Oscillatory Properties of Fractal Diffusion

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Motivation

- Leave boring word of 2D or 3D spaces!
- Embrace dimension as a real value number!
- Observe changes in dynamics!

Diffusion on Fractal Sets

- Let $\mathcal{G} \subset \mathbb{R}^d$ with measure μ_G, $d_f = \dim(\mathcal{G})$ and

$$d_f \neq d, \quad d_f \in \mathbb{R}$$

- Let $\{X_t\}_{t\in\mathcal{T}} \subset \mathcal{G}$ be (well behaved) diffusion process
- Let $R_t = ||X_t - X_0||_2$ be absolute travelled distance

Dimensions

- Fractal dimension d_f
- Walk dimension d_w
- Spectral dimension $d_s = 2d_f/d_w$

Properties

- $\mu_{\mathcal{G}}(B_d(x,r) \cap \mathcal{G}) \propto r^{d_f}$
- $\mathbb{E}\, R_t^\alpha \propto t^{\alpha/d_w}$
- $\Pr(R_t = 0) \propto t^{-d_s/2}$

Diffusion Modelling

Fractal Set Model

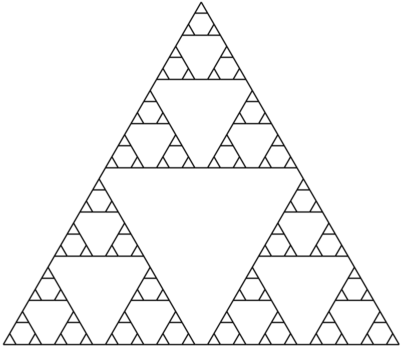
- Finite sub-graph $\mathcal{F} = (\mathcal{V}, \mathcal{E})$ of regular grid
- Recursive construction

Random Walk

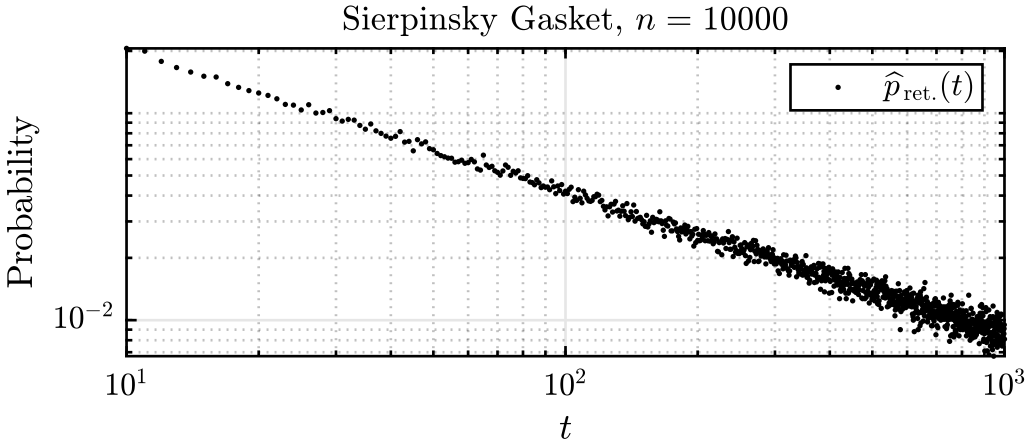
- Random walk X_t over graph $\mathcal{F}$
- $X_{t+1} \sim \mathrm{U}(\mathcal{N}_{\mathcal{E}}(X_t))$

Observables

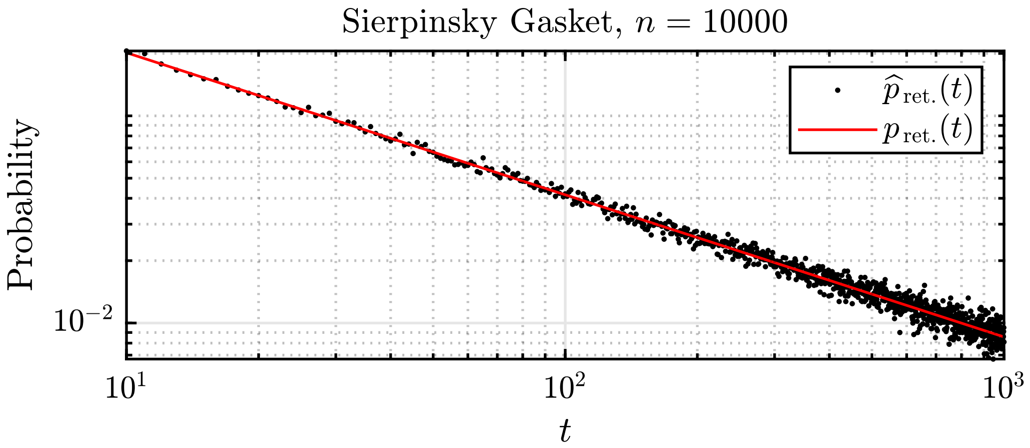
- $r_t = ||x_t - x_0||_2$
- $\widehat{p}_{\text{ret.}}(t) = \frac{1}{n} \sum_j^n 1_{(r_t=0)}$



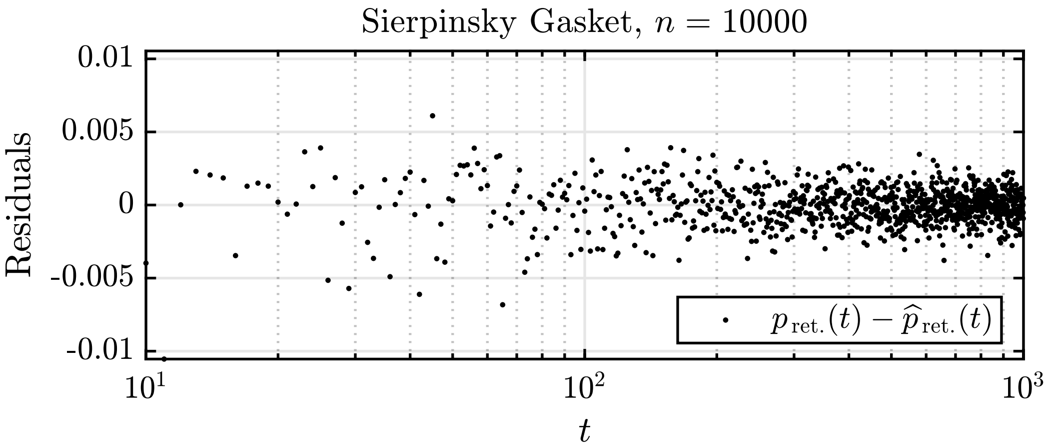
Data From Simulations



Dimension Estimation

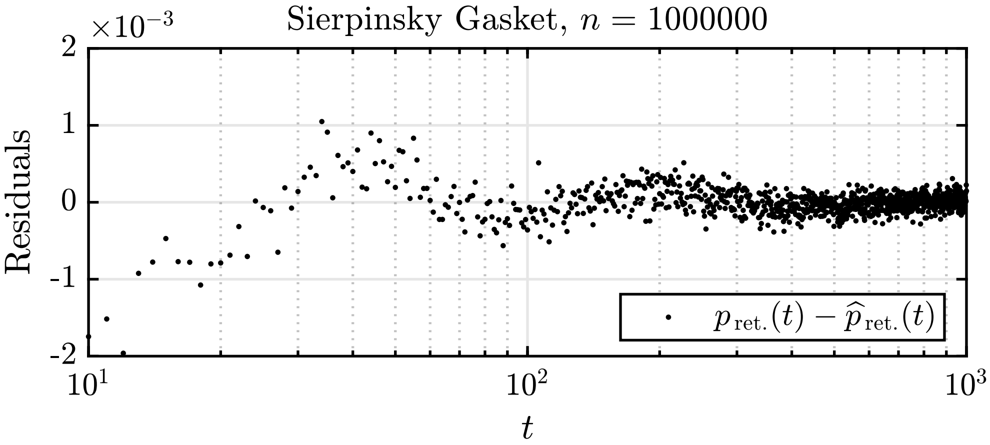


Residuals



Sierpinsky Gasket, $n = 10000$
0.01
0.005
0
-0.005
-0.01
Residuals
10^1
10^2
10^3
t
$p_{\text{ret.}}(t) - \widehat{p}_{\text{ret.}}(t)$

Residuals



Sierpinsky Gasket, $n = 1000000$
$\times 10^{-3}$
2
1
0
-1
-2
Residuals
10^1
10^2
10^3
t
$p_{\text{ret.}}(t) - \widehat{p}_{\text{ret.}}(t)$

Advanced Approaches

Distribution Bounds

$$A_1\, t^{-d_s/2} \exp\left(-Q_1 \left(\frac{r^{d_w}}{t}\right)^{1/(d_w-1)}\right) \leq \Pr(R_t = r) \leq A_2\, t^{-d_s/2} \exp\left(-Q_2 \left(\frac{r^{d_w}}{t}\right)^{1/(d_w-1)}\right)$$

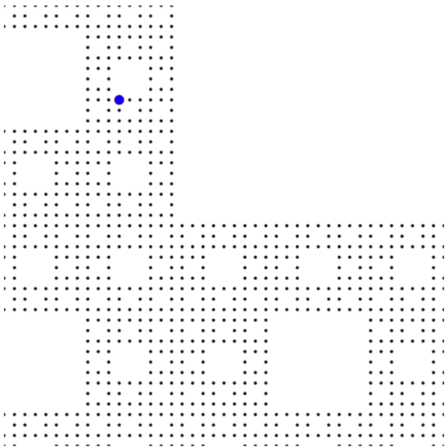
Periodic Model

$$\Pr(R_t = 0) = t^{-d_s/2}\, F\left(\frac{2\,\pi \ln t}{d_w \ln l}\right)$$

Alternative Diffusion Model

Butterfly Flight

- Only point set $\mathcal{V}$ assumed, no edges
- Underlying standard random walk on $\mathbb{N}^d$: Y_τ
- Studied random walk X_t
- $X_0 \sim \mathrm{U}(\mathcal{V})$
- $\Pr(X_{t+1} = x \mid X_t = y) = \Pr(Y_T = x \mid Y_0 = y), \quad x, y \in \mathcal{V}$
- $T = \min\{\tau : Y_\tau \in \mathcal{V}, \tau \in \mathbb{N}\}$



Analytical Study

- Exploration of numerically calculated values, not of estimations from MC simulations
- Focus on the ”1D” point sets e.g. Cantor Dust for computational feasibility

Analytical Approach

- Ordered point coordinates $x_1, \ldots, x_n$
- Unit jump probability p
- $\Pr(X_{t+1} = x_{i\pm1} | X_t = x_i) = \frac{p}{|x_{i\pm1} - x_i|}$

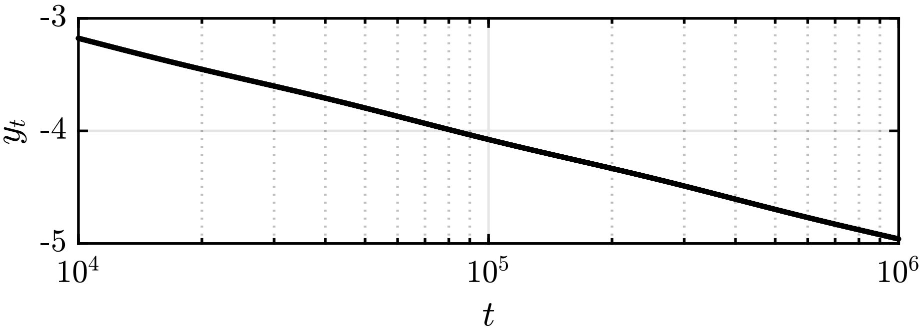
Numerical Tasks

- Recursively calculate points $x_1, \ldots, x_n$
- Construct (sparse) transition matrix $\boldsymbol{P}$
- Calculate return probabs. using $\boldsymbol{d}_t$

Analytical Study

Model with Oscillations

- $y_t = \log \Pr(||X_t - X_0|| = 0), \quad x = \log t$
- $y(t) = a_0 + a_1\, x + \sum_{k=1}^{n} c_k\, \cos(\omega\, k\, x) + s_k\, \sin(\omega\, k\, x)$



Analytical Study

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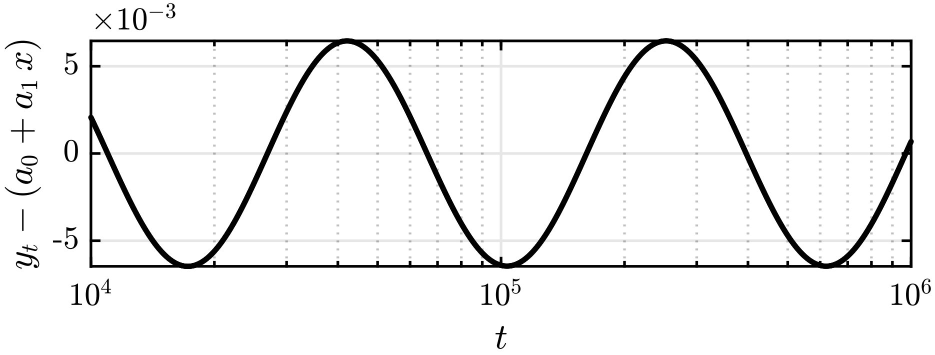

Analytical Study

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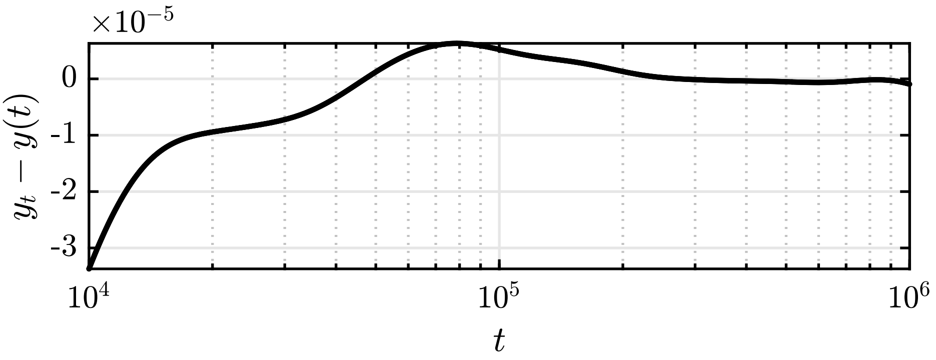

Analytical Study

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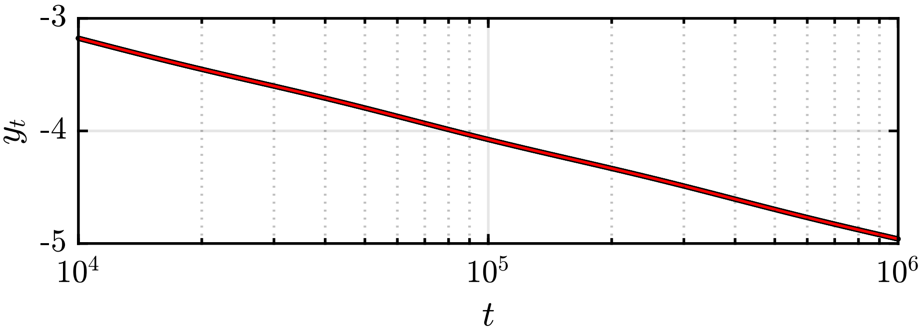

Remarks

- Not only one dimension
- Different ”speed” of diffusion in fractals
- Oscillations are natural occurrence

Thank you for your attention!