Quantum Tomography

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- Basic concepts
- Collins-Soper frame
- Tomography procedure for dilepton events

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Introduction

- **Tomography**: builds up higher dimensional objects from lower dimensional projections
  - CT: 3 dimensional image of the body is constructed from several X-ray scans from various angles
- **Quantum tomography** is a strategy to reconstruct all that can be observed about a quantum physical system
  - Bypassing model-dependent formalisms based on quantum field theory such as the scattering amplitudes or structure functions
  - Uses a known "probe" to explore an unknown system.
  - Data is related directly to matrix elements, with minimal model dependence and optimal efficiency

Basic Concepts

- Quantum state of a system: **Pure** or Mixed
- Pure state: a vector in a complex vector space denoted a Hilbert space, also called ket $|\psi\rangle$

  $\langle\psi|\psi\rangle = 1$
- In two dimensional Hilbert space ket is represented in its basis vectors:

  $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

  $$|\alpha|^2 + |\beta|^2 = 1$$

- In general dimension with basis ψ_j :

  $$|\psi\rangle = \sum_j c_j|\psi_j\rangle$$

Basic Concepts

- Quantum state of a system: Pure or **Mixed**
- The density matrix describes not only pure but also mixed states (an incoherent mixture of pure state) by denoting a $|\psi_k\rangle$ a set of pure states:

$$\rho = \sum_{k=1}^{N} p_k |\psi_k\rangle\langle\psi_k|, \qquad 0 < p_k \leq 1, \qquad \sum_{k=1}^{N} p_k = 1$$

- A projector defined as $\hat{P}_\psi = |\psi\rangle\langle\psi|$ is a type of operator that maps from and to a Hilbert space. It can be used to describe probabilities or expectation values from a measurement of a given state.
- The incoherent mixture can be obtained by interactions between pure states, quantum noise or decoherence.

Basic Concepts

- Density matrix allows to introduce expectation values of measured operators $\hat{A}$

$$\langle A \rangle = \mathrm{Tr}(A\rho)$$

- Density matrix can be further used for construction of Lorentz invariants such as the degree of polarization

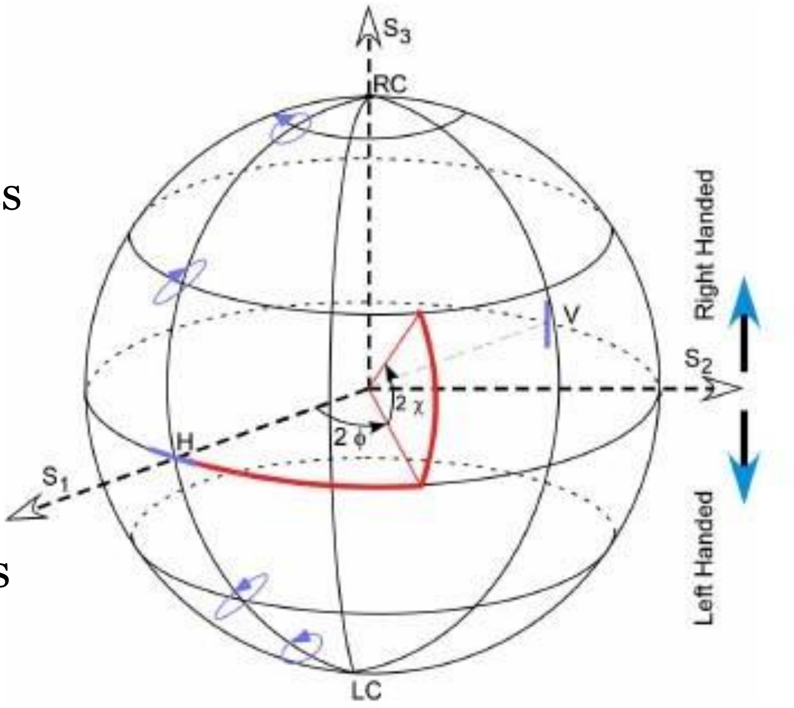
$$d = \sqrt{(3\mathrm{Tr}(\rho^2) - 1)/2}$$

- and entanglement entropy $\mathcal{S}$ that can be interpreted as a measure of order of quantum mechanical system.

$$\mathcal{S}(\rho) = -\mathrm{Tr}(\rho \log \rho)$$

Poincaré sphere

- Showing density matrix, states and polarization
- Sphere has a unit radius and is shown in a three dimensional coordinate space of Stokes parameters
- Points on the surface of the sphere are pure states
- Mixed states are placed within the sphere
- On the equator are places linearly polarized
- Closer to the poles, the polarization becomes circular
- The closer to the center of the sphere, the more unpolarized the states are



Tomography procedure

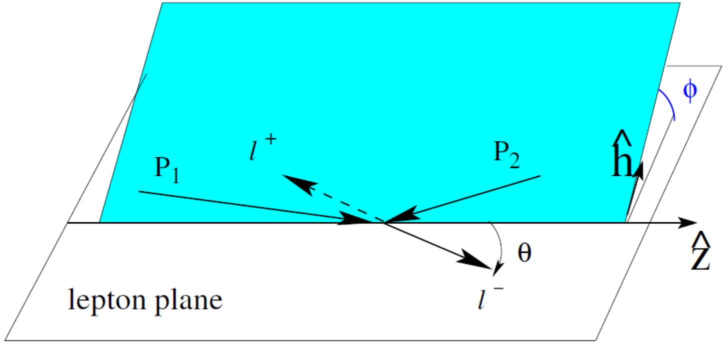
- The tomography procedure reconstructs all that can be observed about a quantum physical system.
- For inclusive lepton pair production
  - invariant mass distribution,
  - lepton pair angular distribution
  - polarization
- The unknown system is parameterized by a certain density matrix, which is model-independent $\rho(X)$
- The probe is described by a known density matrix $\rho(probe)$
- Matrices are represented by numbers generated and fit to experimental data, not abstract operators
- Quantum mechanics predicts an experiment will measure $tr(\rho(probe)\cdot\rho(X))$
  - What will be observed is limited by the dimension and symmetries of the probe, often 3x3 matrix

Collins-Soper frame

- Using vectors P_A and P_B for description of colliding nuclei, one may get axes defining the Collins-Soper (CS) frame

$$P_A P_B \to \ell^+(k)\ell^-(k') + \mathcal{X}$$

P_A=(1, 0, 0, 1), P_B=(1, 0, 0, -1)



- We will use total momentum

$$Q = k + k' \qquad Q^\mu = (\sqrt{Q^2}, \vec{Q} = 0)$$

- Lepton pair angular distributions are described in the pair rest-frame defined event-by-event. The frame orientation depends on the beam momenta and the pair total momentum
- The spacial axes of 4 vectors X^μ, Y^μ, Z^μ

$$Q \cdot X = Q \cdot Y = Q \cdot Z = 0$$

$$\tilde{Z}^\mu = P_A^\mu Q \cdot P_B - P_B^\mu Q \cdot P_A;$$

$$\tilde{X}^\mu = Q^\mu - P_A^\mu \frac{Q^2}{2Q \cdot P_A} - P_B^\mu \frac{Q^2}{2Q \cdot P_B}$$

$$\tilde{Y}^\mu = \epsilon^{\mu\nu\alpha\beta} P_{A\nu} P_{B\alpha} Q_\beta.$$

Probe

- The probe of unknown system, independent on the process yielding dilepton pair, can be parameterized as
  - *k* are momenta of leptons, α,β polarization indices in Dirac density matrix
  - Feynman rules with high energy limit

$$\rho(lep)^{\beta\beta'}_{\alpha\alpha'} \sim (\not{k}_1)_{\alpha\alpha'}(\not{k}_2)_{\beta\beta'}$$

- Density matrix is by default hermitian, we may rewrite

$$\rho(lep) = \rho^{\mu\nu}(lep)\gamma_\mu\gamma_\mu$$

- Then in the rest frame of dilepton pair

$$\rho^{jk}(lep) = \frac{1}{3}\delta^{jk} + a\,J^{jk}_p \ell^p - b\,U^{jk};$$

$$U^{jk}(\hat{\ell}) = \hat{\ell}^j\hat{\ell}^k - \frac{\delta^{jk}}{3}.$$

Probe

$$\rho^{jk}(lep) = \frac{1}{3}\delta^{jk} + a\, J_p^{jk} \ell^p - b\, U^{jk};$$

$$U^{jk}(\hat{\ell}) = \hat{\ell}^j \hat{\ell}^k - \frac{\delta^{jk}}{3}.$$

- J_p^{jk} is a well known rotation generator.
- $a\, J_p^{jk} \ell^p$ is tensor transforming as a spin-1 particle under rotations.
- The first term represents a tensor transforming as spin-0 and the third as spin-2.
- The second and third terms contain scalar variables *a*, *b* dependent on transferred momentum.
- The scalars depend on vertex from which the dilepton pair originates.

Density matrix

- From density matrix of the unknown intermediate system, we observe only part coupling to the lepton density matrix via polarization indices.
- The probe operators are orthogonal and measure a component of interest of the unknown system classified by its transformation properties, for studies of angular distributions, it is the transformation under rotation.
- The generator of a group of rotations is the J. From orthogonality properties:

$$\frac{1}{2}\mathrm{Tr}(\vec{\ell}\cdot\vec{J}\vec{S}\cdot\vec{J}) = \hat{\ell}\cdot\vec{S}.$$

- With this, the density matrix can be written similarly to the probe matrix:

$$\rho_{ij}(X) = \frac{1}{3}\delta_{ij} + \frac{1}{2}\vec{S}\cdot\vec{J}_{ij} + U_{ij}(X);$$
$$U(X) = U^T(X); \quad tr(U(X)) = 0$$

Density matrix parameter estimation

- For estimation of parameters of $\rho(X)$, it is useful to apply Cholesky decomposition
- Decomposes the density matrix via triangular matrix with real parameters on its diagonal, being between ±1

$$\rho = MM^{\dagger}$$

- For a three dimensional case

$$M(m) = \frac{1}{\sqrt{\sum_k m_k^2}} \begin{pmatrix} m_1 & m_4 + im_5 & m_6 + im_7 \\ 0 & m_2 & m_8 + im_9 \\ 0 & 0 & m_3 \end{pmatrix}$$

Connection to angular distribution

- Density matrices of the probe and unknown system need to be connected to the angular distribution in the final state
  - Need connection between angular distribution, cross-section and the density matrices
- From the QM, the cross-section is given my a squared matrix element $\mathscr{M}$

$$d\sigma \sim \sum_{s,s'} \Big| \sum_{J} \mathcal{M}(\chi_J \to f_{s,s'}) \Big|^2 \cdot d\Pi_{LIPS},$$
$$= \mathrm{Tr}\Big[\Big(\sum_{s,s'} T^\dagger \big|f_{s,s'}\big\rangle\big\langle f_{s,s'}\big| T\Big) \cdot \Big(\sum_{J,K} \big|\chi_J\big\rangle\big\langle \chi_K\big|\Big)\Big] \cdot d\Pi_{LIPS}.$$

- The matrix element is rewritten using the transfer matrix T that is used for studies of propagation of wave functions. The $d\Pi_{LIPS}$ denotes Lorentz invariant phase space and originates from the Fermi golden rule.
- The first term in the trace refer solely to the final state particles and by definition define the density matrix of the probe.
- The second term in the trace describes the unknown matrix

Connection to angular distribution

- In order to infer the connection between angular distribution and cross-section, the cross-section can be rewritten as

$$k_0 k'_0 \frac{d\sigma}{d^3k d^3k'} = \frac{d\sigma}{d^4Q d\Omega}$$

- Left term is probability to find a lepton pair with q, ℓ give a set of initial state variables that can be rewritten in terms of conditional probabilities

$$P(Q,\, \ell | \, init) = P(\ell | Q,\, init) P(Q | init)$$

- Yielding a formula for angular distribution

$$\frac{dN}{d\Omega} = \frac{1}{\sigma}\frac{d\sigma}{d\Omega} = P(\ell | Q,\, init) = \frac{3}{4\pi}\mathrm{Tr}(\rho(\ell)\rho(X))$$

Density matrix parametrization

- Writing out the terms from each density matrix and using the parameters from $\rho(X)$ the angular distribution is:
- In order to obtain the parameters one has to fit distribution to the data
- Due to convexity of measured trace

  $\mathrm{Tr}\rho(\ell)\rho(X))$

  any observed local extremum will be also a global extremum

$$\frac{dN}{d\Omega} = \frac{1}{4}(1+m_3^2)$$
$$+ 0.22 m_3 m_9 \sin\theta\cos\phi$$
$$- 0.22 m_3 m_7 \sin\theta\sin\phi$$
$$+ 0.22(m_2 m_5 + m_7 m_8 - m_6 m_9)\cos\theta$$
$$+ \frac{1}{4}(1-3m_3^2)\cos^2\theta$$
$$- \frac{1}{2} m_3 m_6 \sin(2\theta)\cos\phi$$
$$+ \frac{1}{2}\left(m_2^2 + m_8^2 + m_9^2 + \frac{1}{2}m_3^2 - \frac{1}{2}\right)\sin^2\theta\cos(2\phi)$$
$$- \frac{1}{2}(m_2 m_4 + m_8 m_6 + m_7 m_8)\sin^2\theta\sin(2\phi)$$
$$- \frac{1}{2} m_3 m_8 \sin(2\theta)\sin\phi.$$

Density matrix parametrization

- By applying a basic property of density matrix $\mathrm{Tr}(\rho(X)) = 1$ and considering the parity conservation (parity is conserved in QCD and QED), five additional parameters are eliminated

$$
\begin{aligned}
\frac{dN}{d\Omega} = & \frac{1}{4}(1+m_3^2) \\
& + \frac{1}{4}(1-3m_3^2)\cos^2\theta \\
& - \frac{1}{2}m_3 m_6 \sin(2\theta)\cos\phi \\
& + \frac{1}{2}(m_2^2 + m_8^2 + m_9^2 + \frac{1}{2}m_3^2 - \frac{1}{2})\sin^2\theta\cos(2\phi) \\
& - \frac{1}{4}(2m_2^2 + m_3^2 + -1)\sin^2\theta\sin(2\phi).
\end{aligned}
$$

- This can be further simplified by the normalization as $m_3 m_6 = \sqrt{m_3^2(1 - m_2^2 - m_3^2)}$

Summary

- Quantum tomography is a method to experimentally extract all that is observable about a quantum mechanical system
- The method of quantum tomography uses a known “probe” to explore an unknown system.
- Data is related directly to matrix elements, with minimal model dependence and optimal efficiency
- Matrices are represented by numbers generated and fit to experimental data, not abstract operators
- What will be observed is strictly limited by the dimension and symmetries of the probe.
- The description never involves more variables than will actually be measured.