Ising Model of Ferromagnetism

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Content

Physical introduction

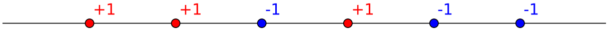

Figure: Example of configuration $\sigma_1 = (+1, +1, -1, +1, -1, -1)$.

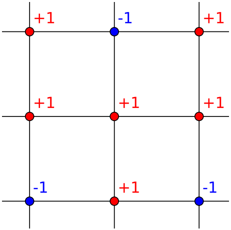

Figure: Example of configuration σ_2.

- first goal: find formula of partition function Z
- $Z = \sum_{\boldsymbol{\sigma} \in \Sigma} \exp\big(-\beta E(\boldsymbol{\sigma})\big)$, where $\beta = \frac{1}{k_B T}$
- $P\big(\boldsymbol{\sigma}\big) = \frac{1}{Z} \exp\big(-\beta E(\boldsymbol{\sigma})\big)$
- $E(\boldsymbol{\sigma}) = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j - \mu \sum_j h_j \sigma_j$
- $f\big(H, T\big) = \lim_{N \to +\infty} f_N\big(H, T\big) = -k_B T \lim_{N \to +\infty} \frac{\ln Z_N\big(H, T\big)}{N}$
- $M\big(H, T\big) = -\frac{\partial}{\partial H} f\big(H, T\big)$
- second goal: find critical temperature

Ising model in 1D

- let's assume N particles on line
- $E(\boldsymbol{\sigma}) = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} - H \sum_{j=1}^{N} \sigma_j$
- $M(H,T) = \frac{e^{\frac{J}{k_B T}} \sinh\left(\frac{H}{k_B T}\right)}{\left(e^{\frac{2J}{k_B T}} \sinh^2\left(\frac{2H}{k_B T}\right) + e^{\frac{-2J}{k_B T}}\right)^{\frac{1}{2}}}$
- there is no critical temperature in one dimension

Ising model in 1D

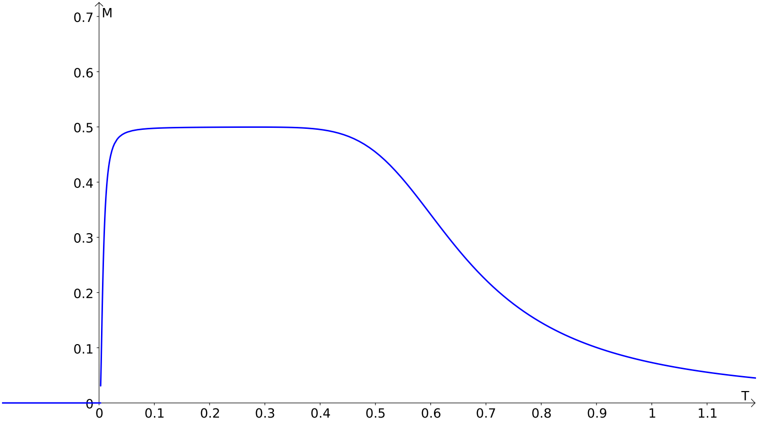

Figure: Magnetization M as a function of temperature T for $k_B = 1$, $J = 1$ a $H = 0.01$.

Ising model in 2D

- assumption $H = 0$
- $E(\boldsymbol{\sigma}) = -\sum_{\langle ij \rangle} J \sigma_i \sigma_j$

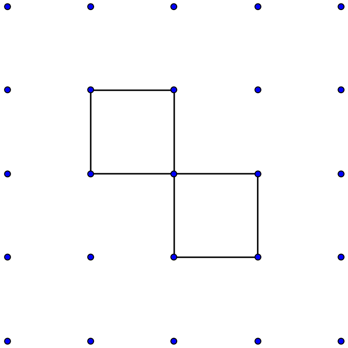

Figure: Example of admissible graph.

Ising model in 2D

- final formula:

$$f(T) = -k_B T \Bigg[\ln\big(2\cosh(2\beta J)\big) + + \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \left[1 - \frac{\sinh(2\beta J)}{\cosh^2(2\beta J)} (\cos\xi + \cos\eta) \right] \mathrm{d}\xi \, \mathrm{d}\eta \Bigg]$$

- critical temperature:

$$T_C = \frac{J}{k_B} \cdot \frac{2}{\ln\left(1+\sqrt{2}\right)}$$

$$T_C \doteq \frac{J}{k_B} \cdot 2.26$$

Ising model in 2D – second method

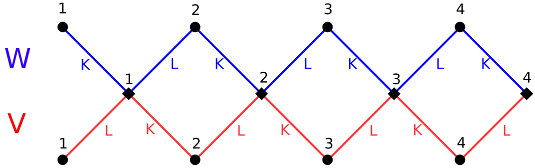

Figure: Example of rotated grid and functions V and W.

$$V(\boldsymbol{\sigma}^{\boldsymbol{i}}, \boldsymbol{\sigma}^{\boldsymbol{j}}) = \exp\left[\sum_{k=1}^{n}\left(K\sigma^i_{k+1}\sigma^j_k + L\sigma^i_k\sigma^j_k\right)\right]$$

$$Z_N = \mathrm{Tr}\left(\mathbb{V}\mathbb{W}\right)^{\frac{m}{2}} = \Lambda_1^{\frac{m}{2}} + \Lambda_2^{\frac{m}{2}} + \dots + \Lambda_{2^n}^{\frac{m}{2}}$$

$$\Lambda(K, L)\Lambda(L + \frac{1}{2}\pi\mathrm{i}, -K) = \left(2\mathrm{i}\sinh\left(2L\right)\right)^n + \left(-2\mathrm{i}\sinh\left(2K\right)\right)^n r$$

Simulations in 1D

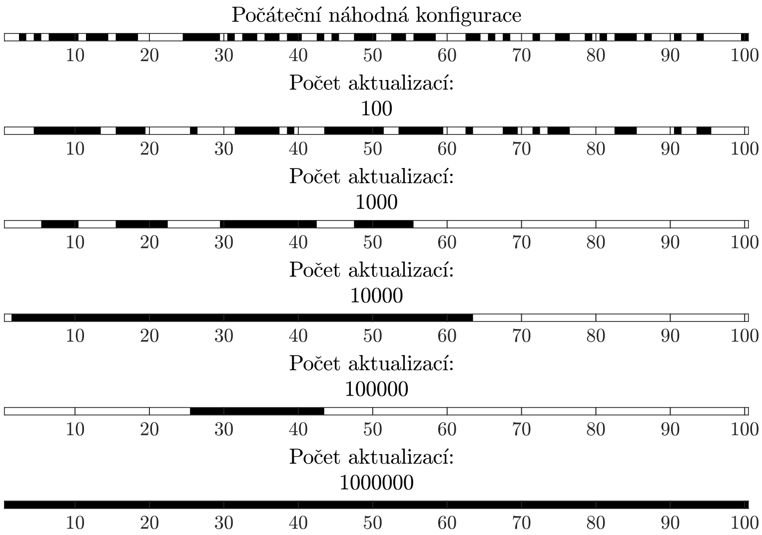

Figure: Simulations of Ising model in 1D for temperature $T = 0.1$.

Simulations in 2D

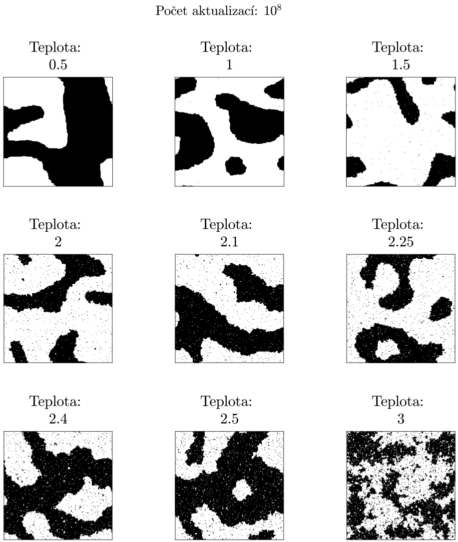

Figure: Simulations of Ising model in 2D – temperatures 0.5 - 3, 10^8 updates.

Critical temperature in 1D

- magnetization:

$$m = \frac{1}{N}\left|\sum_{i=1}^{N}\sigma_i\right|$$

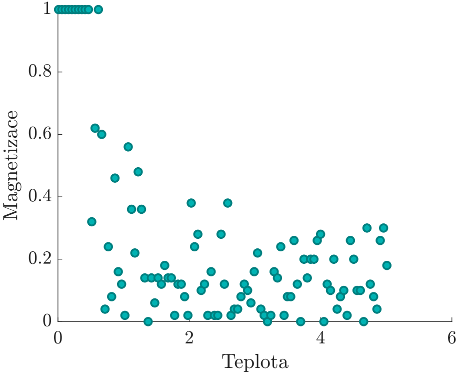

Figure: Critical temperature in 1D.

Critical temperature in 2D

$$T_C = \frac{2}{\ln\left(1+\sqrt{2}\right)}$$

$$T_C \doteq 2.26$$

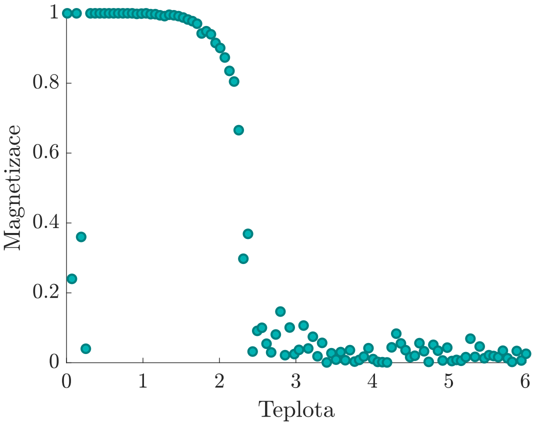

Figure: Critical temperature in 2D.

Critical temperature in 3D

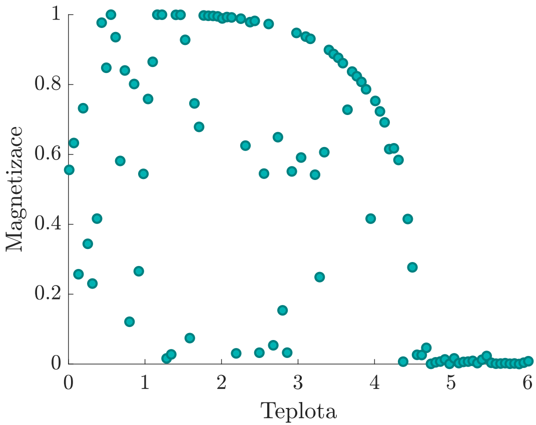

Figure: Critical temperature in 3D.

Critical temperature in 3D - the alternative

- magnetization - m_{TV}:

$$m_{TV} = \frac{1}{N} \sum_{\langle ij \rangle} \max\left(\sigma_i \sigma_j \; ; \; 0\right)$$

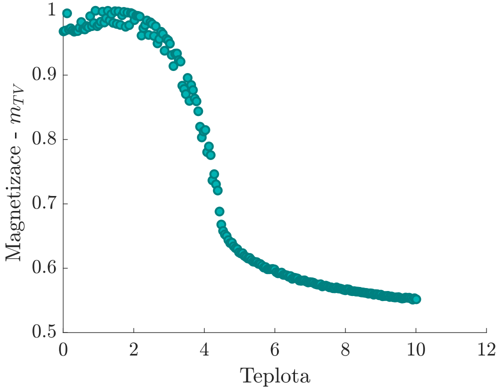

Figure: Critical temperature in 3D - m_{TV}.

Hexagonal grid

- each particle has 3 links

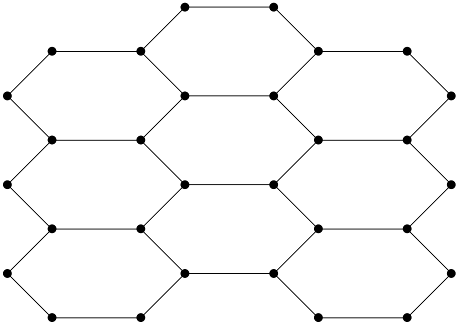

Figure: Example of hexagonal grid.

Critical temperature on hexagonal grid

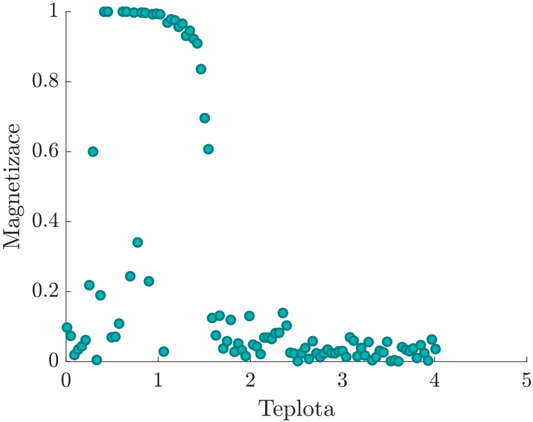

Figure: Critical temperature on hexagonal grid.

Triangular grid

- each particle has 6 links

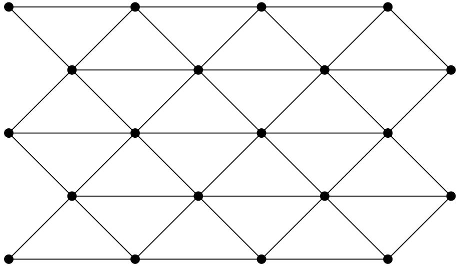

Figure: Example of triangular grid.

Critical temperature on triangular grid

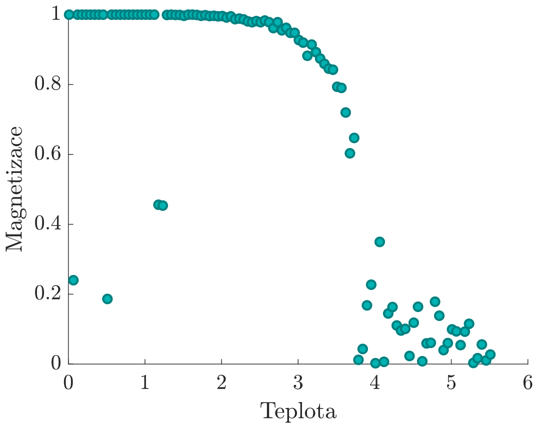

Figure: Critical temperature on triangular grid.

Conclusion

1. Theory
   - Physical introduction
   - Ising model in 1D
   - Ising model in 2D
2. Simulations

Thank you for your attention.