-1))} \\ &= \frac{\beta\Gamma(1/\beta+\alpha/(\alpha-1))}{s\Gamma(1/\beta)\Gamma(\alpha/(\alpha-1))}, \end{aligned} \tag{28}$$

Case $\alpha > 1$ with Finite Support

$$\begin{aligned} \mathrm{H}(z,\alpha,\beta) &= C\cdot \mathrm{Q}(z,0,\beta,1/(\alpha-1),s) \\ &= \mathrm{B}\left(\left(\frac{z}{s}\right)^{\beta},\frac{1}{\beta},\frac{\alpha}{\alpha-1}\right), \end{aligned} \tag{29}$$

$$\begin{aligned} \mathrm{M}_{\gamma^*} &= C\cdot \mathrm{Q}(+\infty,\gamma,\beta,1/(\alpha-1),s) \\ &= \frac{\mathrm{B}((\gamma+1)/\beta,\alpha/(\alpha-1))s^{\gamma}}{\mathrm{B}(1/\beta,\alpha/(\alpha-1))} \\ &= \frac{\Gamma((\gamma+1)/\beta)\Gamma(\alpha/(\alpha-1)+1/\beta)s^{\gamma}}{\Gamma(1/\beta)\Gamma(\alpha/(\alpha-1)+(\gamma+1)/\beta)}. \end{aligned} \tag{30}$$

Using standardization condition $M_2^* = 1$, we obtain the scaling factor

$$s = \left(\frac{\Gamma(1/\beta)\Gamma(\alpha/(\alpha-1)+3/\beta)}{\Gamma(3/\beta)\Gamma(\alpha/(\alpha-1)+1/\beta)}\right)^{1/2} > 0. \tag{31}$$

The formula (29) states that $T = (Z/s)^{\beta}$ has Beta distribution with parameters $p = 1/\beta, q = \alpha/(\alpha-1)$.

Case $\alpha \in (0,1)$ Generates Heavy Tails

It is the most interesting case as

$$+\int_{-\infty}^{+\infty} \mathrm{g}^{\alpha}(y)\mathrm{d}y = \max \tag{32}$$

wrt (8),(9). Using Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$, we design

$$\mathscr{L} = \int_{-\infty}^{+\infty} \left(\mathrm{g}^{\alpha}(y) + \lambda_1 \mathrm{g}(y) + \lambda_2 |y|^{\beta} \mathrm{g}(y) \right) \mathrm{d}y, \tag{33}$$

which must be maximized via g and minimized via λ_1, λ_2. The FOCs produce

$$\alpha \mathrm{g}^{\alpha-1}(y) + \lambda_1 + \lambda_2 |y|^{\beta} = 0. \tag{34}$$

together with the constrains (8), (9). The optimal density is

$$\mathrm{g}(y) = \left(-\frac{\lambda_1}{\alpha} - \frac{\lambda_2}{\alpha} |y|^{\beta} \right)^{\frac{1}{\alpha-1}}. \tag{35}$$

The left hand side integral in (8) is finite only when $\lambda_1 < 0, \lambda_2 < 0, 1-\beta < \alpha < 1$.

Case $\alpha \in (0,1)$ Generates Heavy Tails

The resulting density formula is

$$\mathrm{g}(y) = \frac{C}{2}\left(1+\left(\frac{|y|}{s}\right)^{\beta}\right)^{-\frac{1}{1-\alpha}}, \tag{36}$$

where $C, s > 0$ are a normalizing factor and the scale. The random variable $Z = |Y|$ has the density

$$\mathrm{h}(z,\alpha,\beta) = C\left(1+\left(\frac{z}{s}\right)^{\beta}\right)^{-\frac{1}{1-\alpha}} \tag{37}$$

for $z \geq 0$. The normalizing factor C, the CDF $\mathrm{H}(z)$, and M_{γ}^{*} of Y can be evaluated via

$$\mathrm{R}(z,a,b,c,s) = \int_0^z x^a\left(1+\left(\frac{x}{s}\right)^{b}\right)^{-c}\mathrm{d}x, \tag{38}$$

Finally, we obtain explicit formulas

$$\begin{aligned} C &= \frac{1}{\mathrm{R}(+\infty, 0, \beta, 1/(1-\alpha), s)} \\ &= \frac{\beta}{s\mathrm{B}(1/\beta, 1/(1-\alpha) - 1/\beta)} \\ &= \frac{\beta\Gamma(1/(1-\alpha))}{s\Gamma(1/\beta)\Gamma(1/(1-\alpha) - 1/\beta)}, \end{aligned} \tag{39}$$

$$\begin{aligned} \mathrm{H}(z, \alpha, \beta) &= C \cdot \mathrm{R}(z, 0, \beta, 1/(1-\alpha), s) \\ &= \mathrm{B}\left(\frac{(z/s)^{\beta}}{1+(z/s)^{\beta}}, \frac{1}{\beta}, \frac{1}{1-\alpha} - \frac{1}{\beta}\right), \end{aligned} \tag{40}$$

Case $\alpha \in (0,1)$ Generates Heavy Tails

$$
\begin{aligned}
M_\gamma^* &= C \cdot \mathrm{R}(+\infty, \gamma, \beta, 1/(\alpha-1), s) \\
&= \frac{\mathrm{B}((\gamma+1)/\beta, 1/(1-\alpha)-(\gamma+1)/\beta)s^\gamma}{\mathrm{B}(1/\beta, 1/(1-\alpha)-1/\beta)} \\
&= \frac{\Gamma((\gamma+1)/\beta)\Gamma(1/(1-\alpha)-(\gamma+1)/\beta)s^\gamma}{\Gamma(1/\beta)\Gamma(1/(1-\alpha)-1/\beta)}.
\end{aligned} \tag{41}
$$

Using standardization condition $M_2^* = 1$, we obtain the scaling factor

$$
s = \left(\frac{\Gamma(1/\beta)\Gamma(1/(1-\alpha)-1/\beta)}{\Gamma(3/\beta)\Gamma(1/(1-\alpha)-3/\beta)} \right)^{1/2} > 0. \tag{42}
$$

The formula (40) states that $T = \dfrac{(Z/s)^\beta}{1+(Z/s)^\beta}$ has Beta distribution with parameters $p = 1/\beta, q = 1/(1-\alpha)-1/\beta$.

The General Model

Replacing Z by original X we obtain general form of the novel distribution. The final PDF is

$$\mathrm{f}(x,\alpha,\beta,\mu,\sigma) = \frac{1}{2\sigma}\mathrm{h}(|x-\mu|/\sigma,\alpha,\beta). \quad (43)$$

Corresponding CDF is

$$\mathrm{F}(x,\alpha,\beta,\mu,\sigma) = \frac{1}{2} + \frac{\mathrm{sign}(x-\mu)}{2}\mathrm{H}(|x-\mu|/\sigma,\alpha,\beta). \quad (44)$$

The central absolute moment of X of the order $\gamma > 0$ is

$$\mathrm{E}|X-\mu|^{\gamma} = M_{\gamma}^{*}\sigma^{\gamma}. \quad (45)$$

The model parameters α,β,μ,σ can be estimated using:

- maximum likelihood method, which is valid when the density $\mathrm{f}(x,\alpha,\beta,\mu,\sigma)$ is smooth function,
- χ^2 method, which suppose the existence of $\mathrm{F}(x,\alpha,\beta,\mu,\sigma)$,
- moment method, which suppose the existence of moments up to order γ.

The General Model

Under the general supposition of non-degeneracy i.e. $\sigma > 0$ we can perform:

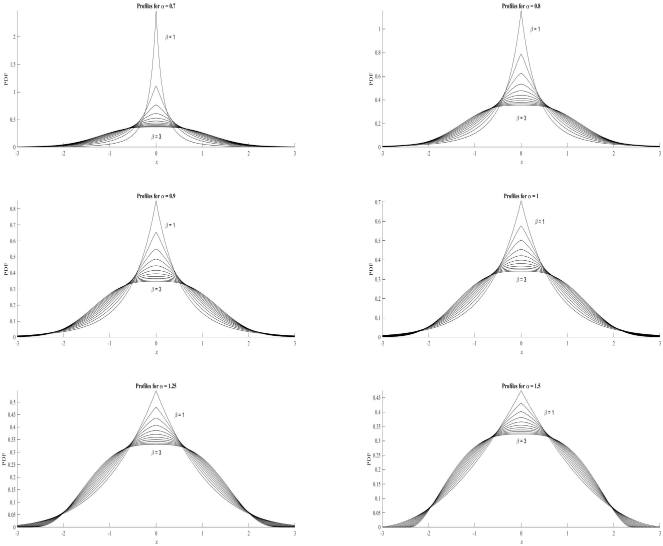
- maximum likelihood method only for $\beta > 1, \alpha > 1 - \beta$,
- χ^2 method only for $\alpha > 1 - \beta, \beta > 0$,
- moment method only for $\alpha > 1 - \beta/(\gamma + 1), \beta, \gamma > 0$.

The novel distribution:

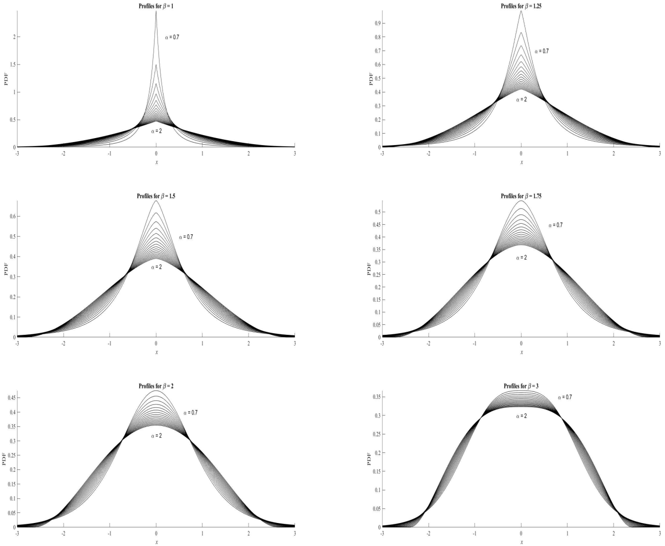
- is generalized normal distribution for $\alpha = 1$,
- has finite support for $\alpha > 1, \beta > 0$,
- has heavy tails for $\alpha \in (1 - \beta, 1), \beta > 0$.

All evaluations of f, F, and the moments are numerically unstable for $0 < |\alpha - 1| < \varepsilon = 0.02$. In this case we apply linear interpolation instead of direct evaluation for $\alpha \in (1 - \varepsilon, 1)$ or for $\alpha \in (1, 1 + \varepsilon)$, respectively.

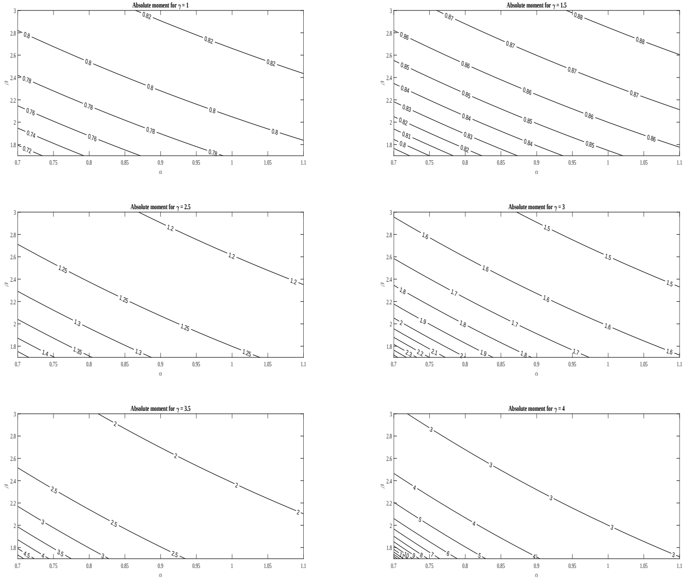
The role of parameter α in the new distribution



The role of parameter β in the new distribution



The role of parametera α and β on its moments



Data and Descriptive statistics

We select five different types of financial assets to perform our analysis. They are American stock price index S&P 500, gold price, stock price of three companies: Apple Inc., Boeing, and McDonald, exchange rate EUR/USD, and Bitcoin price in USD. All series are daily close of a period of five years from 22nd of March 2017 to 4th of February of 2022.

Characteristic	SP500	Gold	Apple	EURUSD	BTCUSD	BOEING	McDONALD
Mean	5.016e-4	3.224e-4	1.256e-3	-1.747e-05	1.552e-3	2.50e-4	3.63e-4
Median	8.982e-4	5.281e-4	1.140e-3	2.302e-05	1.294e-3	9.48e-4	4.916e-4
1st quartile	-3.355e-3	-3.798e-3	-7.534e-3	-2.500e-3	-0.0147	-0.0112	-0.0052
3rd quartile	6.106e-3	4.820e-3	0.0113	2.562e-3	0.0188	0.0115	0.0069
Maximum	0.0897	0.0578	0.1131	0.0160	0.1718	0.2161	0.1666
Minimum	-0.1277	-0.0511	-0.1377	-0.0281	-0.4647	-0.2404	-0.1729
St. deviation	0.0125	9.143e-3	0.0195	4.140e-3	0.0392	0.0291	0.0150
Skewness	-1.0573	-0.1113	-0.30156	-0.28698	-1.2577	-0.7489	-0.2904
Kurtosis	22.69	9.1335	9.339	5.6112	20.565	18.325	34.671
Num of obs	1259	1259	1259	1259	1259	1259	1259

Estimation of parameters of the new distribution

Parameters of the new distribution are estimated jointly using maximum likelihood estimation method (MLE). The MLE procedure is performed as follows:

$$\hat{\theta} = -\arg\min_{\theta \in \Theta} \sum_{i=1}^{n} \ln f(X_i; \theta) = -\arg\min_{\theta \in \Theta} \ln L(\theta), \qquad (46)$$

where Θ is the set of all admissible parameters and $f(X_i; \theta)$ is the density function of the corresponding distribution. The sign "-" is added so that the minimalization procedure can be applied. The MLE estimator has the following property

$$\sqrt{n}(\hat{\theta} - \theta_{true}) \sim N(0, \mathscr{I}^{-1}),$$

where $\mathscr{I} = -\mathbb{E}\left[\frac{\partial^2 \ln L}{\partial \theta \partial \theta'}\right]$ is the so called Fisher information matrix.

Estimation Results

Table 1: The log-likelihood values of all distributions

Distribution	SP500	Gold	Apple	EURUSD	BTCUSD	BA	MCD
t	4056.10	**4265.40**	3302.40	5157.10	2467.70	2964.0	**3818.4**
GED	4051.60	4255.20	3295.30	5147.30	**2481.60**	2951.3	3780.9
NIG	4059.60	4258.50	**3303.30**	5065.10	2476.10	**2965.4**	3808.6
ALPHA STABLE	4049.40	4257.70	3294.50	5151.30	2452.10	2957.2	3817.1
OUR NEW	**4061.70**	**4265.70**	**3303.60**	**5157.10**	**2482.70**	**2966.0**	**3819.7**

Table 2: The estimated values of parameters of new distribution for seven assets

Parameter	SP500	Gold	Apple	EURUSD	BTCUSD	BA	MCD
α	0.7080	0.6084	0.6821	0.7740	0.9198	0.6112	0.2545
asymp. S.E.	0.0717	0.0168	0.729	0.0.0519	0.0634	0.0488	0.0790
β	1.2708	1.7642	1.6210	1.9692	0.9786	1.5169	2.4953
asymp. S.E.	0.1134	0.0595	0.1310	0.1194	0.1203	0.0964	0.1886
σ	0.0127	0.0095	0.0198	0.0041	0.0386	0.0309	0.0180
asymp. S.E.	6.91e-4	3.98e-4	9.94e-4	1.09e-4	0.0015	0.0021	0.0034
μ	0.0010	0.0005	0.0014	2.39e-6	0.0013	0.0009	0.0006
asymp. S.E.	1.68e-4	1.80e-4	3.93e-4	9.65e-5	1.20e-5	8.97e-4	0.0054

Density comparison 1a

Figure 2: The density of returns of stock price index SP500

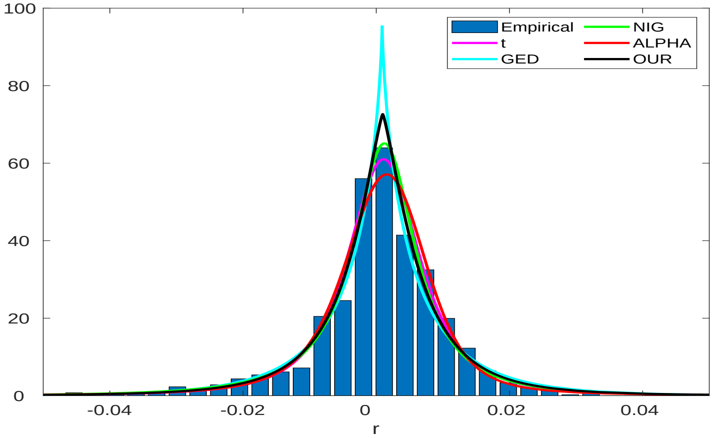
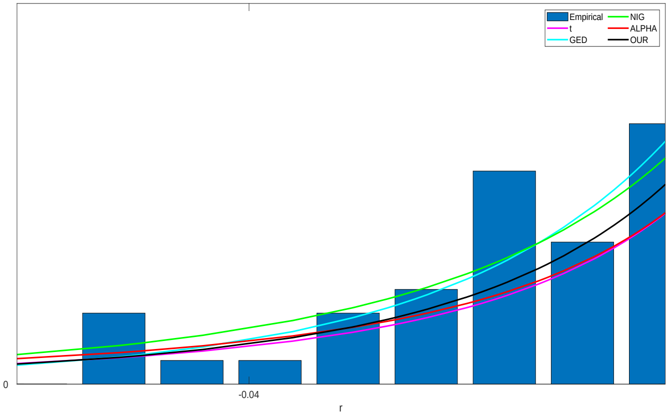
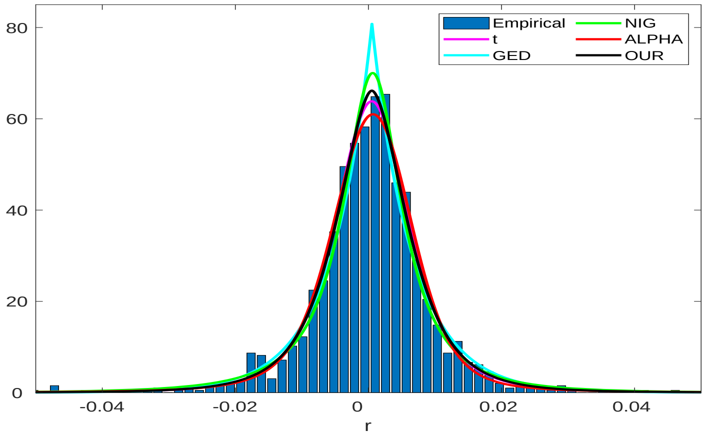


Figure 3: The density of returns of stock price index SP500



Density comparison 2a

Figure 4: The density of returns of gold price



Density comparison 2b

Figure 5: The density of returns of gold price

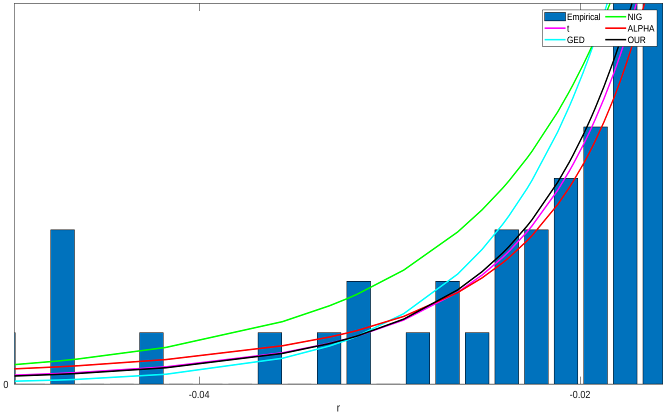


Figure 6: The density of returns of stock price of Apple Inc.

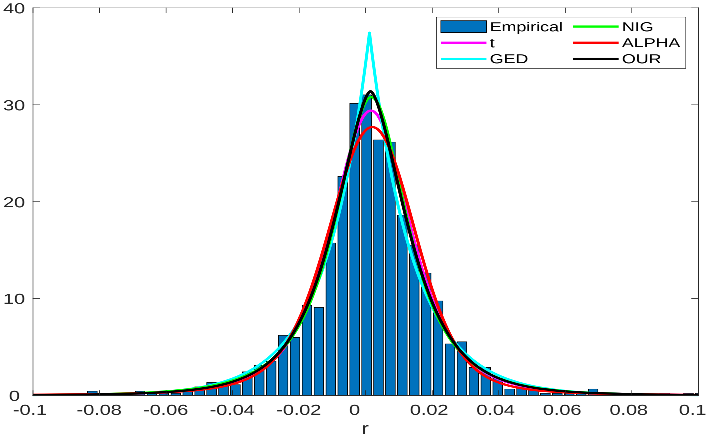
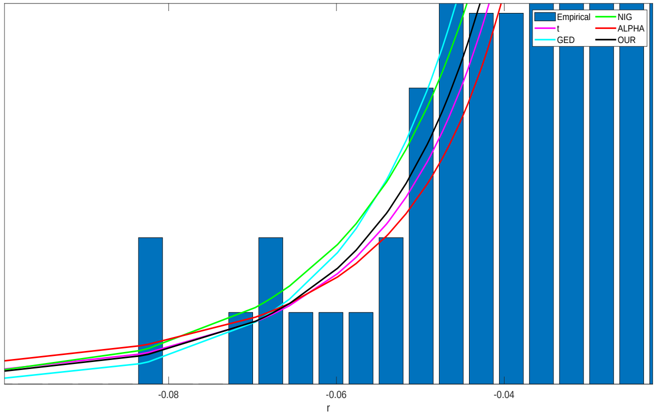


Figure 7: The density of returns of stock price of Apple Inc.



Density comparison 4a

Figure 8: The density of returns of EURUSD exchange rate

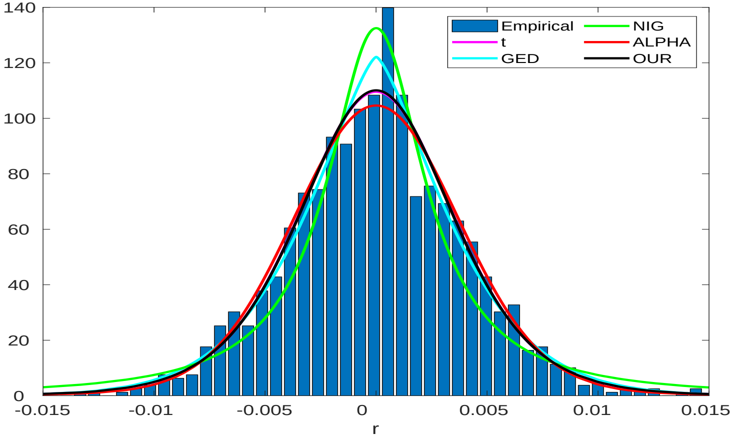
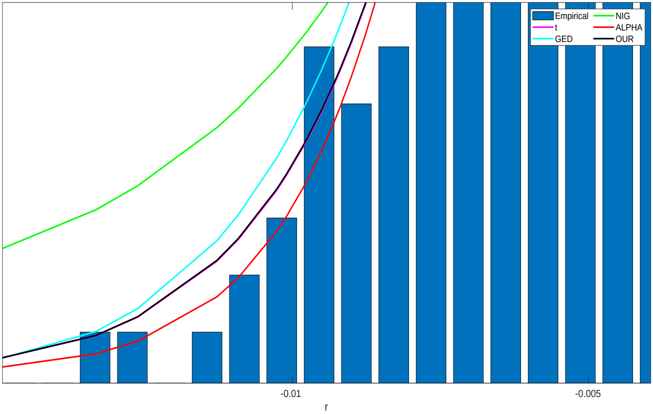


Figure 9: The density of returns of EURUSD exchange rate



Density comparison 5a

Figure 10: The density of returns of Bitcoin

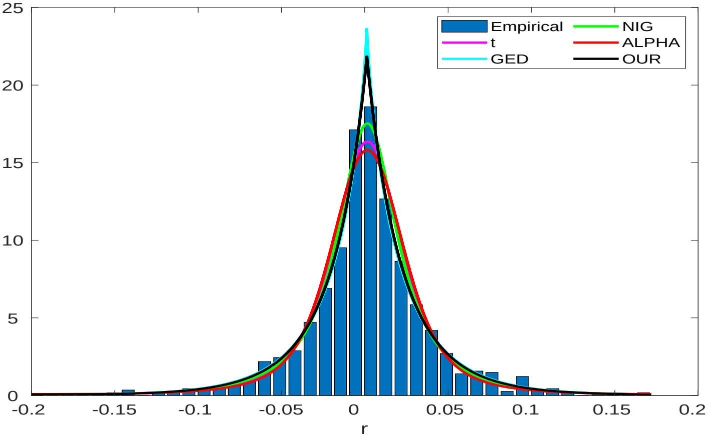
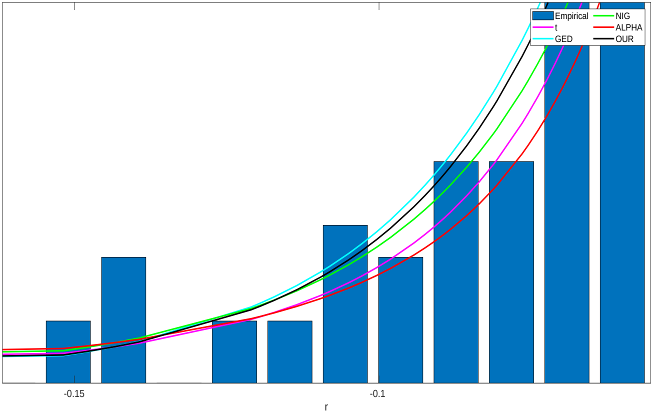


Figure 11: The density of returns of Bitcoin



Conclusion

- We derive a new distribution using Renyi entropy maximization principle with moments constraints
- In fact, it is a family of distributions depending on the values of their shape parameters
- We apply the newly obtainded distribution to model the distribution of five different types of financial instruments
- WE estimate parameters of this distribution, compare its goodness of fit with the one of existing heavy tail distribution alternatives
- The results show that it is dominant alternative to the existing variants of heavy tail distributions

Thank you for your attention!