Numeric Inverse Laplace Transform for Likelihood Evaluation

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June 24, 2023

Motivation

- Positive continuous random variable $X \in \mathbb{R}^+$
- Unknown but continuous density $\mathrm{f}(x)$
- Known characteristic function $\psi(t)$
- Frequently $X = \sum_{k=1}^{m} X_k$
- Therefore $\psi(t) = \prod_{k=1}^{m} \psi_k(t)$
- Sample $\{x_j\}_{j=1}^{n}$
- Negative log likelihood $\Phi = -\sum_{j=1}^{n} \ln \mathrm{f}(x_j) = \min$
- Numeric inversion of Laplace transform instead FFT

Traditional Solution via Fast Fourier Transform

- Employing Fourier transform for $x, t \in \mathbb{R}$
- $\mathrm{f}(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} \psi(t) \exp(-\mathrm{j}xt) \mathrm{d}t$
- Approximation via $N = 2^q$ point FFT
- Time complexity of overture $\mathrm{T}(N) = N \log_2 N$
- Repeated interpolation in lookup table
- Acceptably accurate for $N > 10^7$
- Therefore very slow for MLE

Relationship to Laplace Transform

- Random variable $X \in \mathbb{R}^+$
- Characteristic function $\psi(t) = \int_0^\infty \mathrm{f}(x) \exp(\mathrm{j}tx) \mathrm{d}x$
- Laplace transform $\mathrm{F}(s) = \int_0^\infty \mathrm{f}(x) \exp(-sx) \mathrm{d}x$
- Therefore $s = -\mathrm{j}t,\ t = \mathrm{j}s$
- Conversion formula $\mathrm{F}(s) = \psi(\mathrm{j}s)$
- Employing inverse Laplace transform
- Numerical methods for Laplace transform inversion
- Time complexity $\mathrm{T}(n) = Mn$
- Acceptably accurate for $M < 30$

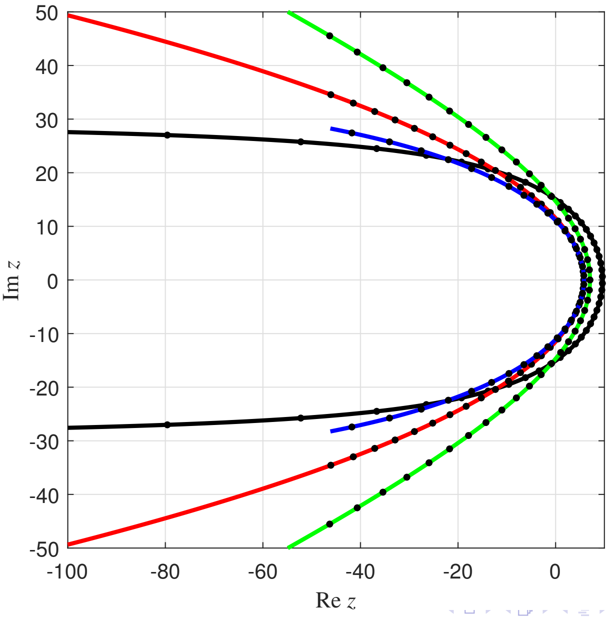
Conversion Results

- Gamma distribution:
- $\mathrm{F}(s) = (1 + s/\beta)^{-\alpha}$
- Beta distribution:
- $\mathrm{F}(s) = {}_1\mathrm{F}_1(\alpha, \alpha + \beta, -s)$
- Inverse Gaussian:
- $\mathrm{F}(s) = \exp\left(\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2 s}{\lambda}}\right)\right)$
- Semi-infinite wall:
- $\mathrm{F}(s) = (1 + \sqrt{Ts})^{-1}$
- Thin pipe echo:
- $\mathrm{F}(s) = \frac{\exp((1 + Ts)^{-1}) - 1}{\exp(1) - 1}$

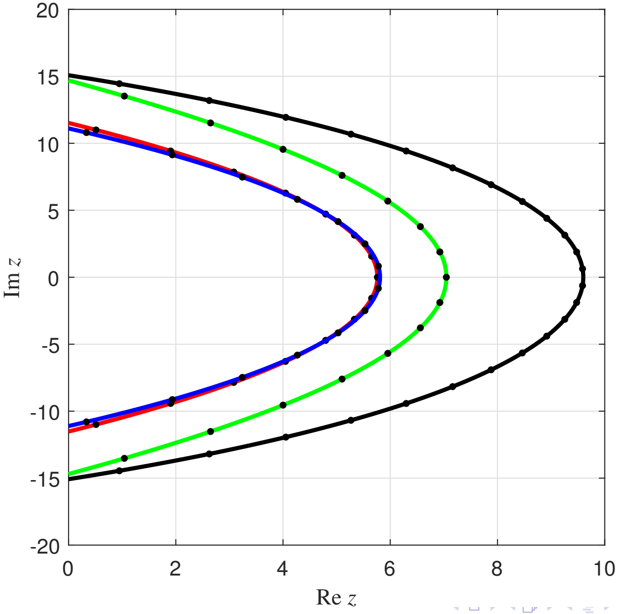
Bromwich Formula for Inverse Laplace Transform

- Inverse Laplace transform $\mathcal{L}^{-1}$
- $\mathrm{f}(x) = \dfrac{1}{2\pi \mathrm{j}} \displaystyle\int_{\mathcal{C}} \mathrm{F}(s) \exp(xs) \mathrm{d}s$
- $\mathrm{f}(0+) = \lim\limits_{\mathrm{Re}s \to +\infty} s\mathrm{F}(s)$
- Traditional Hankel contour $\mathcal{C}$: $s = \xi_0 + \mathrm{j}\omega$, $\omega \in \mathbb{R}$
- Analytically: Integration and/or singularity surrounding
- General numerical scheme for $x > 0$:
- $\mathrm{f}(x) \approx \dfrac{1}{x} \mathrm{Re} \left(\displaystyle\sum_{k=1}^{M} w_k \mathrm{F}\left(\dfrac{s_k}{x}\right) \right)$
- The points $z_k = s_k/x$ are placed on any contour $\mathcal{C}$
- Singularities and/or branches only for $s \leq 0$

Contour Sampling Around Negative Real Axis



Zoom for $\operatorname{Re} z \geq 0$



Method Comparison

- Required approximation error $\epsilon = 10^{-14}$
- $M = 24$ for original Talbot contour
- $M = 22$ for optimal parabolic contour
- $M = 20$ for optimal hyperbolic contour
- $M = 17$ for optimal Wiedeman contour

Two Infinite Parametrizations

- Infinite parameter range $u \in \mathbb{R}$
- Parametrization $s = \mathrm{z}(u)$
- $\mathrm{f}(x) = \dfrac{1}{2\pi\mathrm{j}} \displaystyle\int_{-\infty}^{+\infty} \exp(\mathrm{z}(u)x)\mathrm{F}(\mathrm{z}(u))\mathrm{z}'(u)\mathrm{d}u$
- Sampling $u_k = kh$
- $\mathrm{f}(x) \approx \dfrac{h}{2\pi\mathrm{j}} \displaystyle\sum_{k=-N}^{+N} \exp(\mathrm{z}(u_k)x)\mathrm{F}(\mathrm{z}(u_k))\mathrm{z}'(u_k)$
- Therefore $M = N + 1$
- Parabola $\mathrm{z}(u) = \mu(1 - u^2 + 2\mathrm{j}u)$
- $h = 3/N$, $\mu = \pi/12 \cdot N/x$
- Hyperbola $\mathrm{z}(u) = \mu(1 - \sin(\alpha)\cosh(u) + \mathrm{j}\cos(\alpha)\sinh(u))$
- $h = 1.0818/N$, $\mu = 4.4921N/x$, $\alpha = 1.1721$

Two Finite Parametrizations

- Finite parameter range $\theta \in (-\pi, +\pi)$
- Parametrization $s = \mathrm{z}(\theta)$
- $\mathrm{f}(x) = \dfrac{1}{2\pi\mathrm{j}} \displaystyle\int_{-\pi}^{+\pi} \exp(\mathrm{z}(\theta)x)\mathrm{F}(\mathrm{z}(\theta))\mathrm{z}'(\theta)\mathrm{d}\theta$
- Sampling $\theta_k = -\pi + (k - 1/2)h,\ h = \pi/M$
- $\mathrm{f}(x) \approx \dfrac{1}{2M\mathrm{j}} \displaystyle\sum_{k=1}^{2M} \exp(\mathrm{z}(\theta_k)x)\mathrm{F}(\mathrm{z}(\theta_k))\mathrm{z}'(\theta_k)$
- Talbot contour:
- $\mathrm{z}(\theta) = \dfrac{2M}{x}(-\sigma + \mu\theta\cot(\alpha\theta) + \mathrm{j}\nu\theta)$
- $\sigma = 0,\ \alpha = 1,\ \mu = \nu = 1/5$
- Weideman contour:
- $\sigma = 0.6122,\ \alpha = 0.6407,\ \mu = 0.5017,\ \nu = 0.2645$

Thank you for your attention.