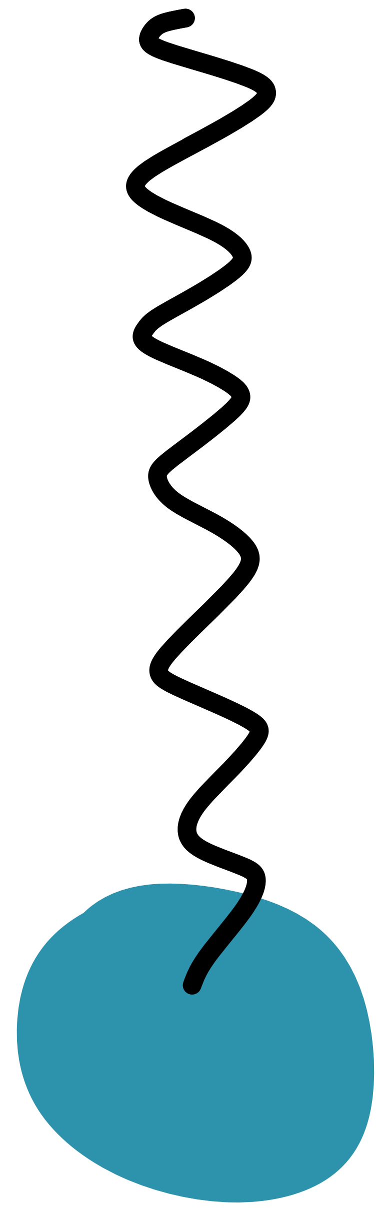
The Balitsky-Kovchegov equation and dipole orientation

Matěj Vaculčiak, 19. 9. 2024 https://doi.org/10.1016/j.physletb.2023.138360 CTU in Prague

Intro



- low-x hadron structure, gluon saturation
- probing hadron (target) with photon (projectile)
  - ep (HERA), eI (EIC), pp, pPb, PbPb (LHC)
- Balitsky-Kovchegov equation
  - gluon evolution ~ dipole evolution

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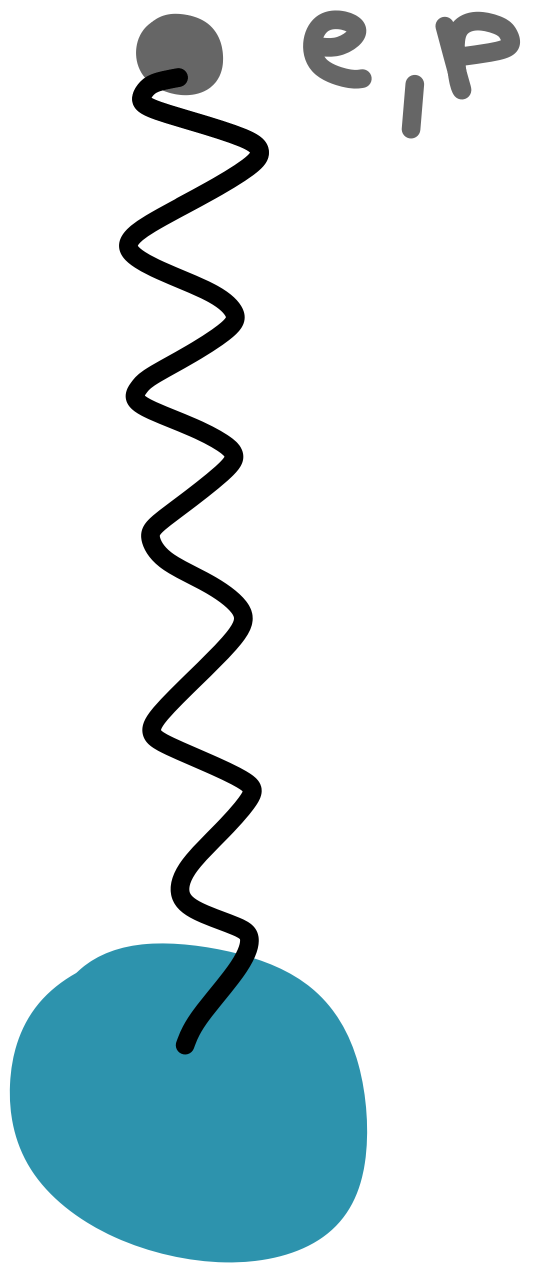

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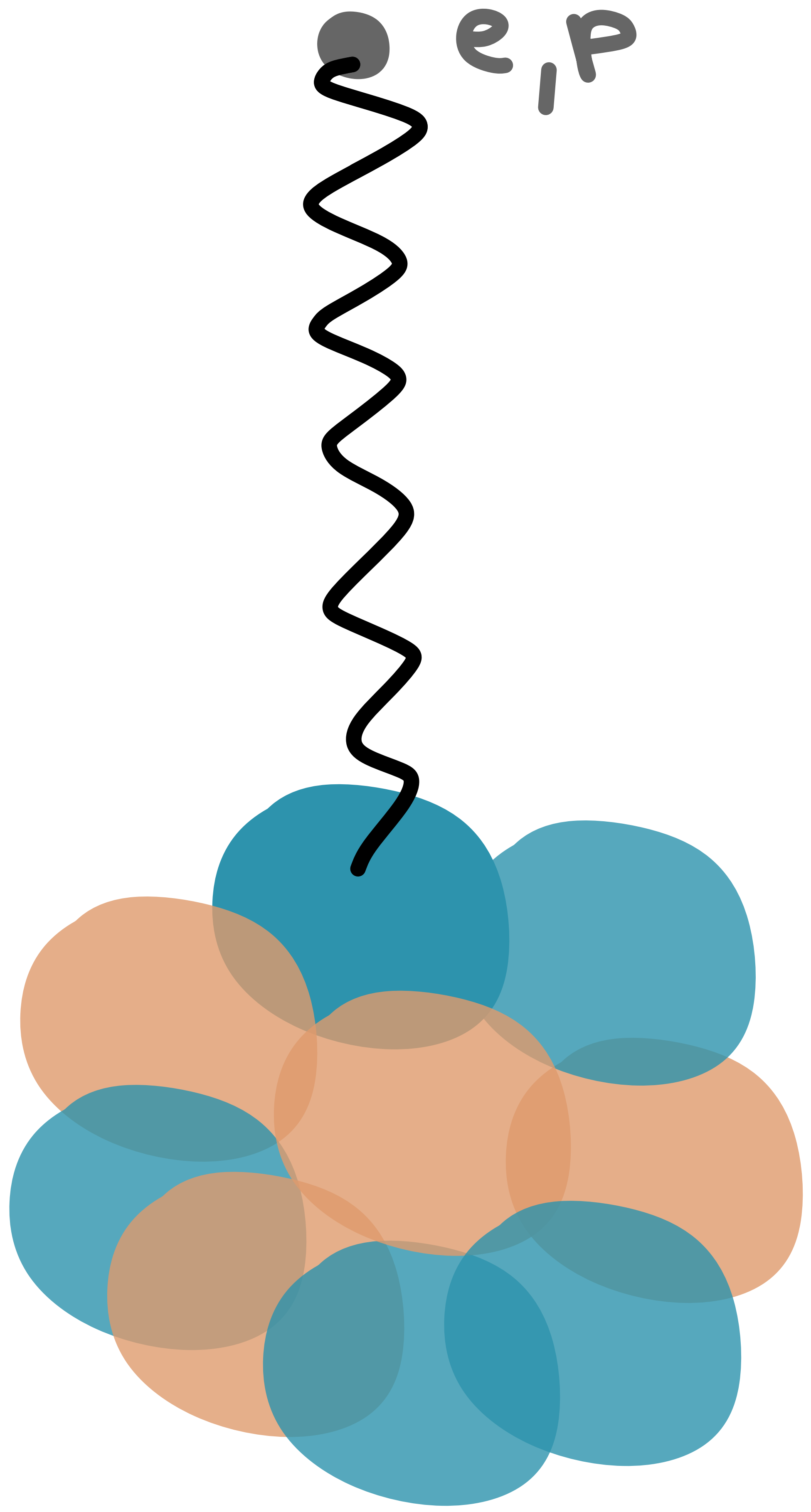

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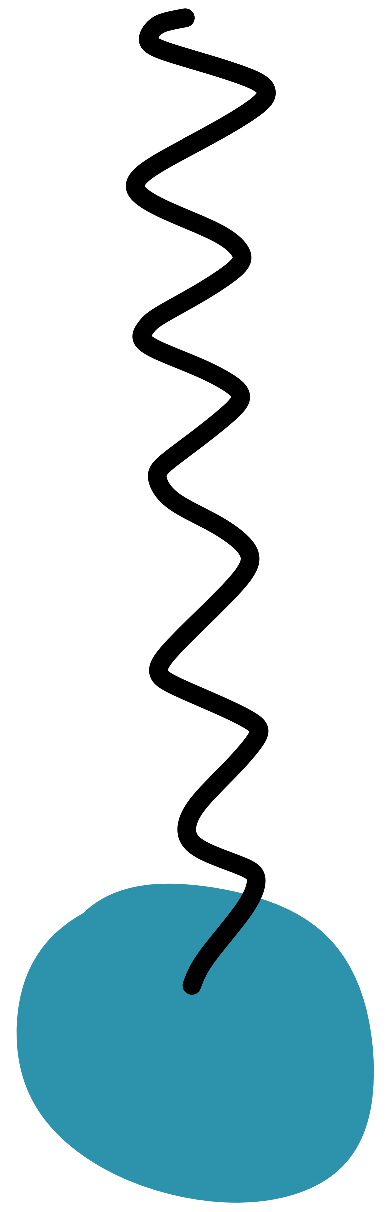

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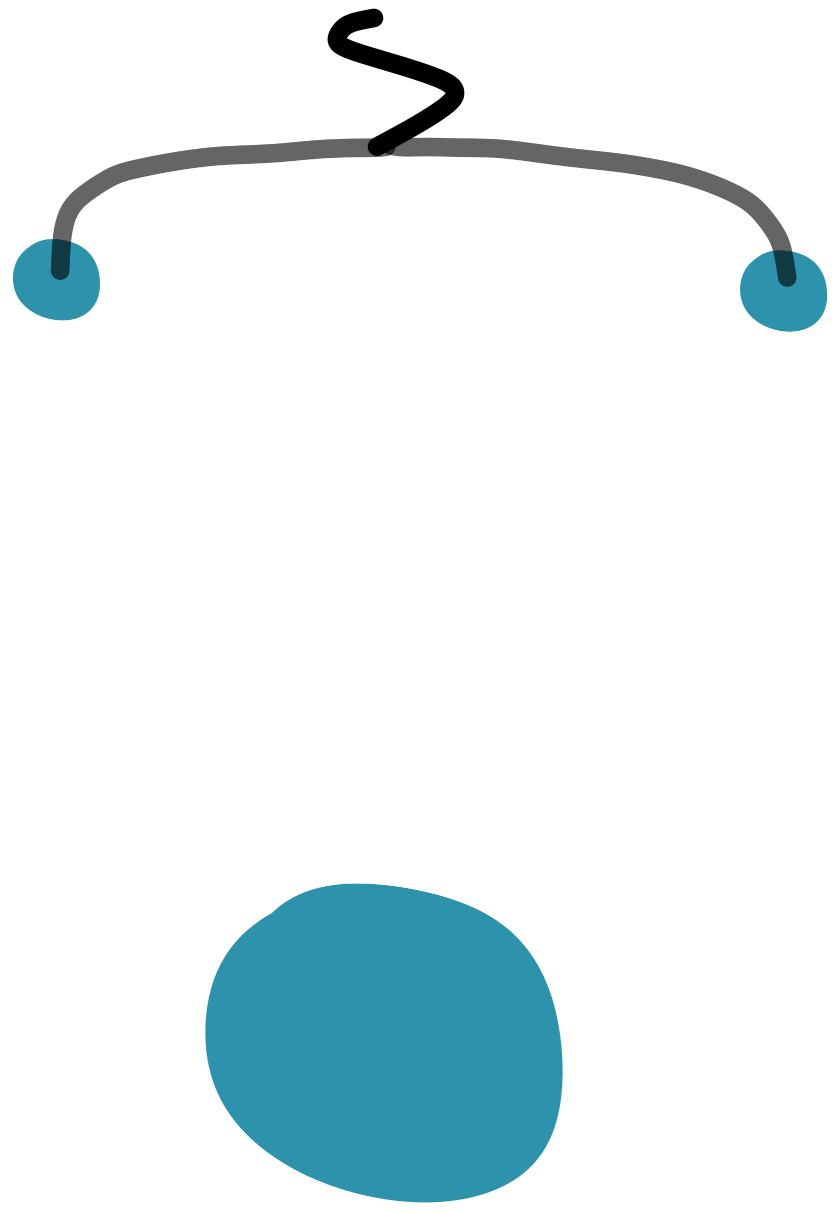

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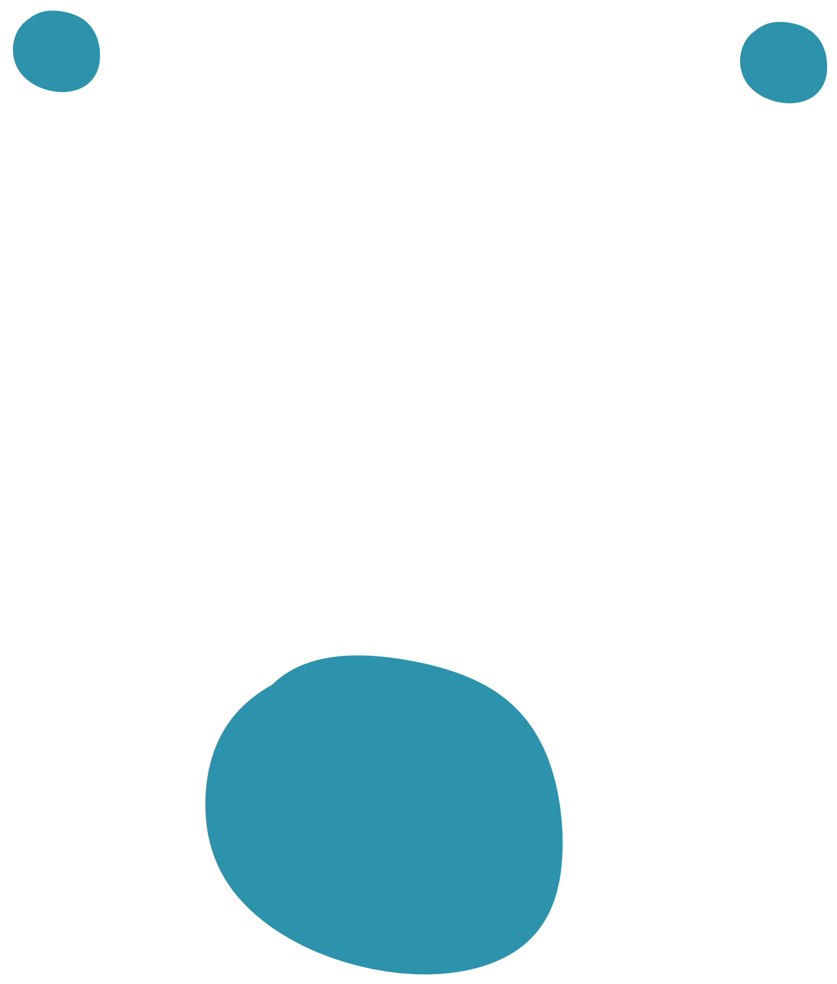

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- dipole amplitude

$$N(\eta, \underline{x}, \underline{y}) \rightarrow N(\eta, r, b, \theta, \varphi)$$

- collinearly improved kernel

$$K = \frac{\bar{\alpha}_s}{2\pi}\frac{r^2}{r_1^2 r_2^2}\left[\frac{r^2}{\min\{r_1^2, r_2^2\}}\right]^{\pm\bar{\alpha}_s A_1}$$

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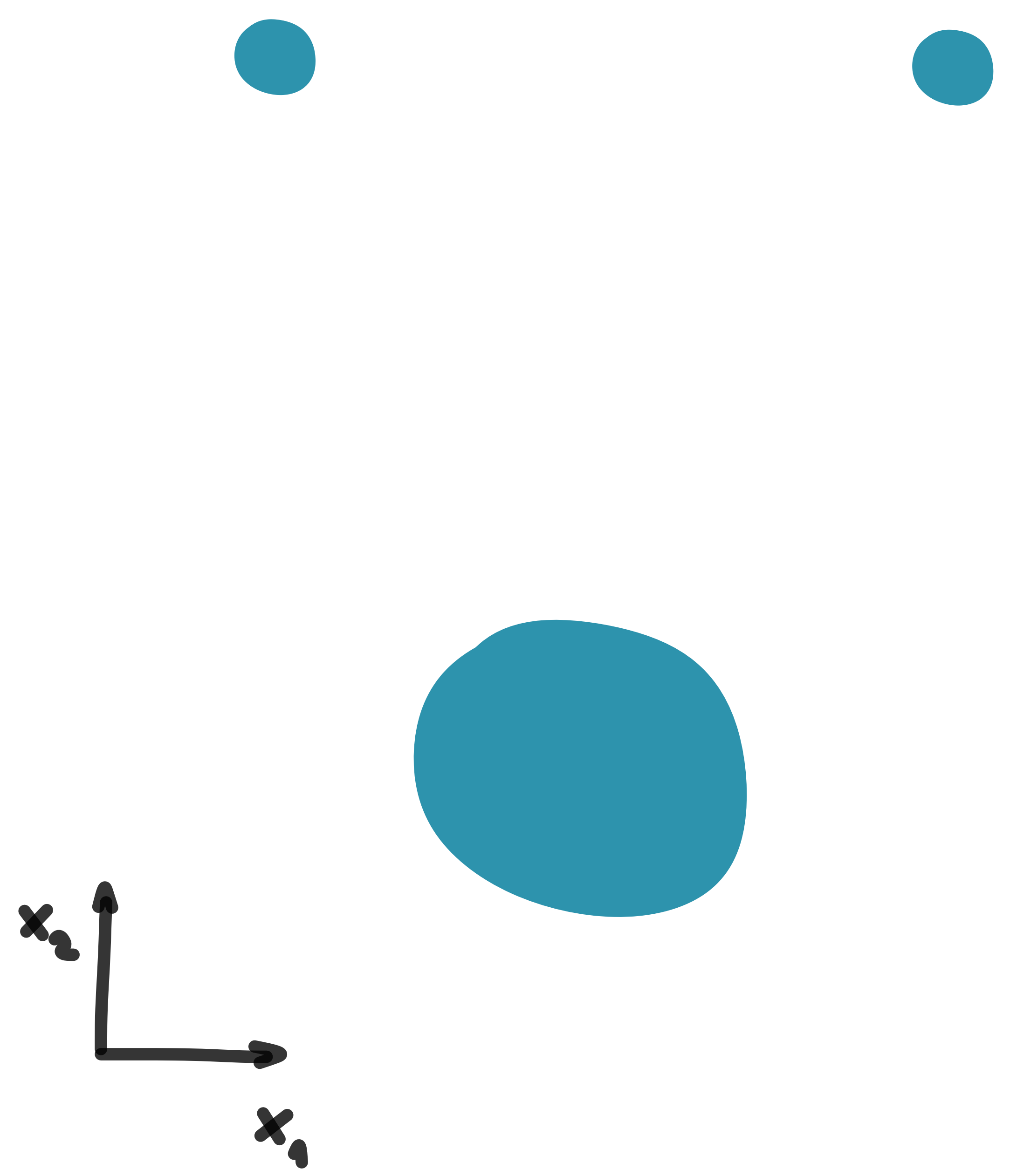

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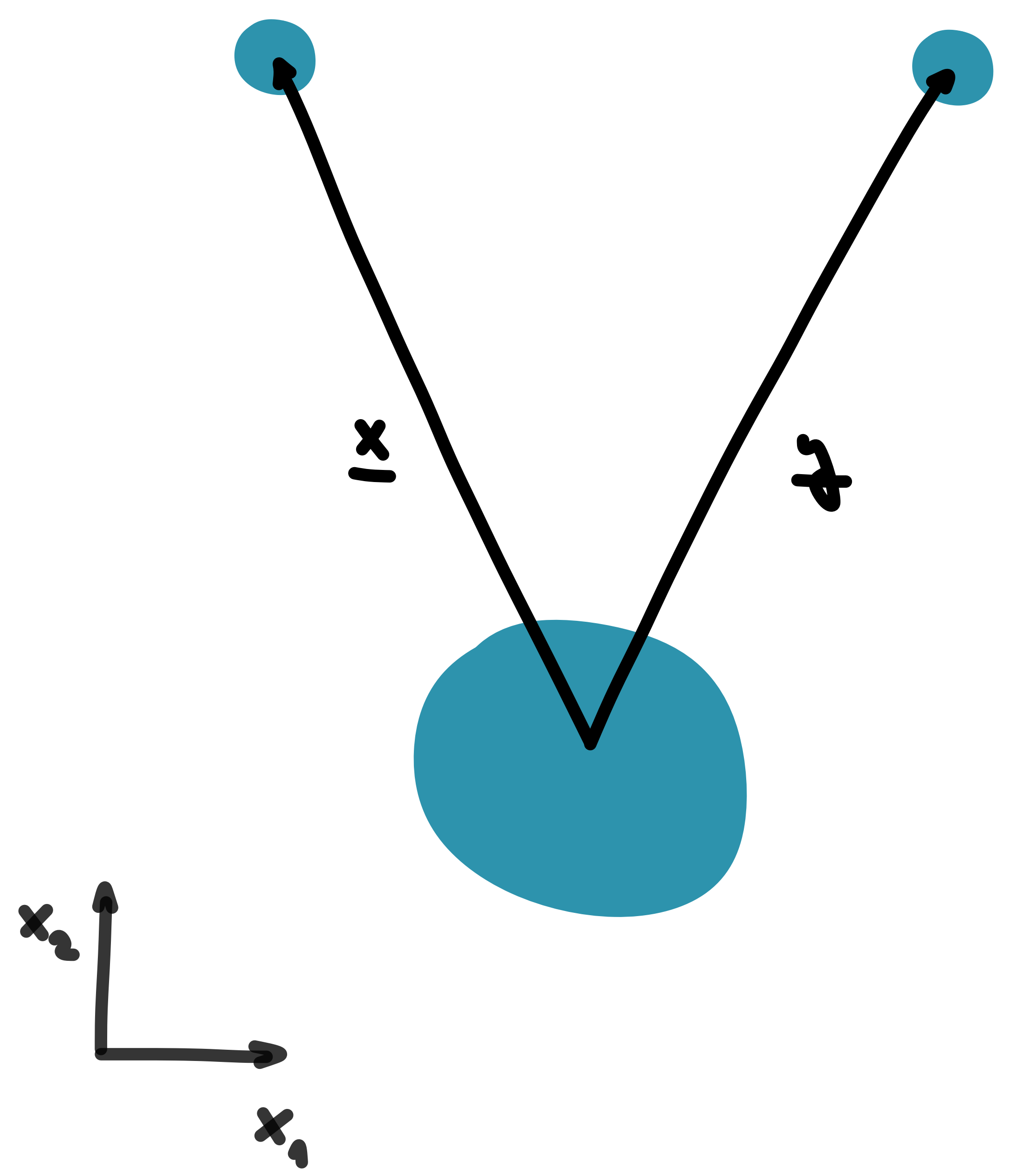

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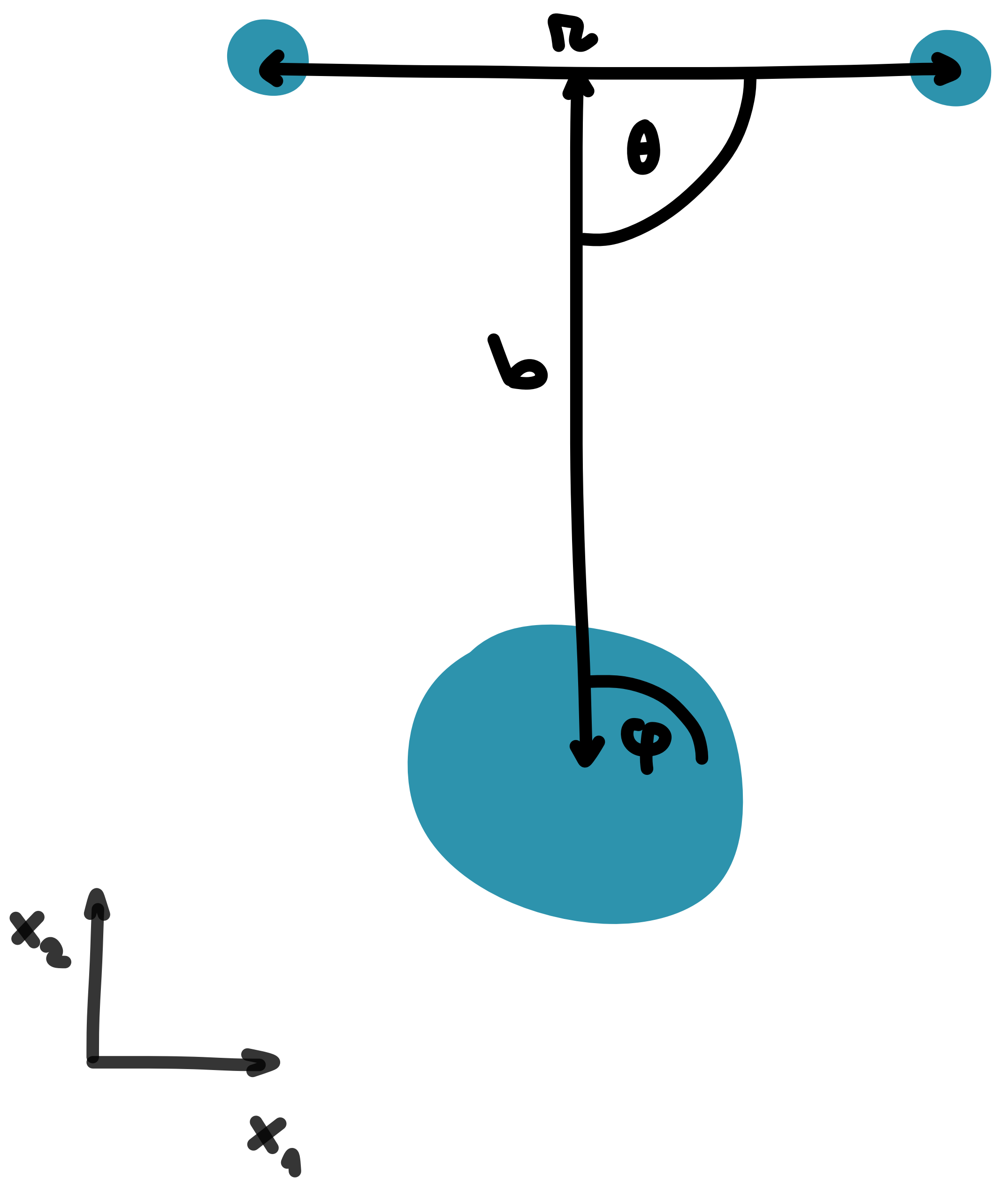

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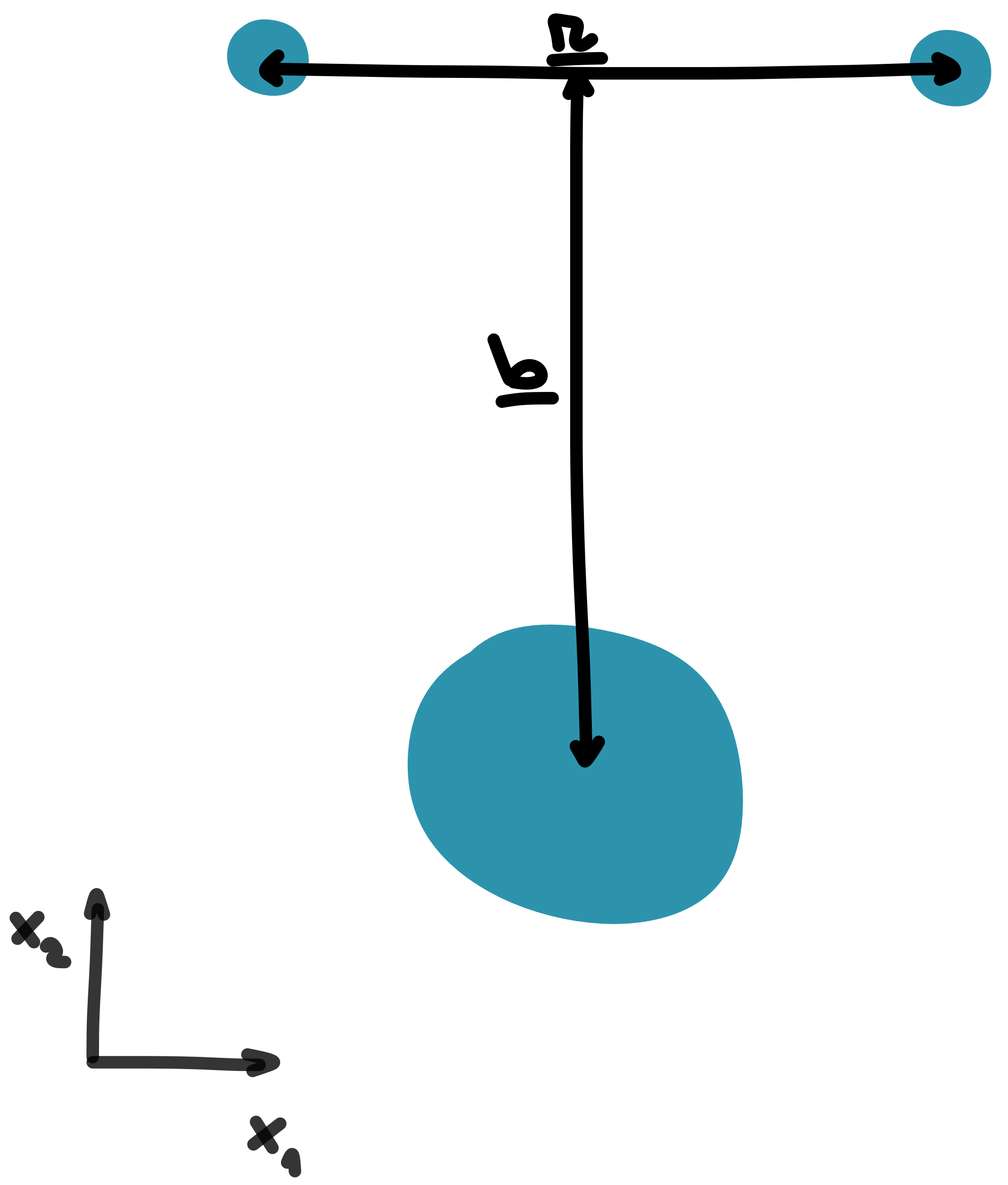

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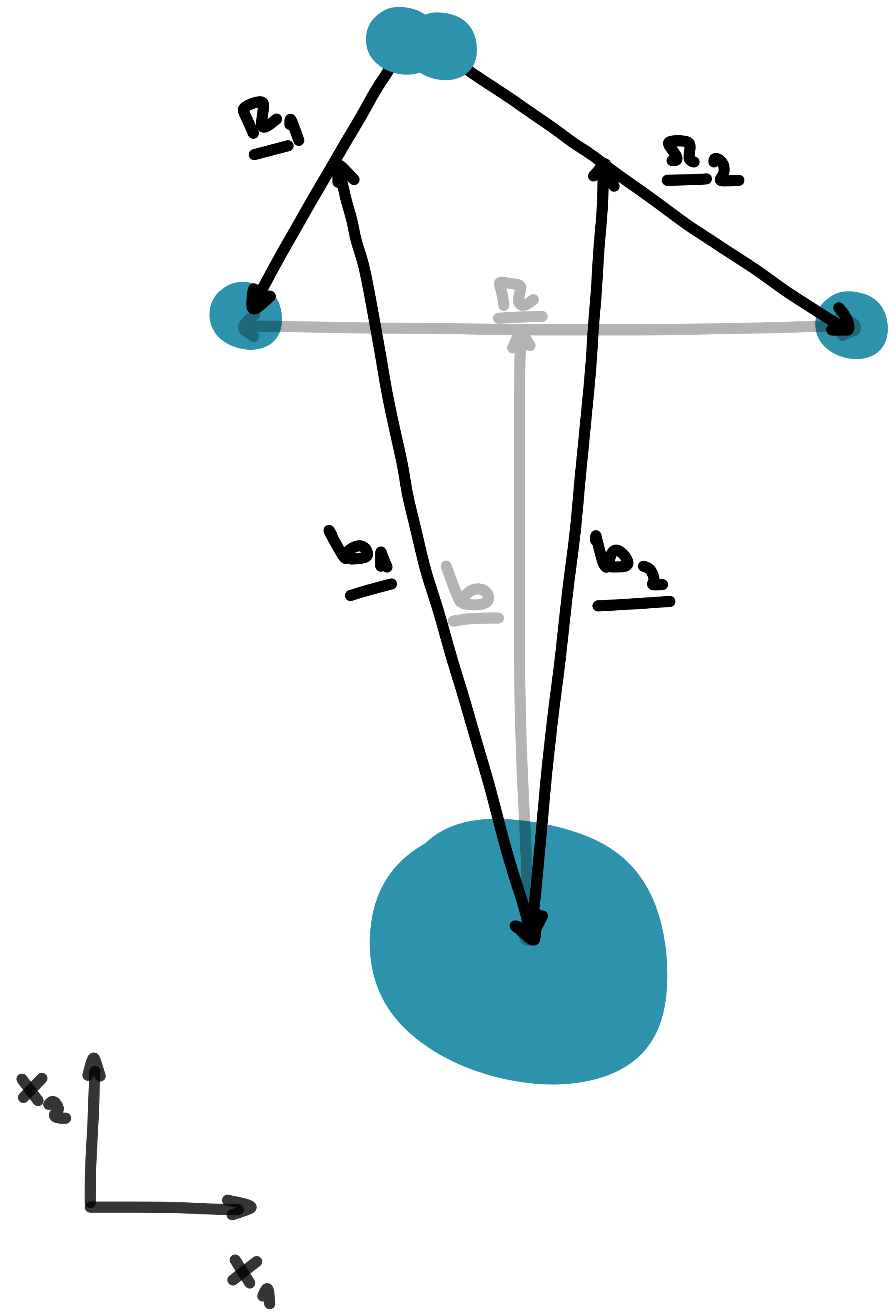

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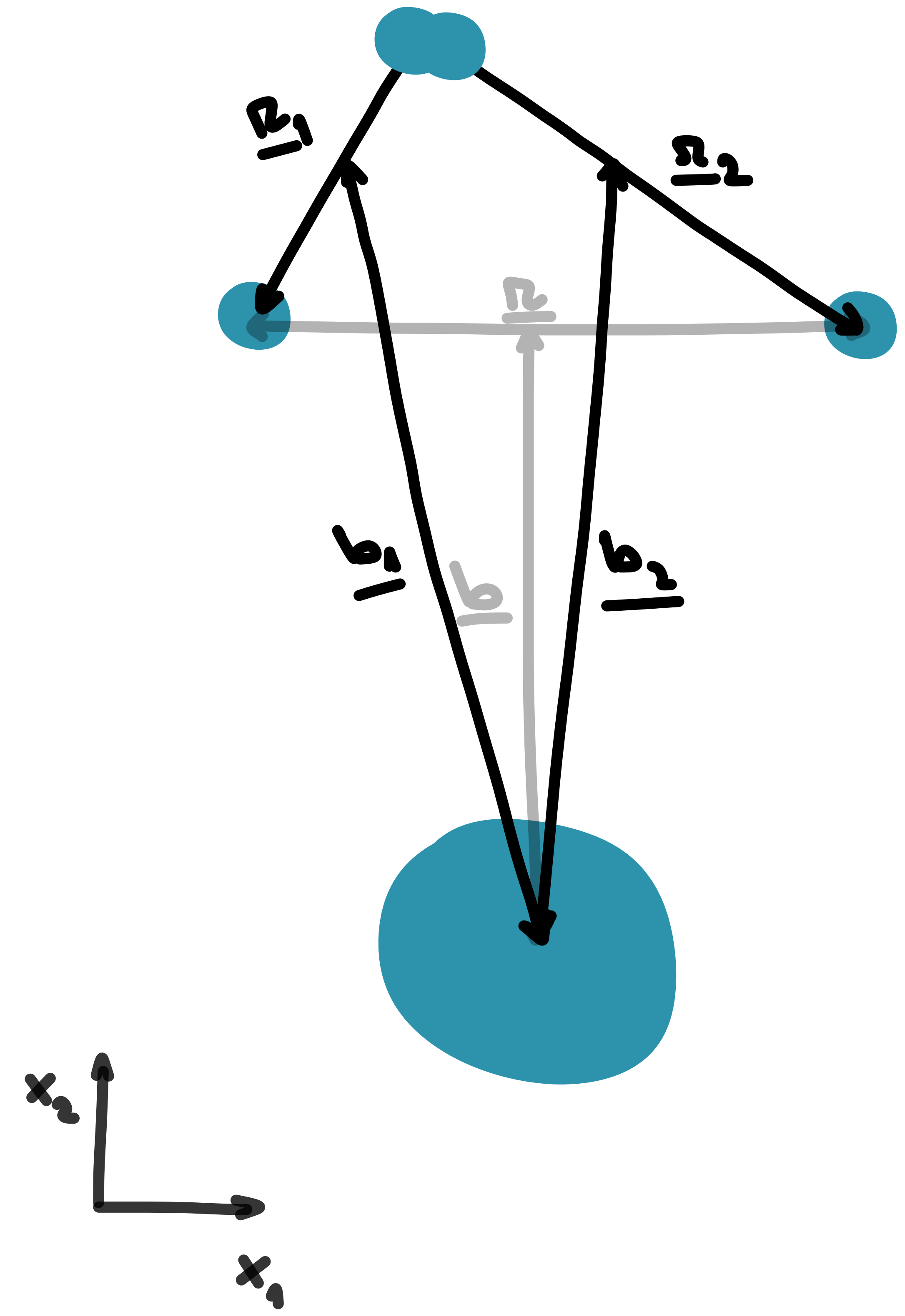

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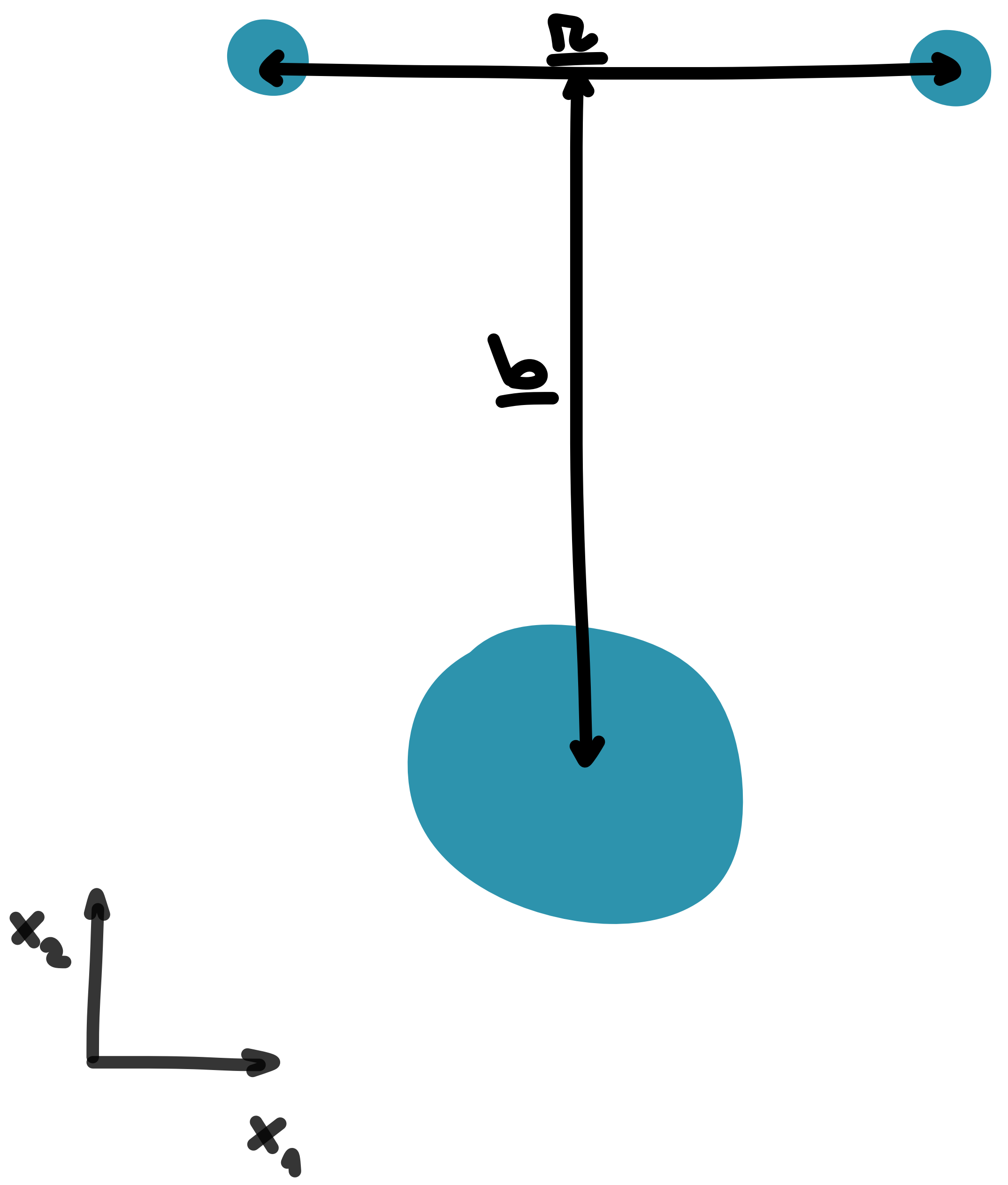

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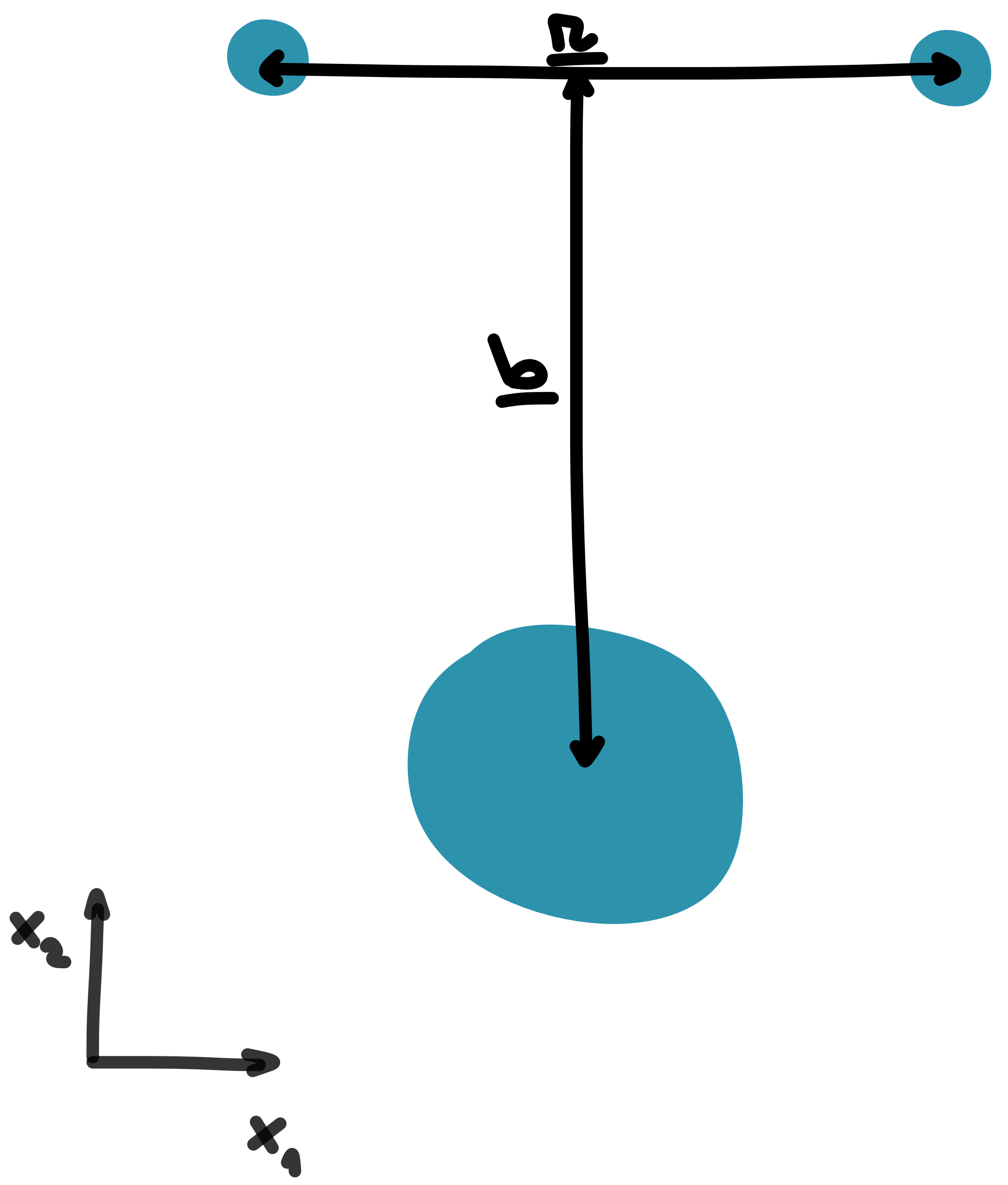

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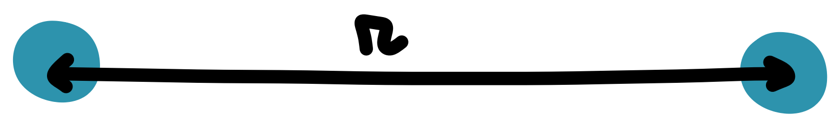
1D BK - amplitude

- infinite target approximation

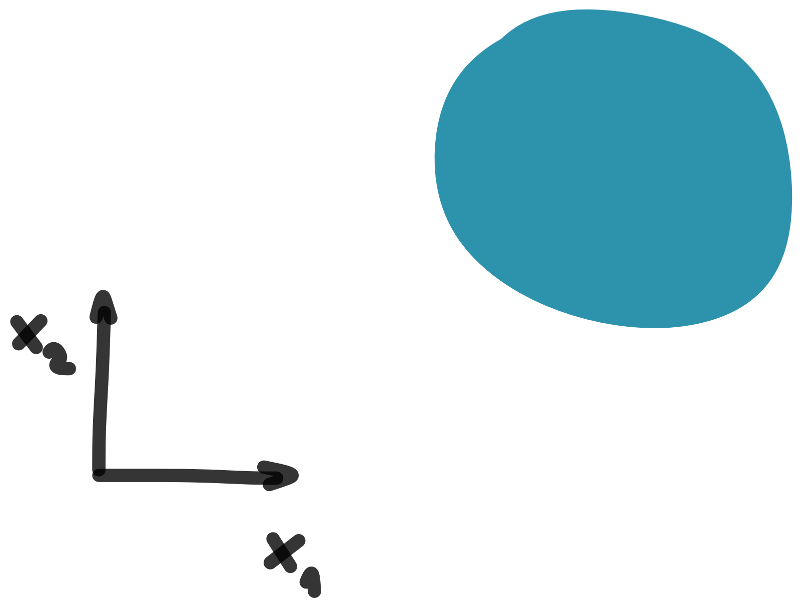
$$2\int \mathrm{d}\underline{b} N(\eta, \underline{r}, \underline{b}) \approx \sigma_0 \; N(\eta, r)$$



- MV initial condition

$$N(\eta \leq 0,\; r) = 1 - e^{-\frac{1}{4}(r^2 Q_{s0}^2)^{\gamma} \ln(\frac{1}{r\Lambda} + e)}$$

$$N(x > 0,\; r) = 0$$



[McLerran, Venugopalan, 1998]

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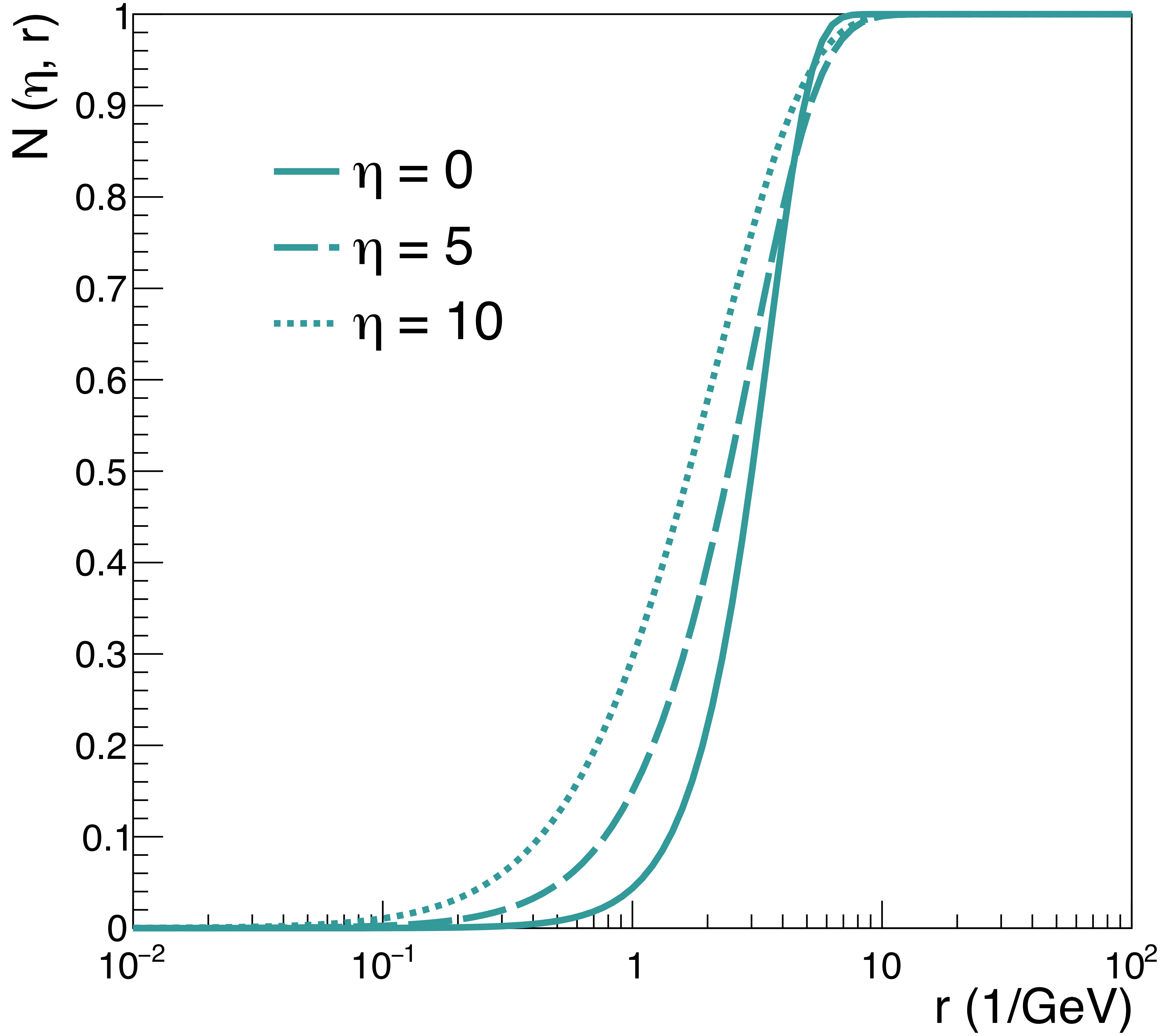


1D BK - data

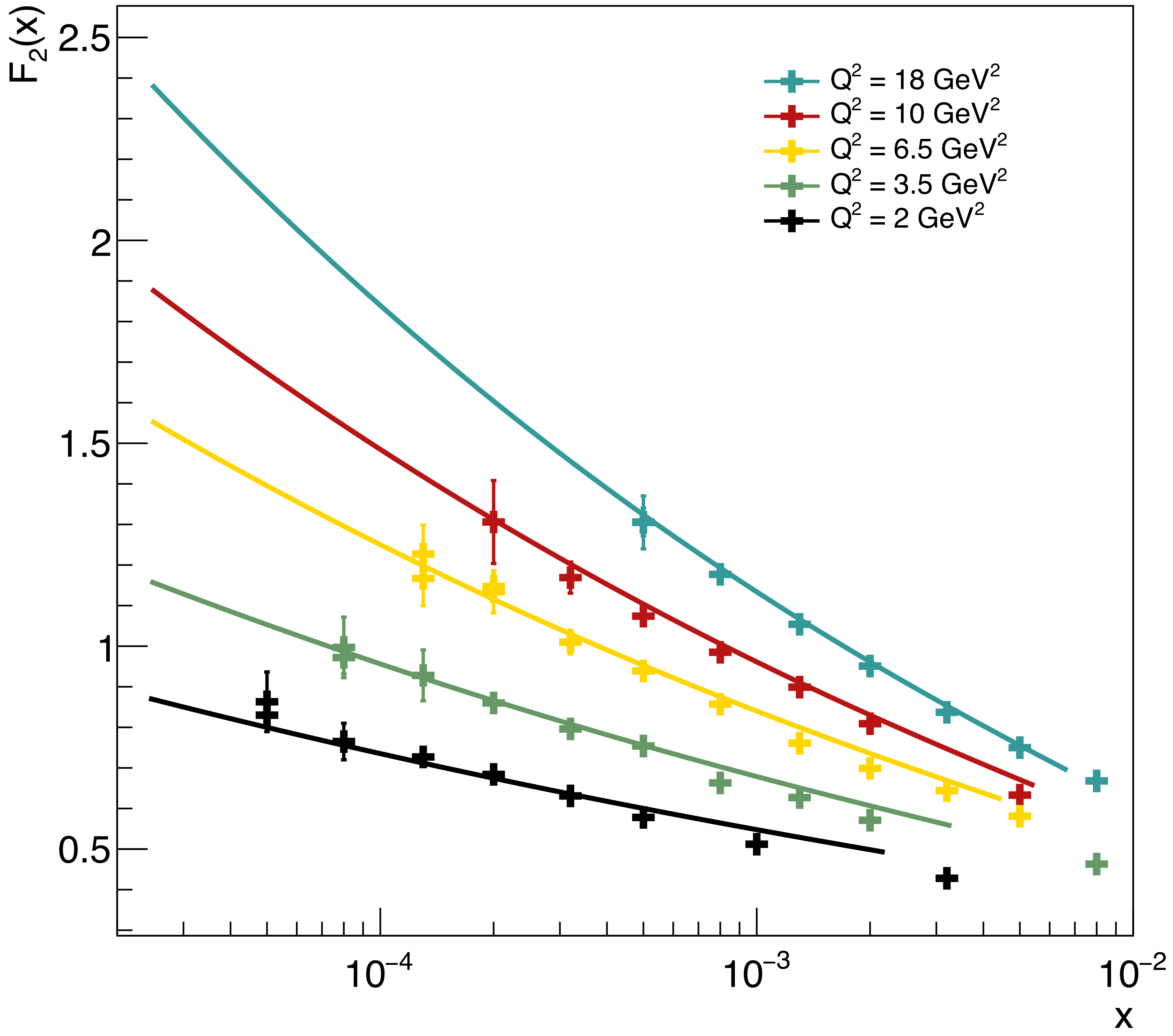
- proton structure functions

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}\left(\sigma_L^{\gamma^* p}\left(x, Q^2\right) + \sigma_T^{\gamma^* p}\left(x, Q^2\right)\right)$$

$$\sigma_{L,T}^{\gamma^* p}(x, Q^2) = \sum_f \int \mathrm{d}^2\underline{r} \int_0^1 \mathrm{d}z \, |\psi_{T,L}^{(f)}(\underline{r}, Q^2, z)|^2 \, 2 \int \mathrm{d}^2\underline{b} N(x_f, \underline{r}, \underline{b})$$

[Golec-Biernat, Wüsthoff, 1998]

1D BK - data



- HERA data described

[HERA, 2010]

2D BK - amplitude

- impact parameter dependence

$$2\int \mathrm{d}\underline{b}N(\eta,\underline{r},\underline{b}) \approx 4\pi\int \mathrm{d}bN(\eta,r,b)$$

- initial condition

  - GBW (*r*)

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$$N(\eta \leq 0,\ r,\ b) = 1 - e^{-\frac{Q_s^2}{4}r^2 e^{-\frac{b^2}{2B}-\frac{r^2}{8B}}}$$

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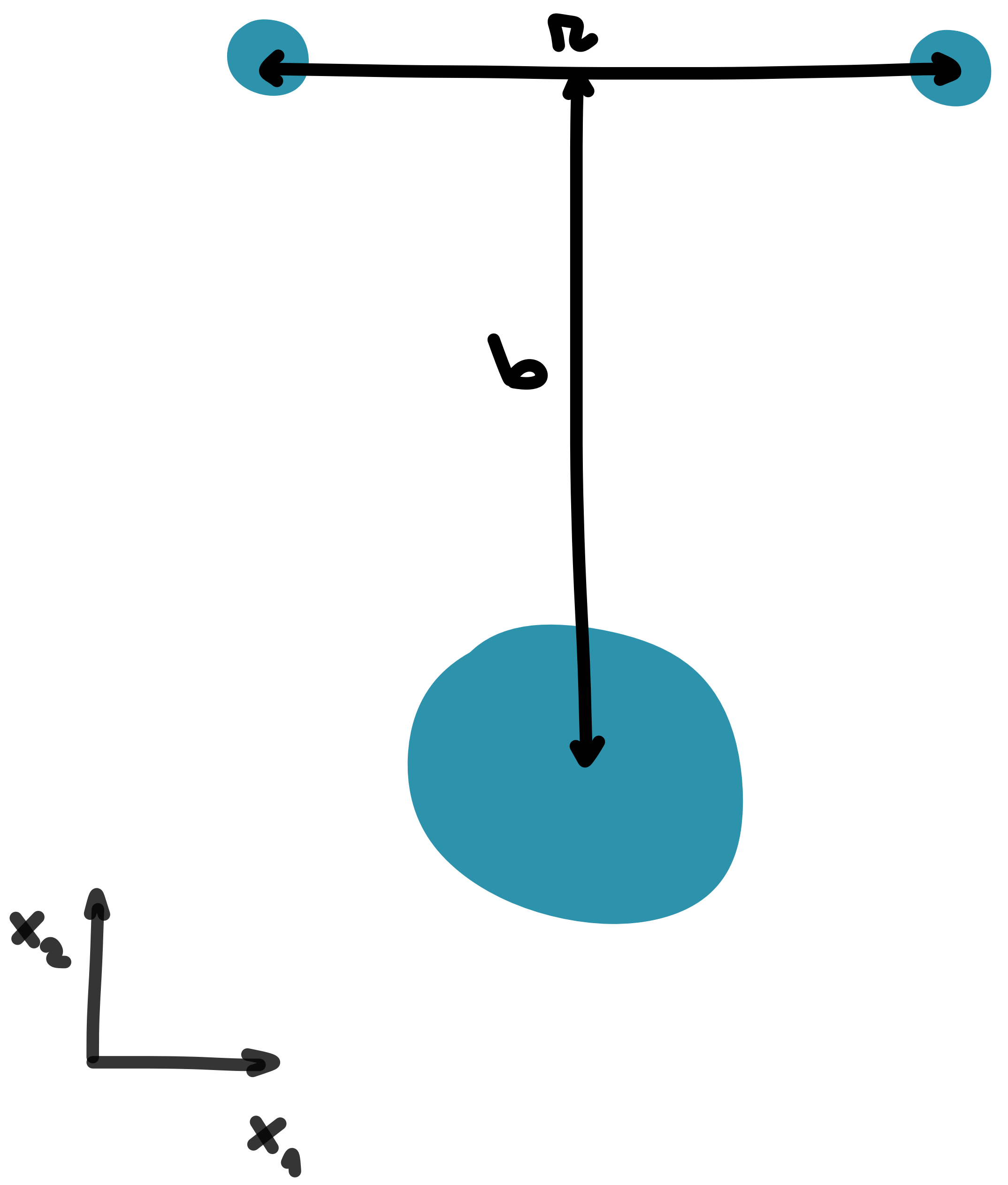

[Cepila, Contreras, Matas, 2018]

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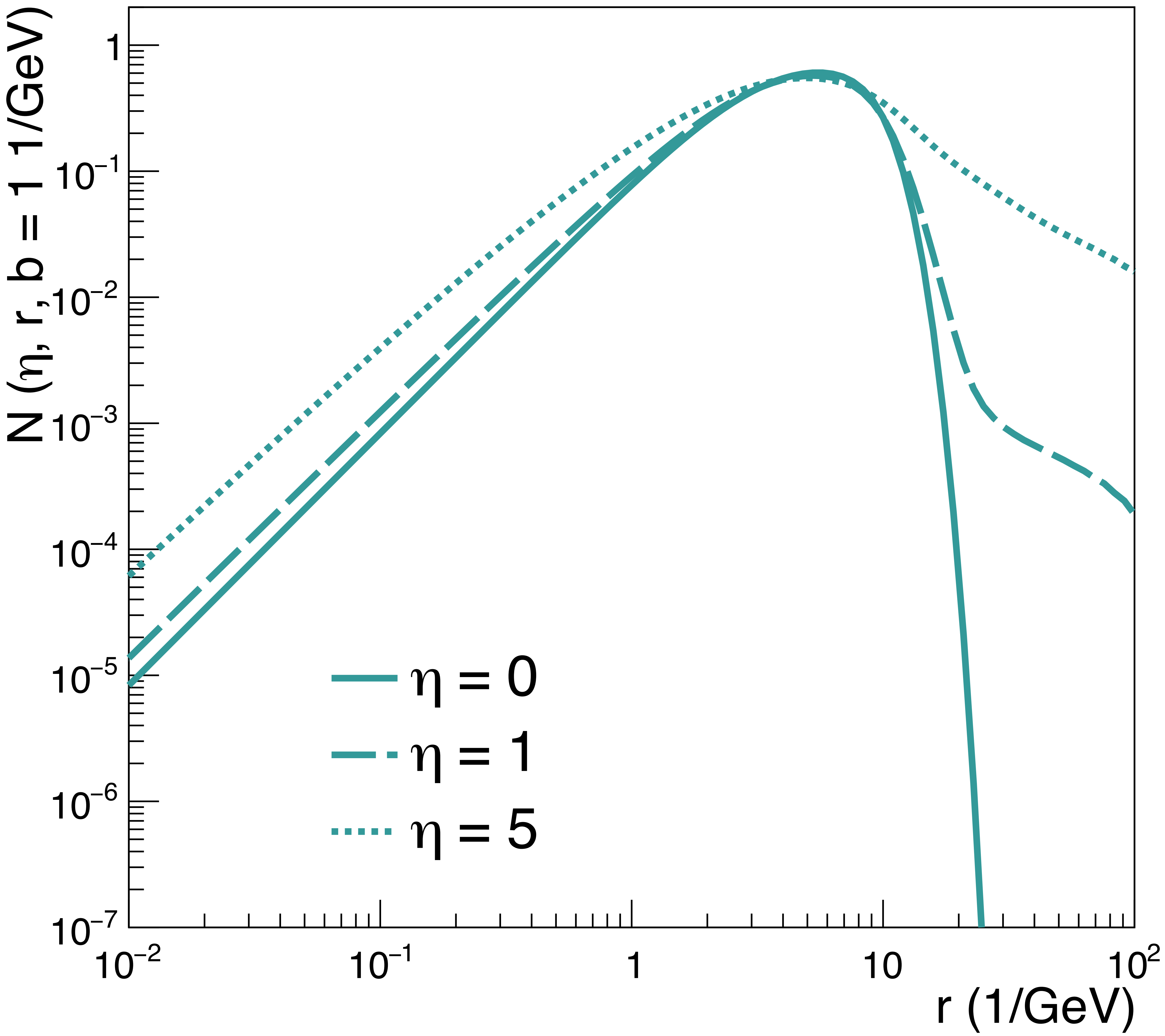

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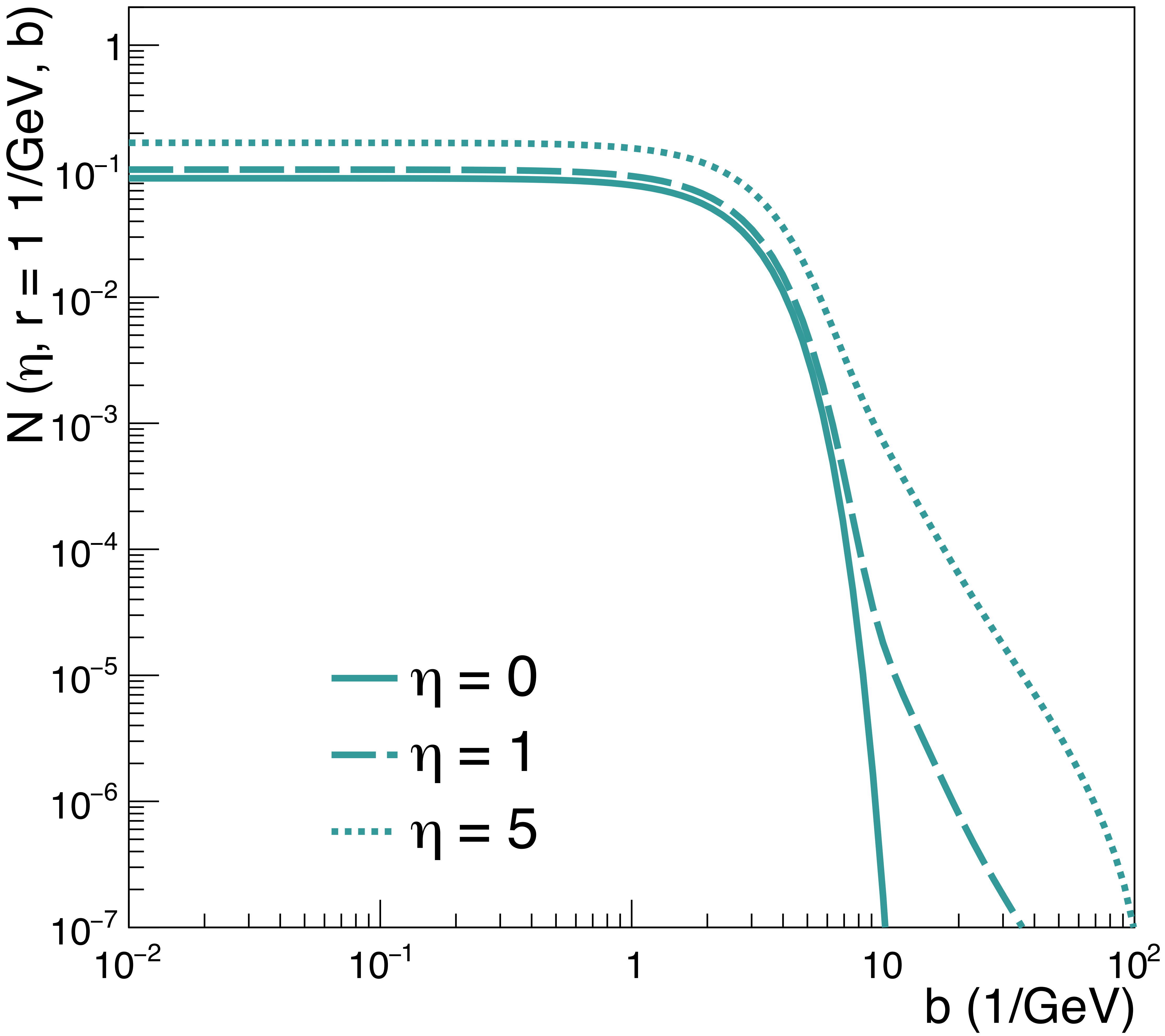

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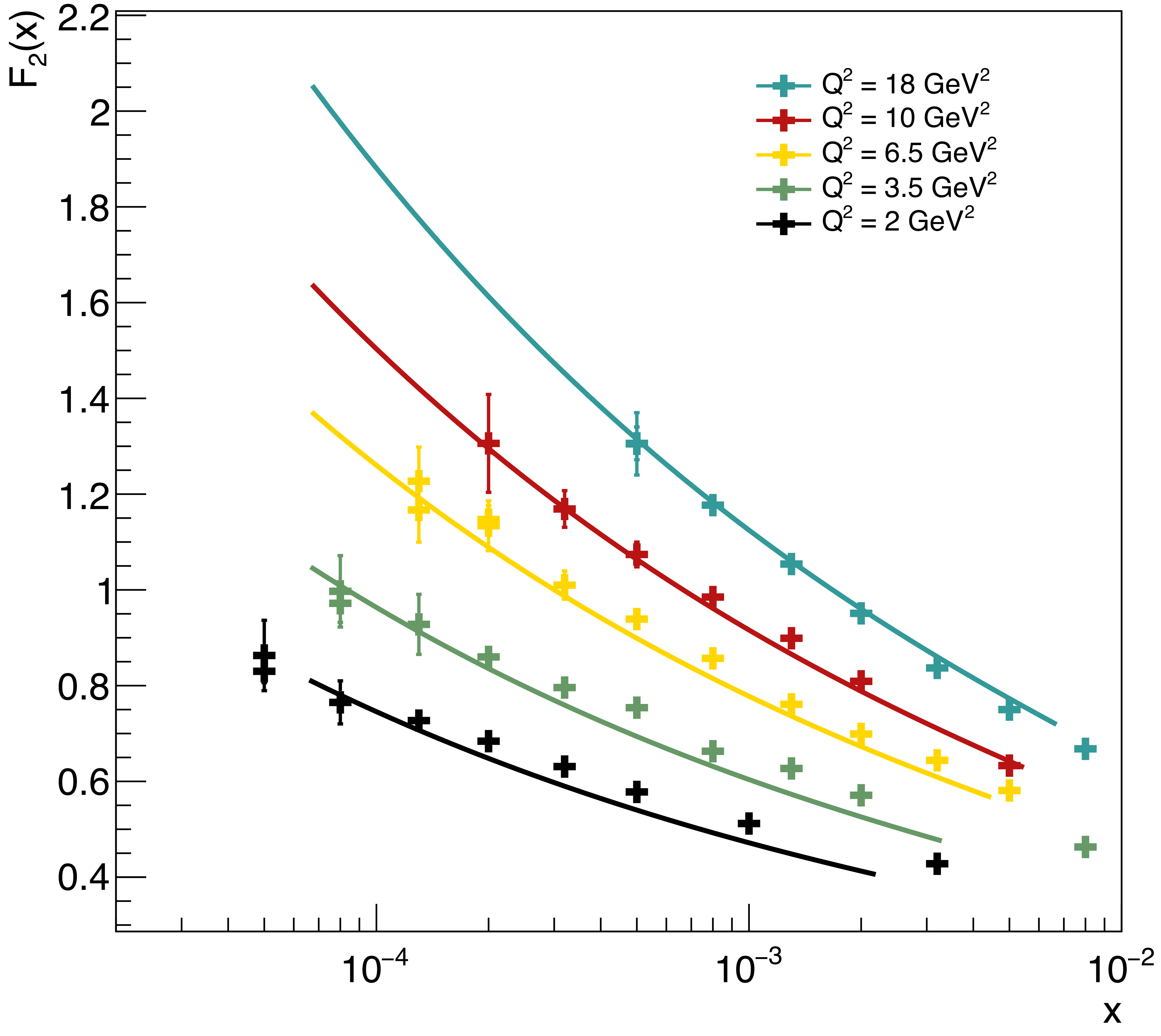


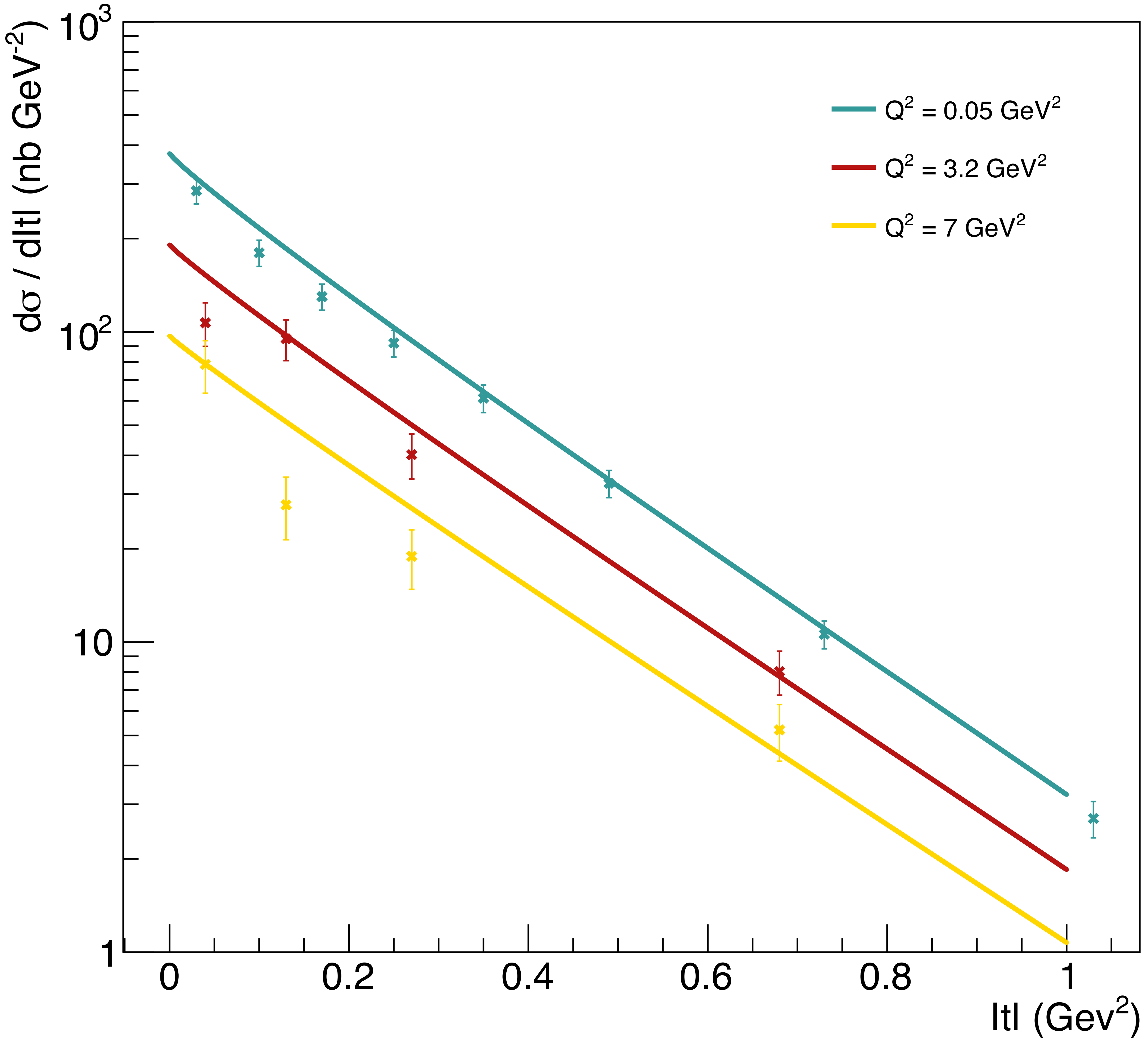
- HERA data still described

2D BK - data

- coherent vector meson production

$$\frac{\mathrm{d}\sigma_{\mathrm{T,L}}}{\mathrm{d}t} = \frac{1}{16\pi}\left|\int \mathrm{d}\underline{r}\int_0^1 \frac{\mathrm{d}z}{4\pi}\int \mathrm{d}^2\underline{b}\left(\Psi_E^{\dagger}\Psi\right)_{T,L}(Q^2,z,r)e^{-i[\underline{b}-(\frac{1}{2}-z)\underline{r}]\underline{\Delta}}2N(\eta,r,\underline{b})\right|^2$$

2D BK - data



- HERA data described
- J/ψ, W=100 GeV

[HERA, 2006]

3D BK - amplitude

- dipole orientation dependence

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$$N(\eta = 0) = 1 - e^{-\frac{1}{4}(Q_s^2 r^2)^\gamma e^{-\frac{b^2}{2B} - \frac{r^2}{8B}}(1+c\cos(2\theta))}$$

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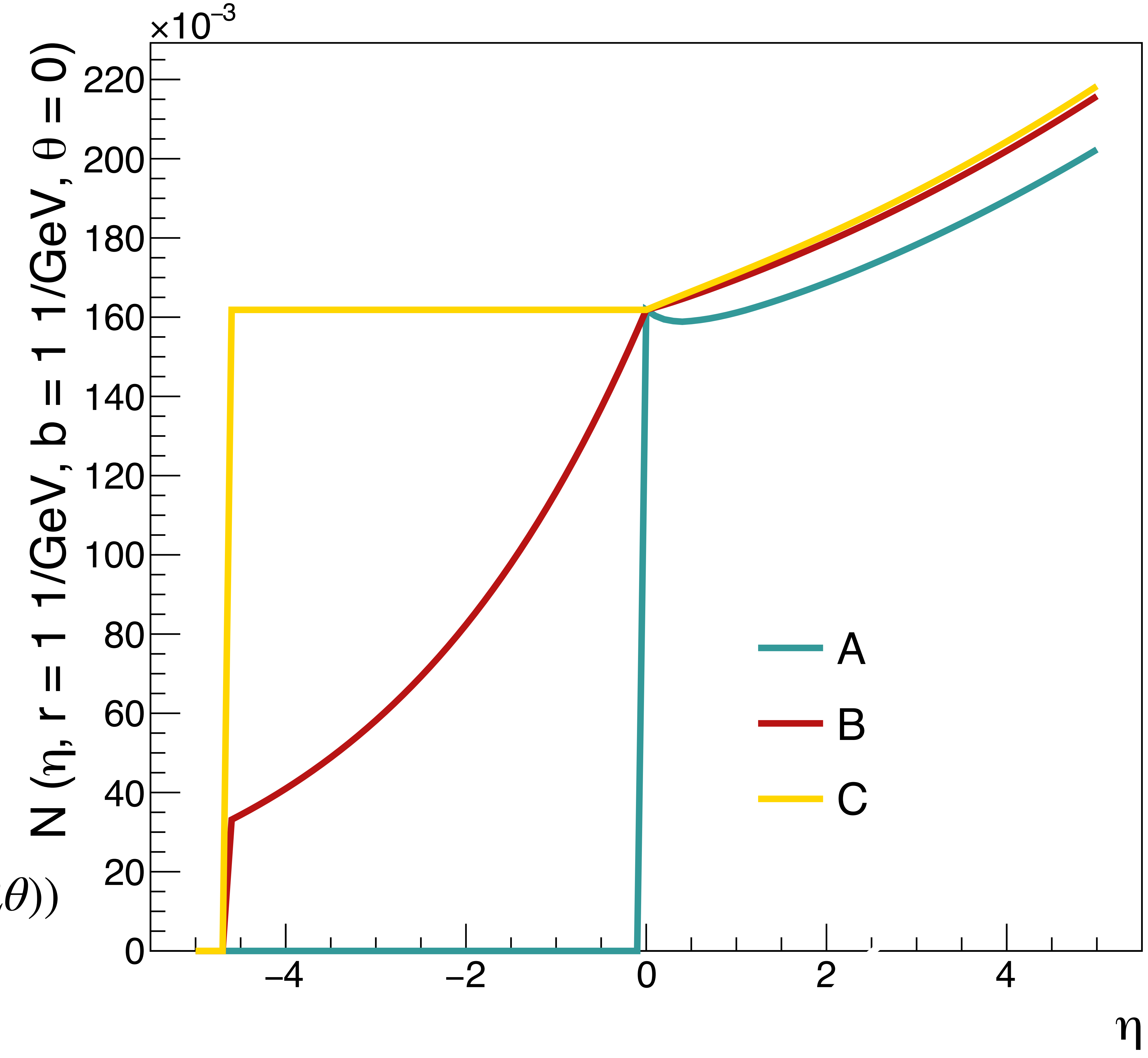

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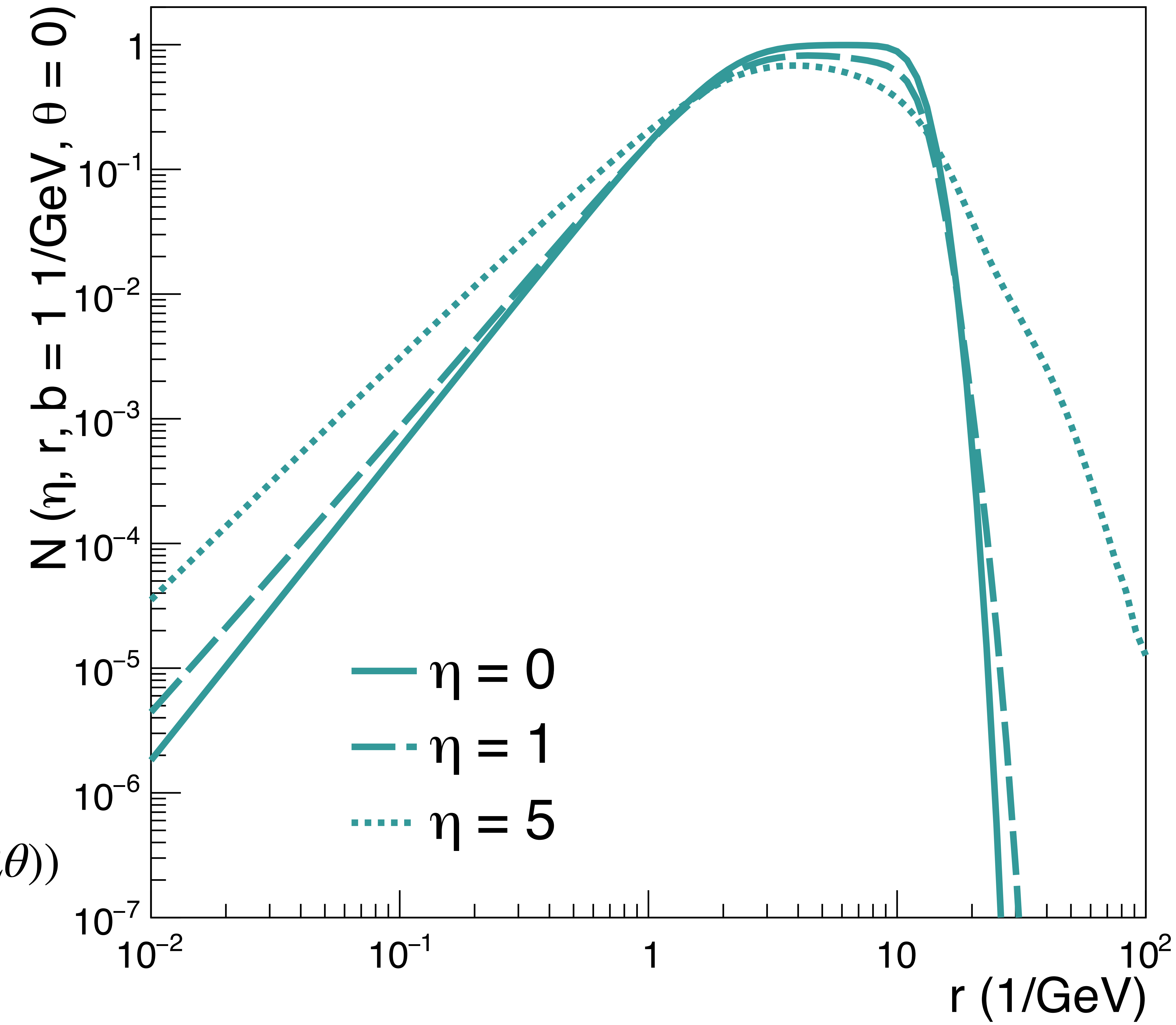

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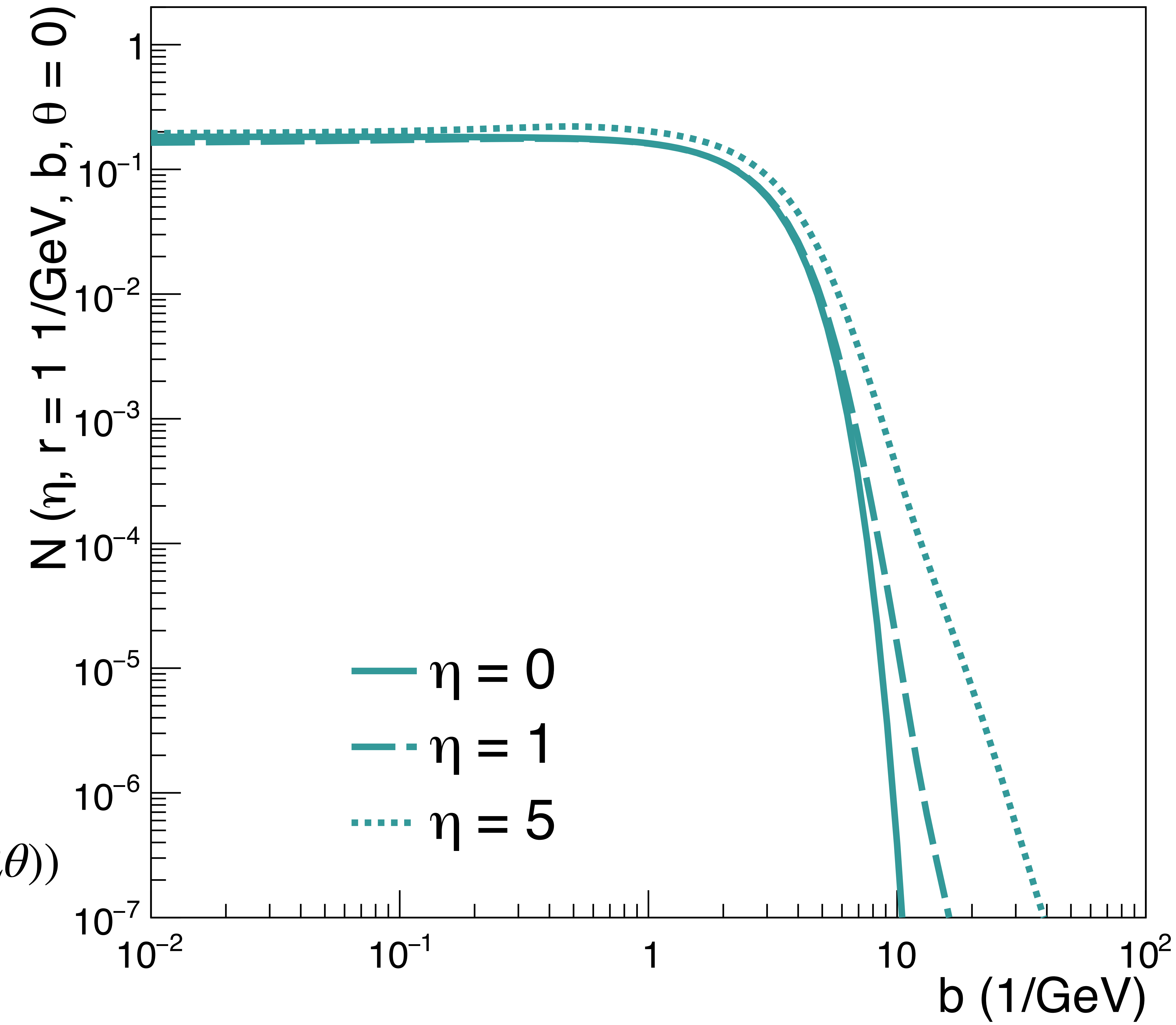

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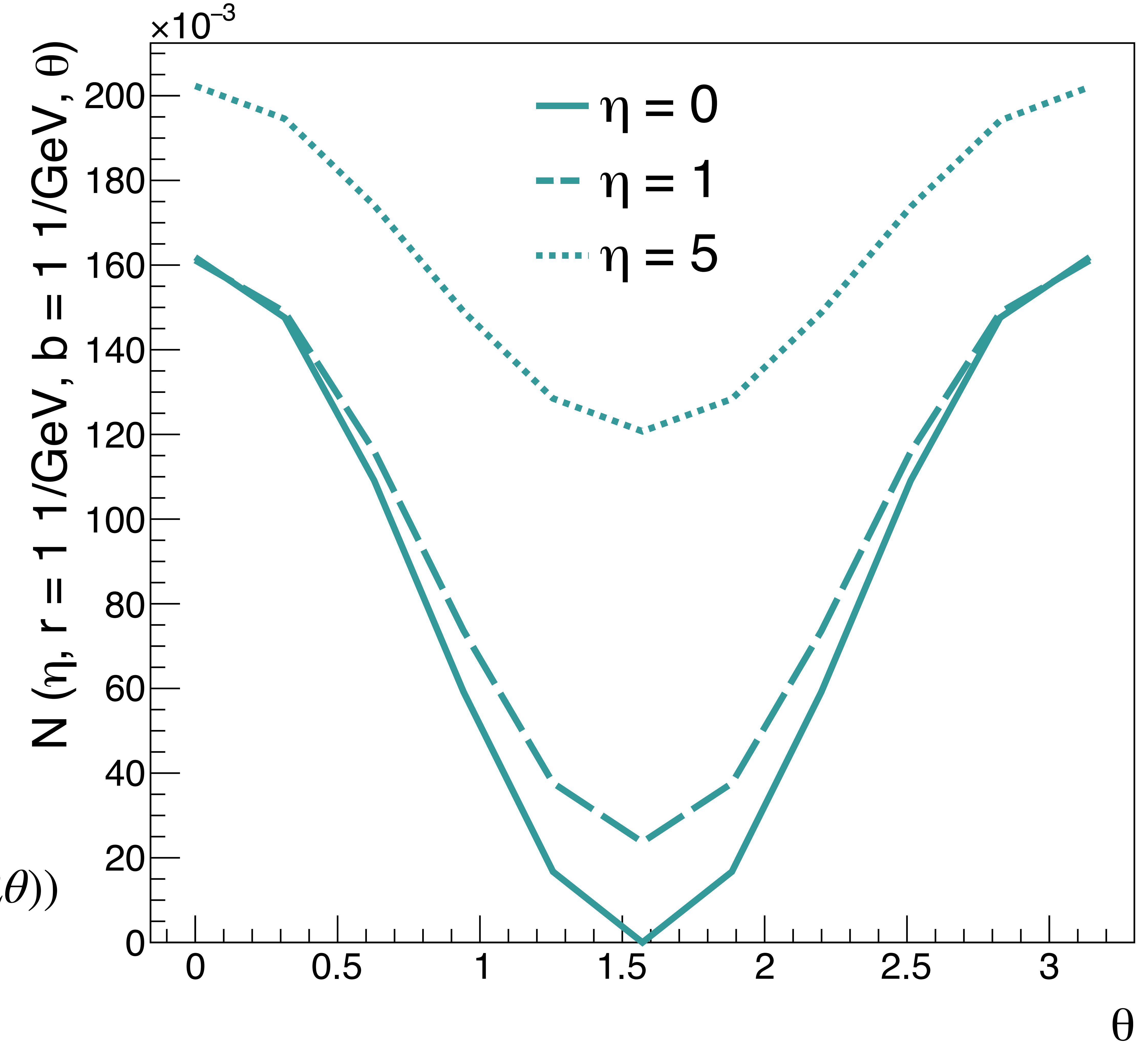

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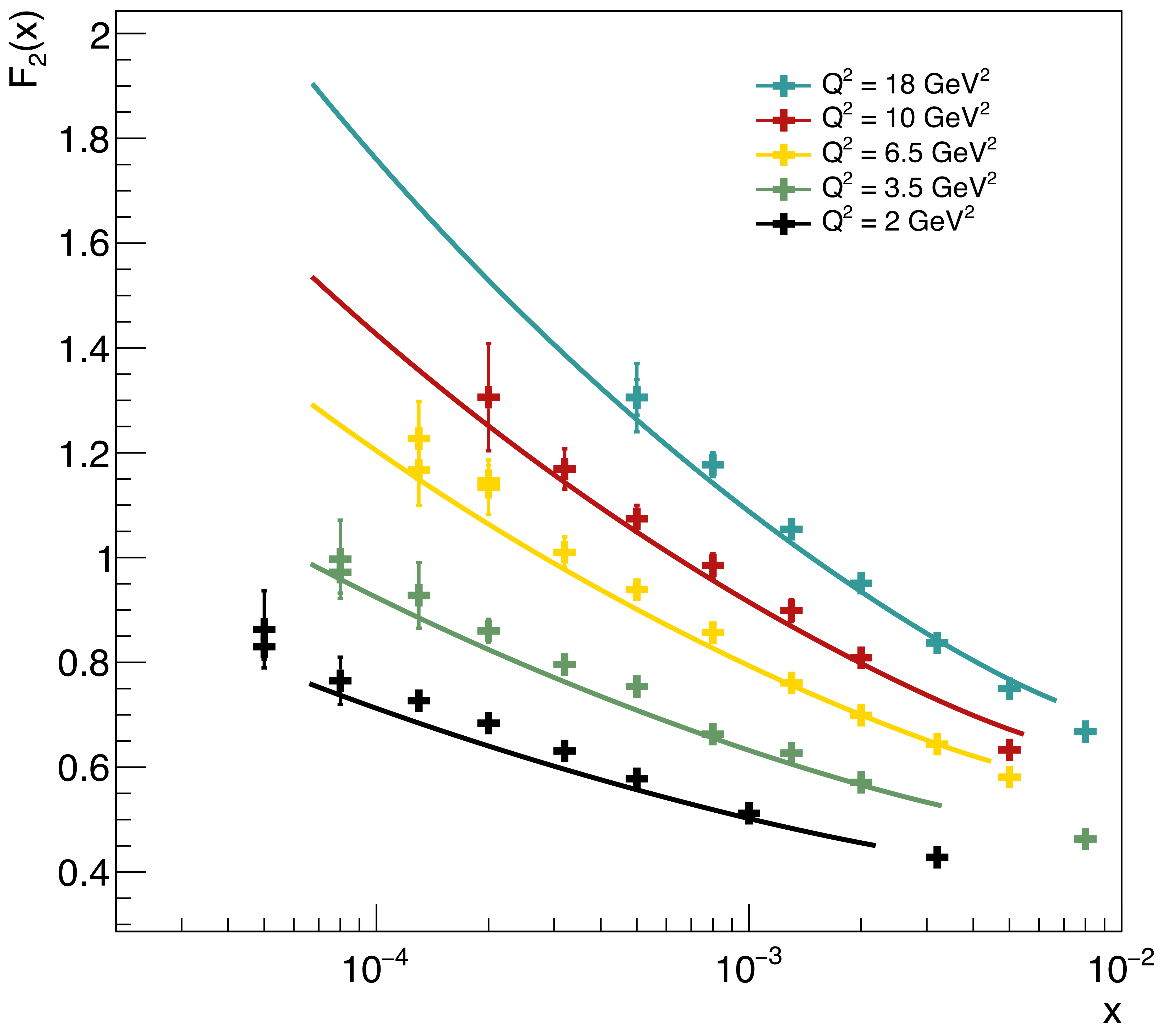
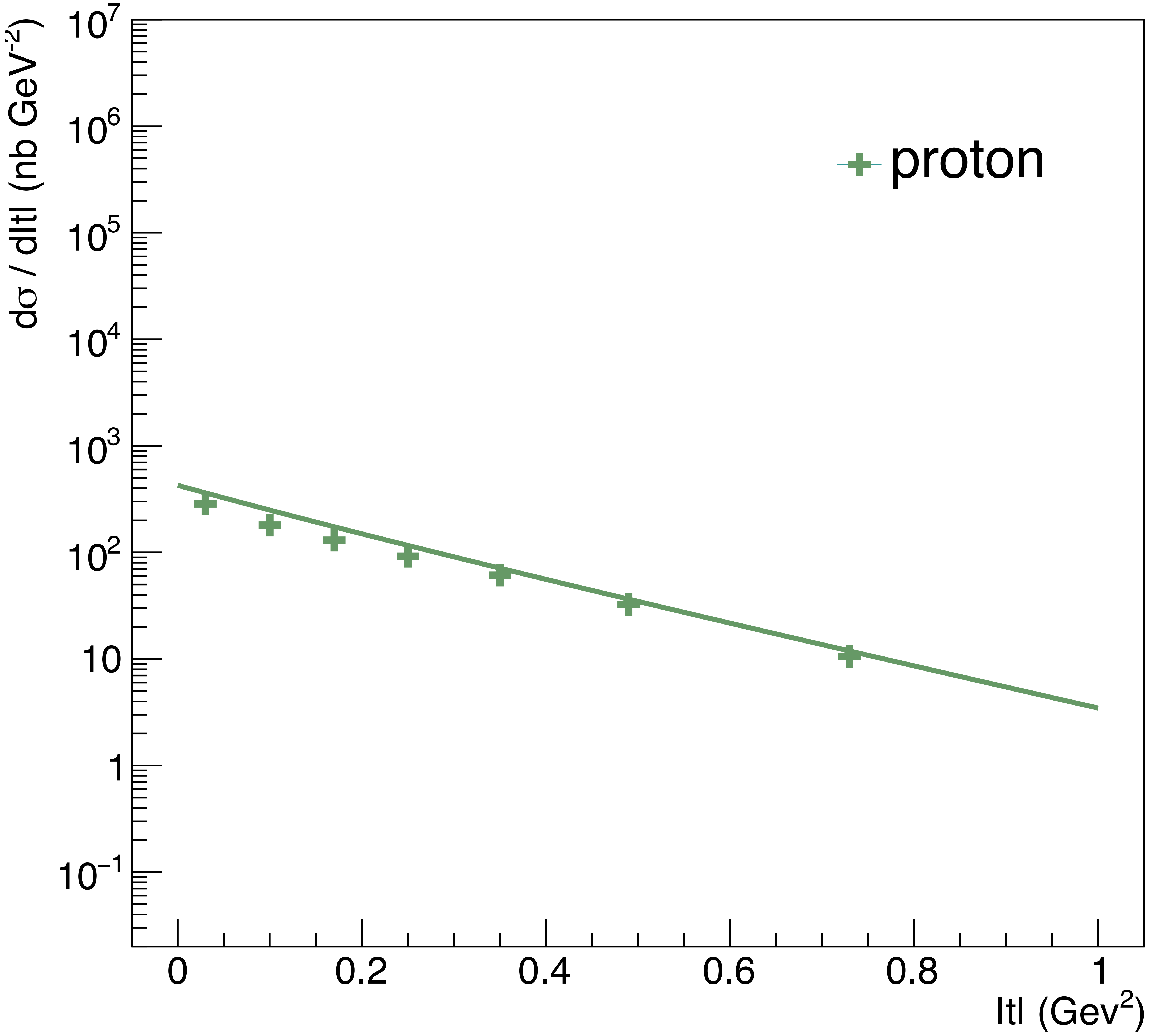
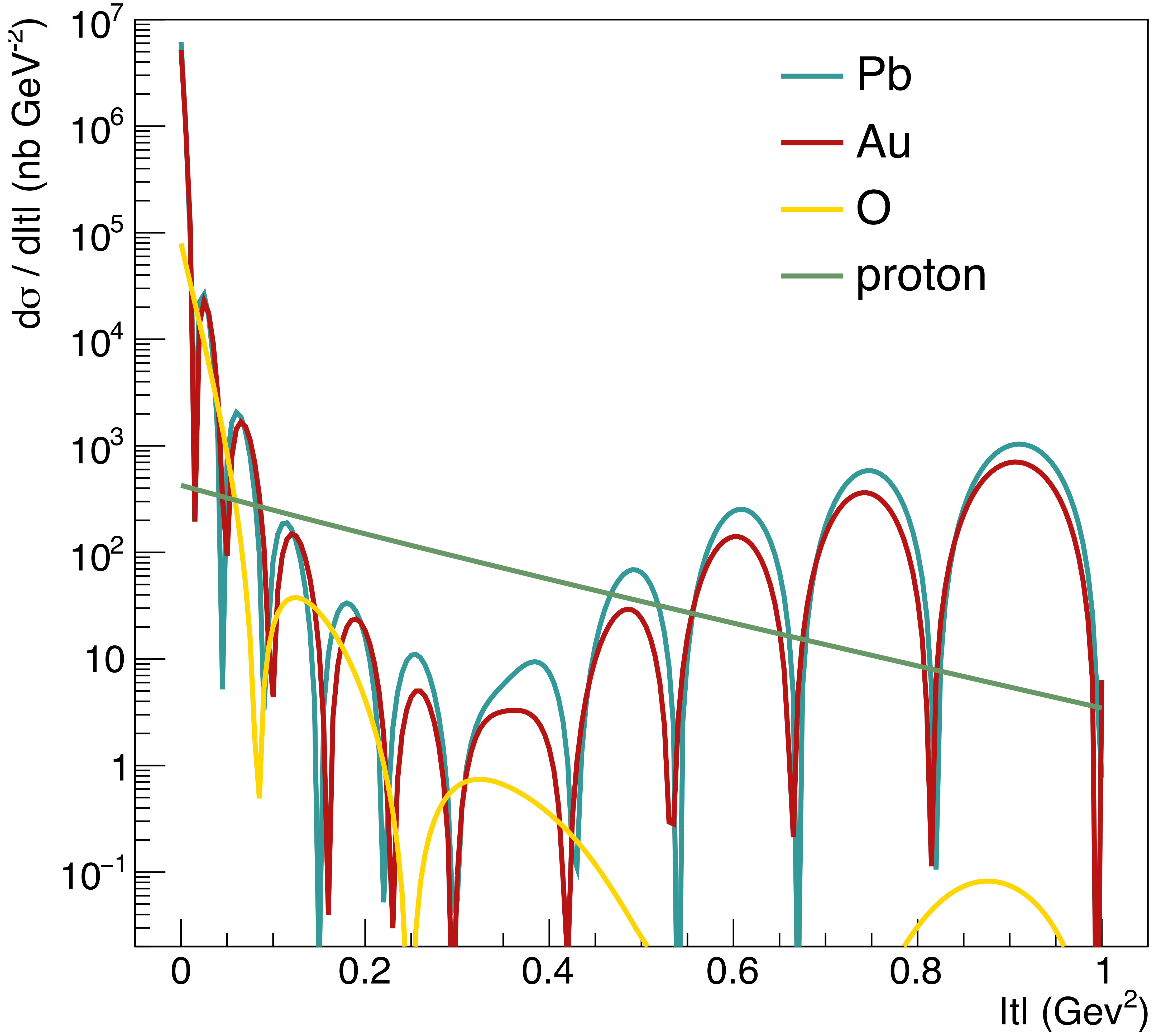


$F_2(x)$
$Q^2 = 18\ GeV^2$
$Q^2 = 10\ GeV^2$
$Q^2 = 6.5\ GeV^2$
$Q^2 = 3.5\ GeV^2$
$Q^2 = 2\ GeV^2$
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
10^{-4}
10^{-3}
10^{-2}
x



$d\sigma / d|t|$ (nb GeV^{-2})
proton
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
0
0.2
0.4
0.6
0.8
1
|t| (Gev^2)

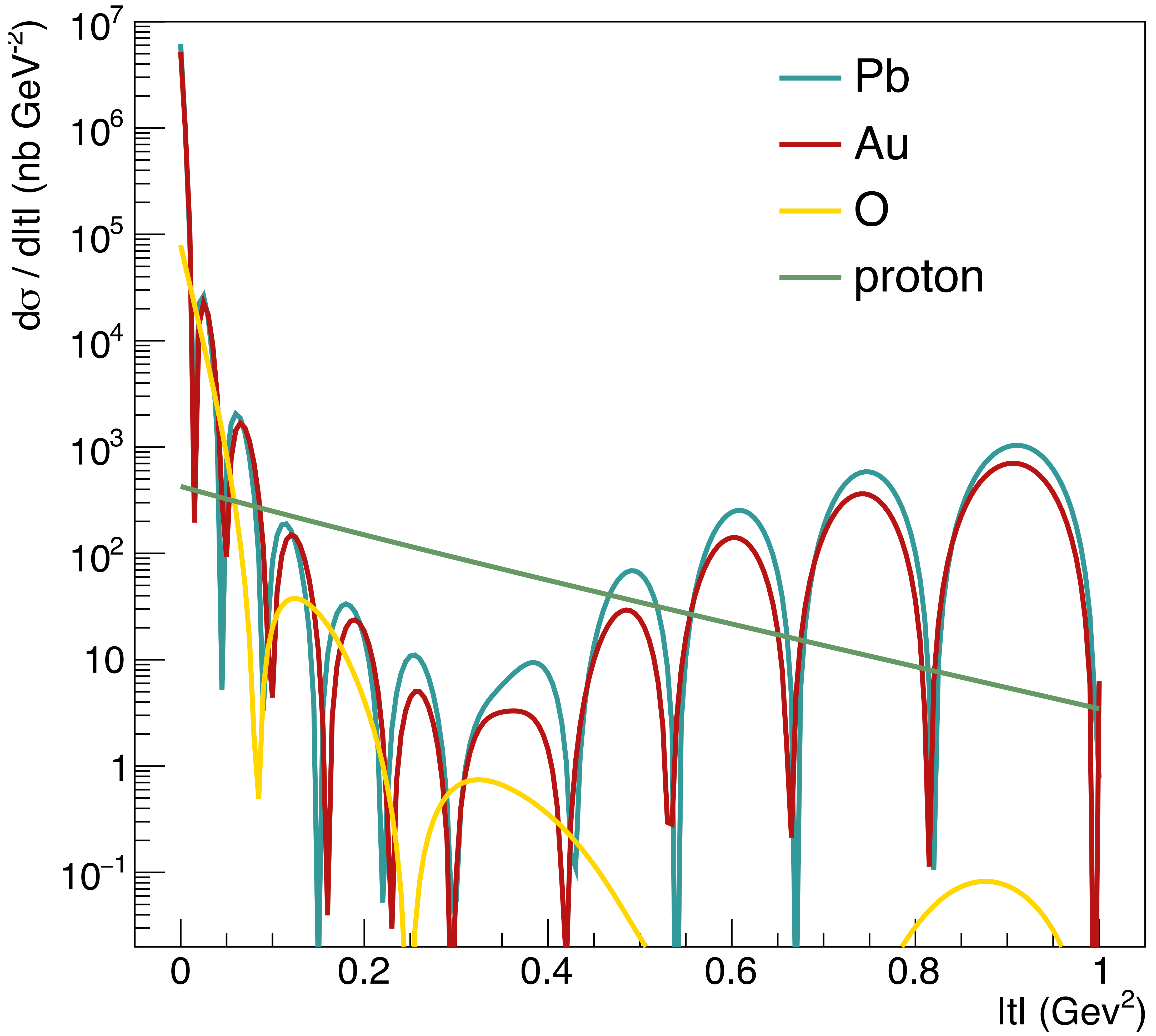
3D BK - data



dσ / d|t| (nb GeV⁻²)
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^-1
0
0.2
0.4
0.6
0.8
1
|t| (Gev²)
Pb
Au
O
proton

3D BK - data

- EIC predictions
- coherent nuclear J/ψ production
- nuclear initial condition
    - Gaussian to Woods-Saxon



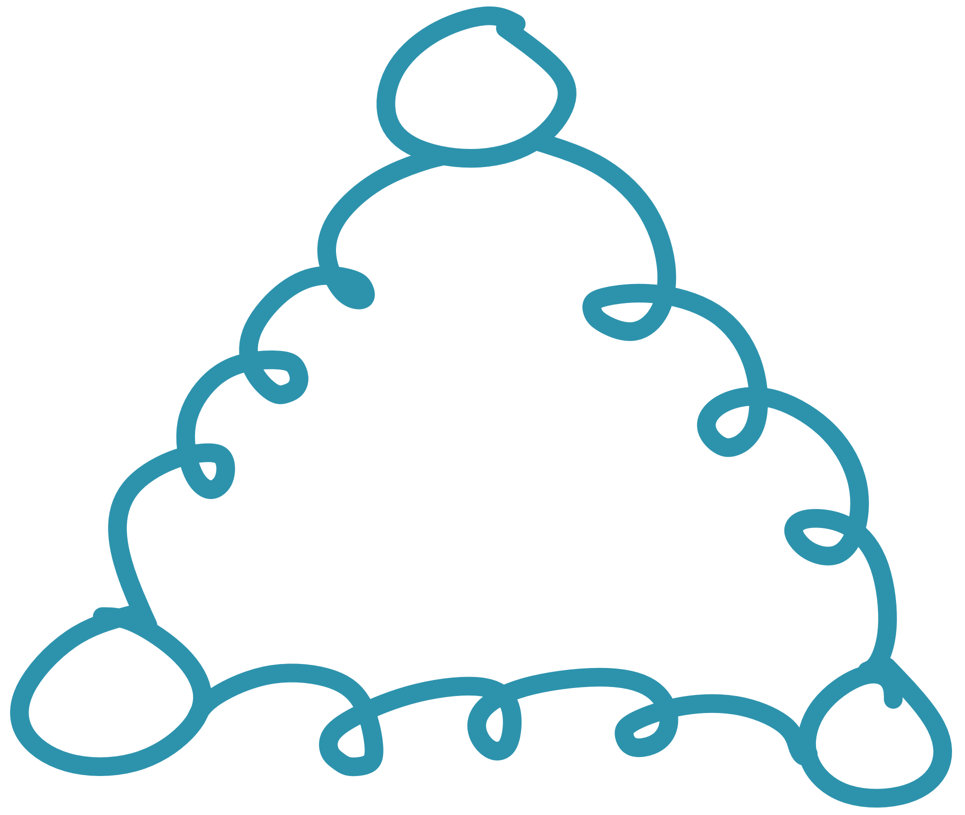
3D BK - summary

- successfull reconstruction of former data description
- EIC predictions for vector meson production
- tool ready for potential
    - modeling TMDs, GTMDs, ...
    - calculating DVCS, dijets, ...

thank you

hadron structure & saturation

- open question of modern physics
- QCD
    - complex → effective theories
    - rich hadron structure evolution
    - saturation → Balitsky-Kovchegov equation



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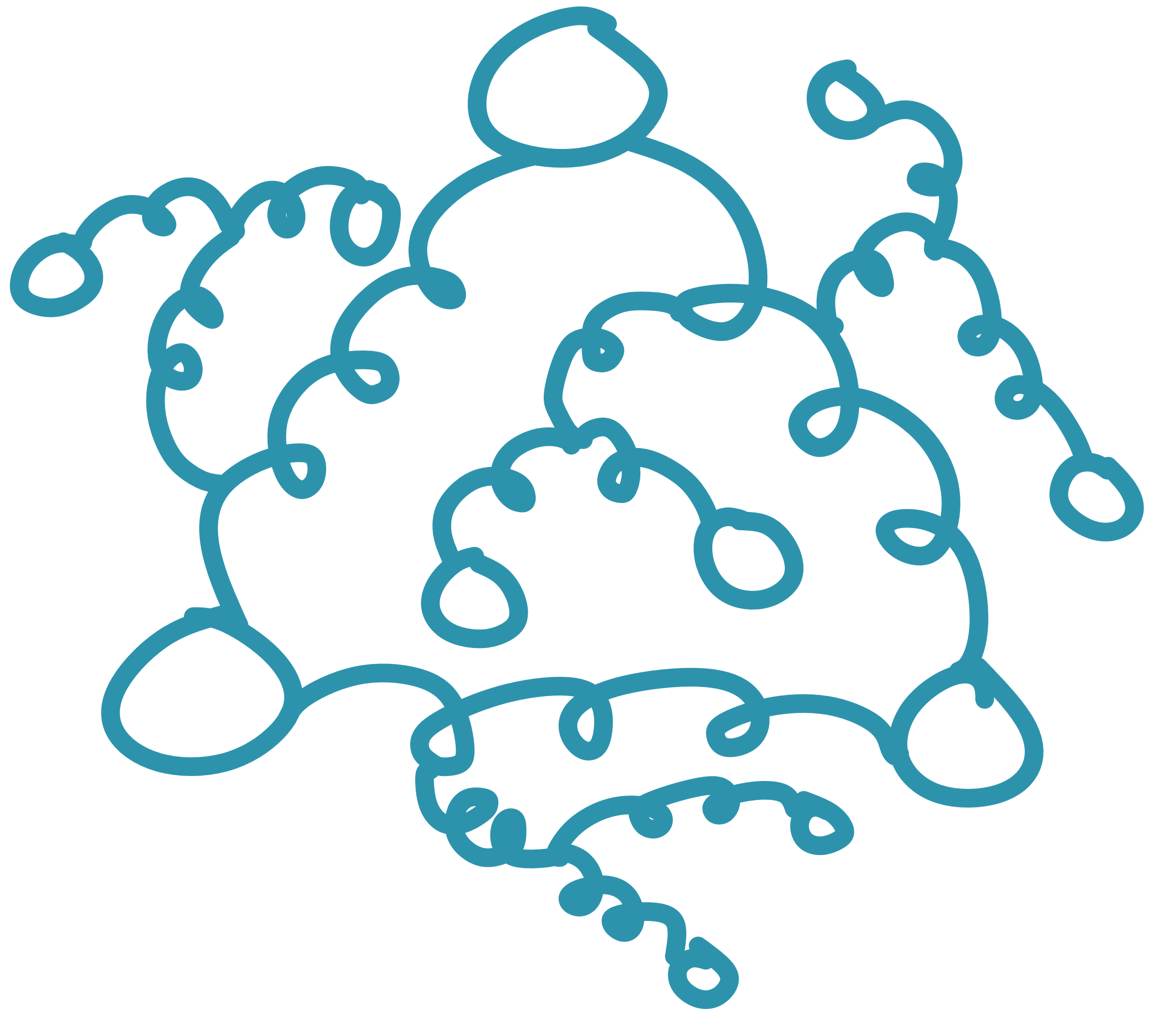
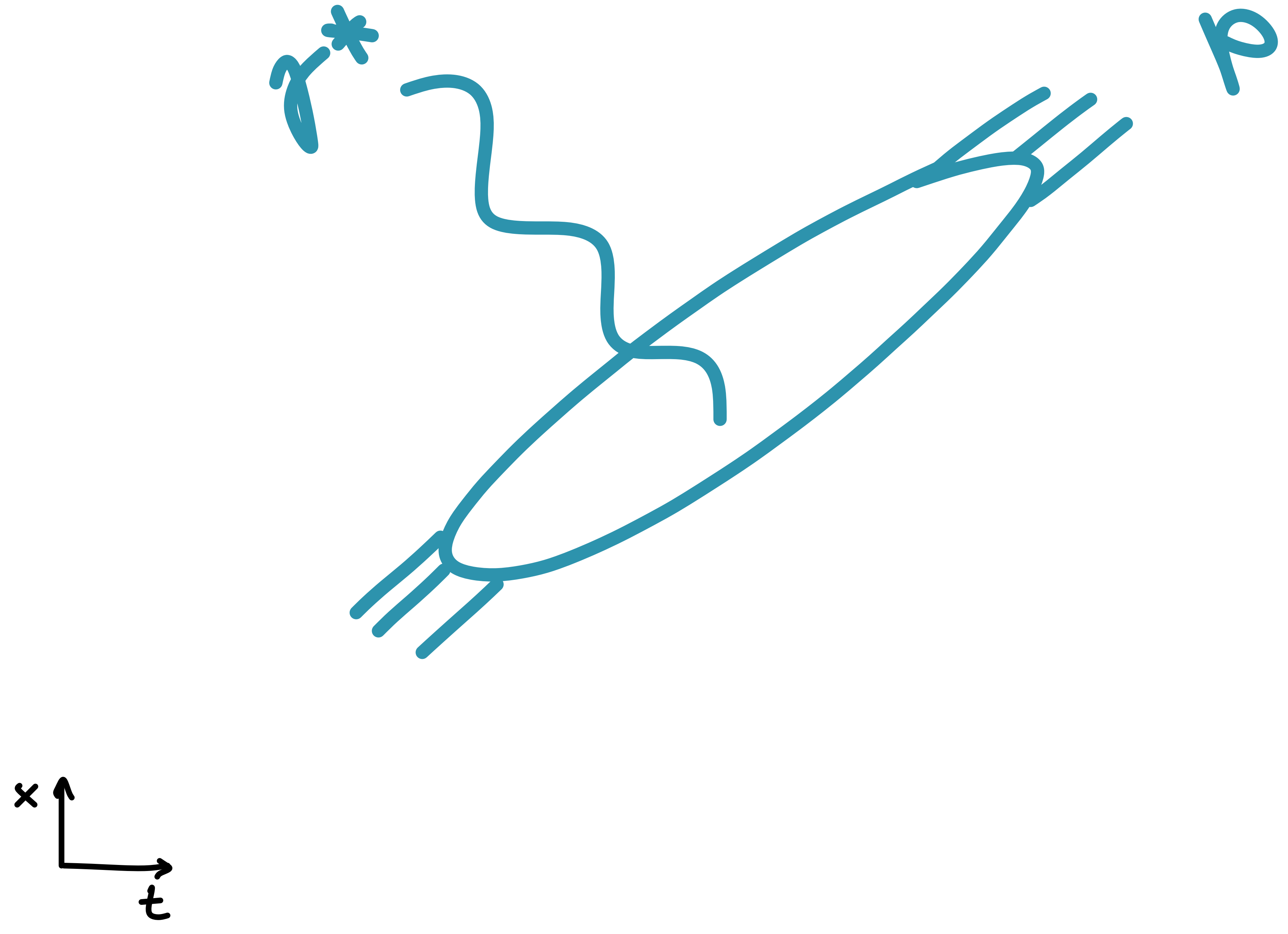


hadron structure & experiment

- colliding proton with a projectile at high E
  - typically electron-proton DIS
- interest in photon-proton cross section

$\sigma_{L,T}^{\gamma^* p}(x, Q^2)$



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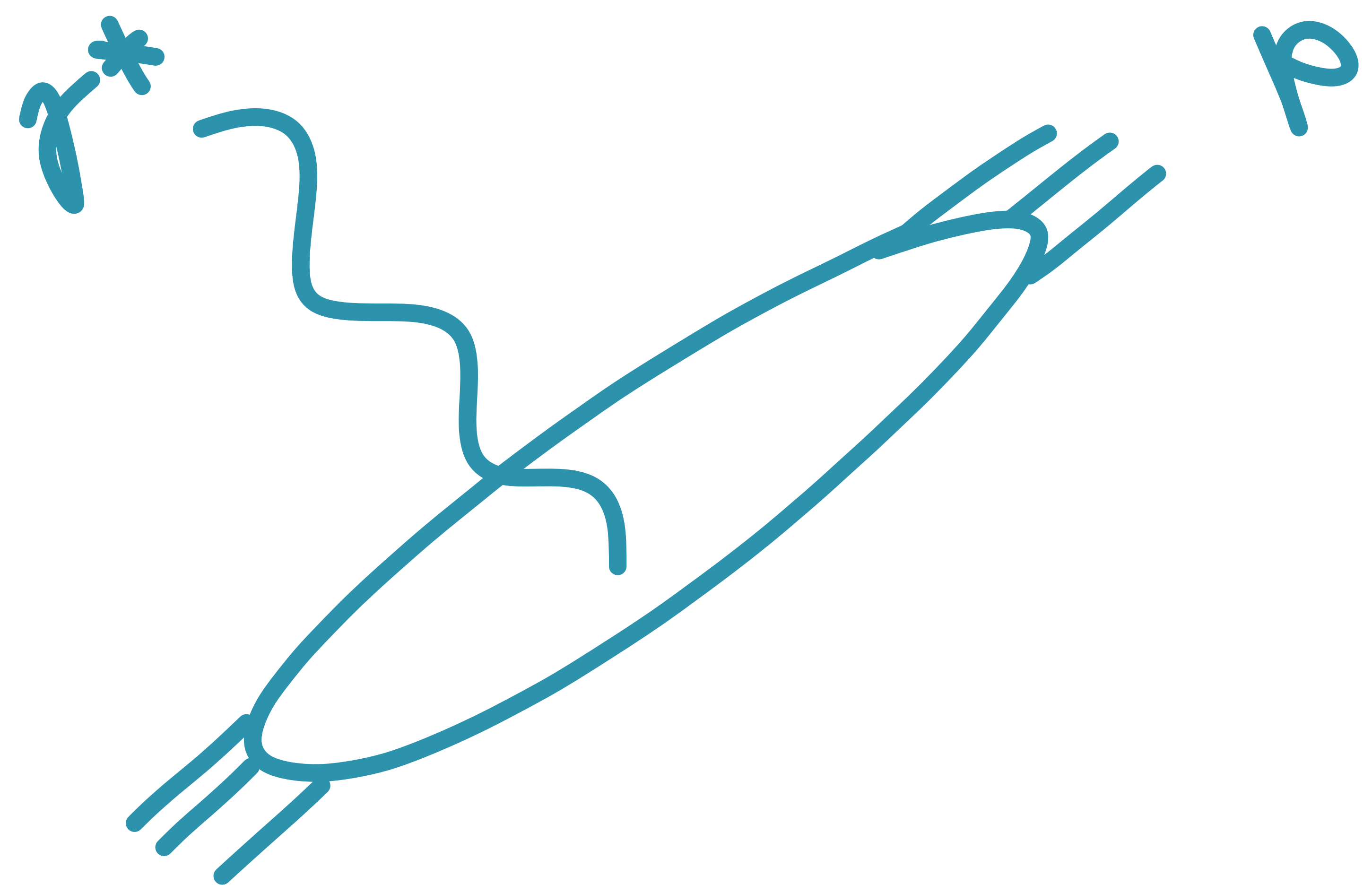
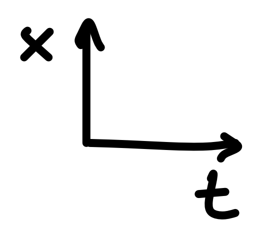
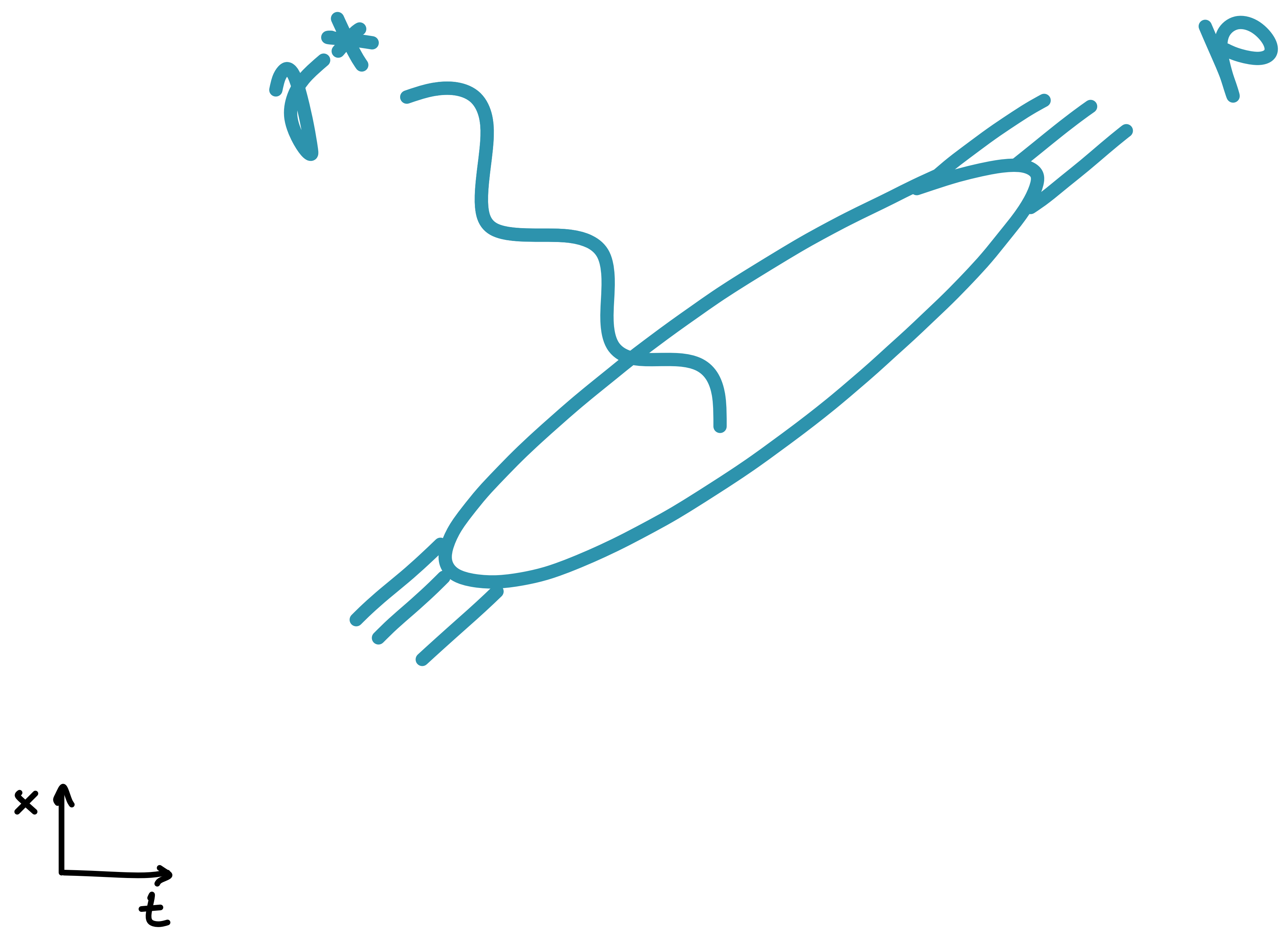




the colour dipole model

- replace γ* with the colour dipole
    - probability ~ light cone wave function $|\psi^{(f)}_{T,L}(\vec{r}, Q^2, z)|^2$
- shockwave approximation



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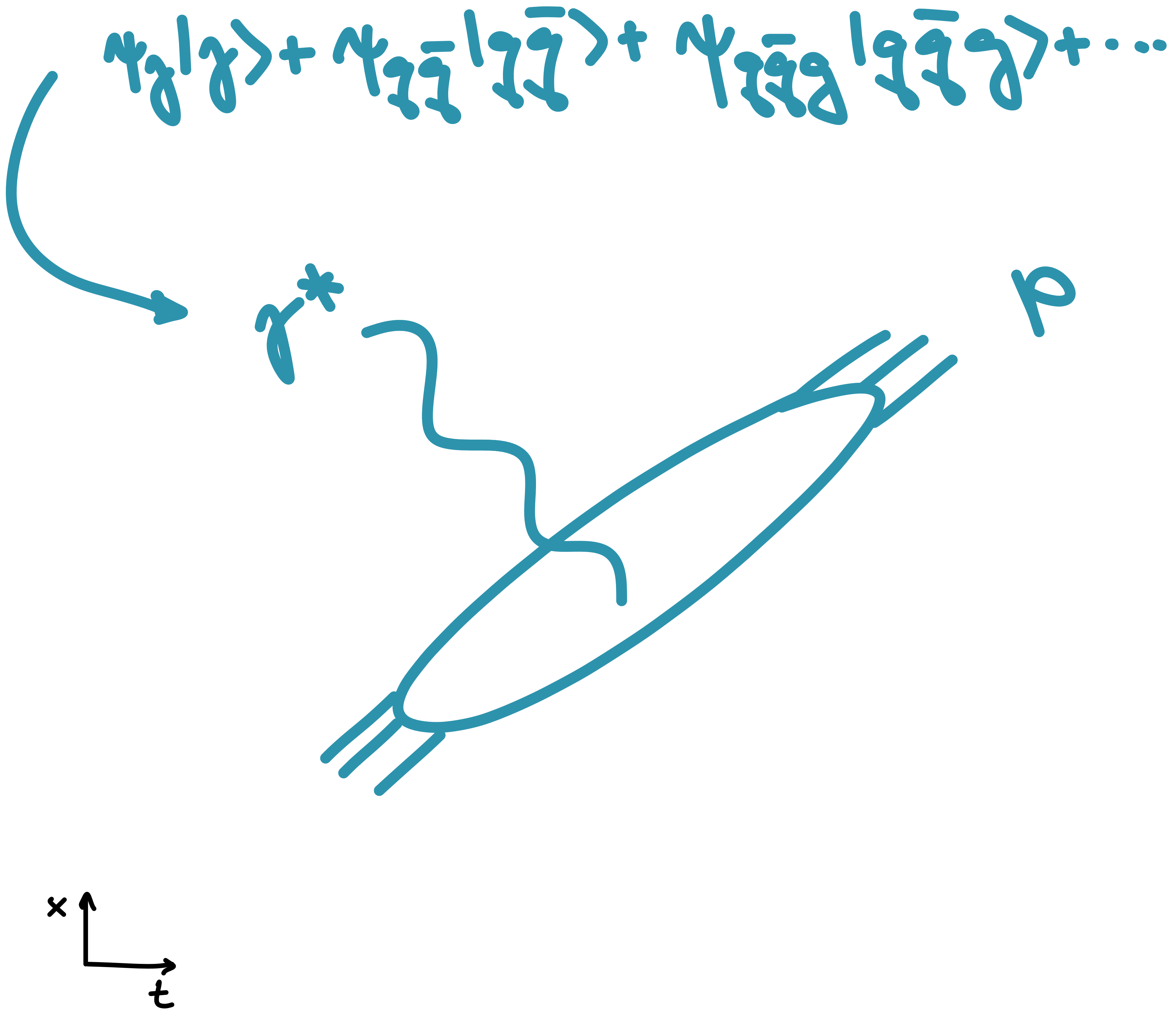

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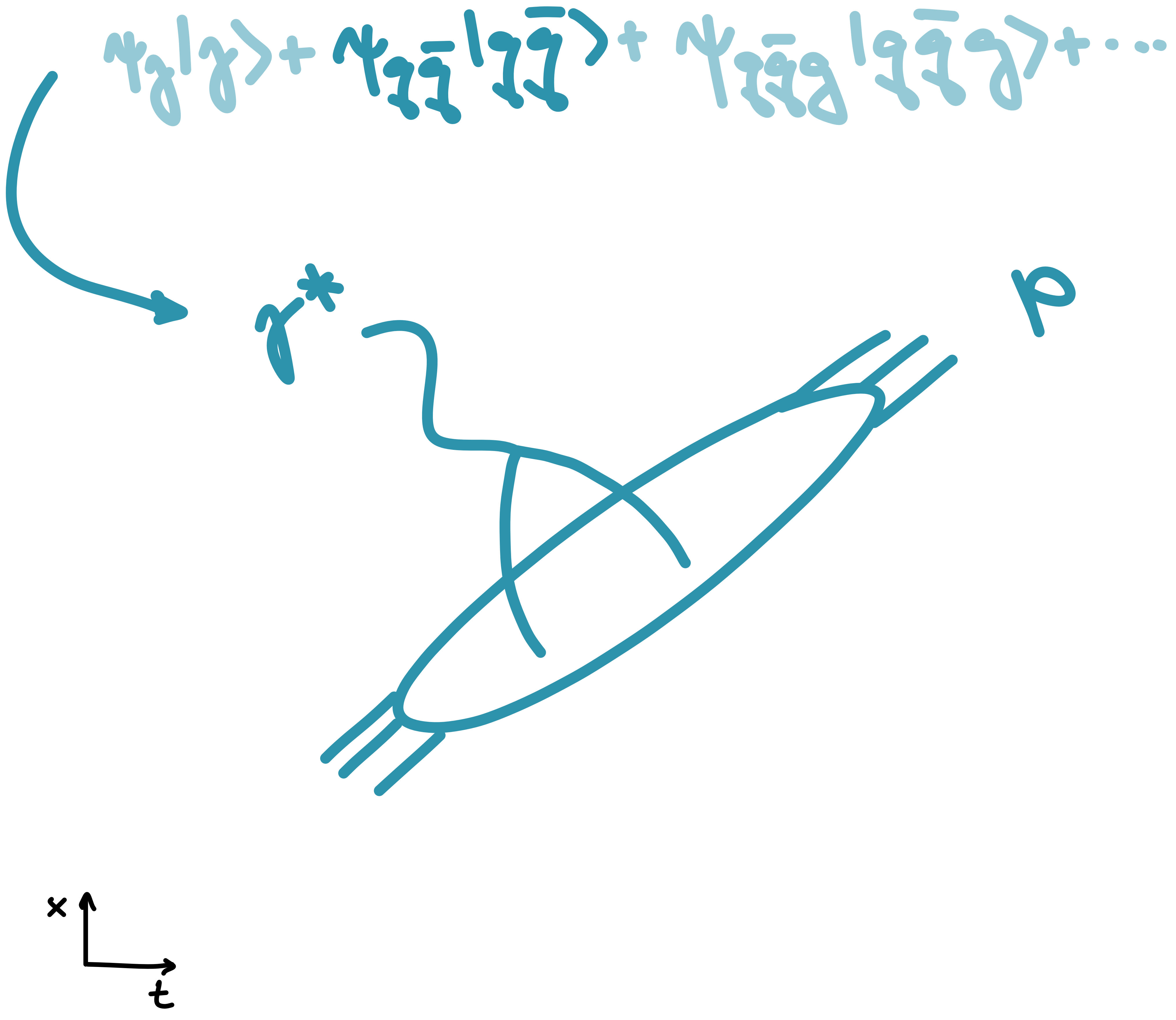
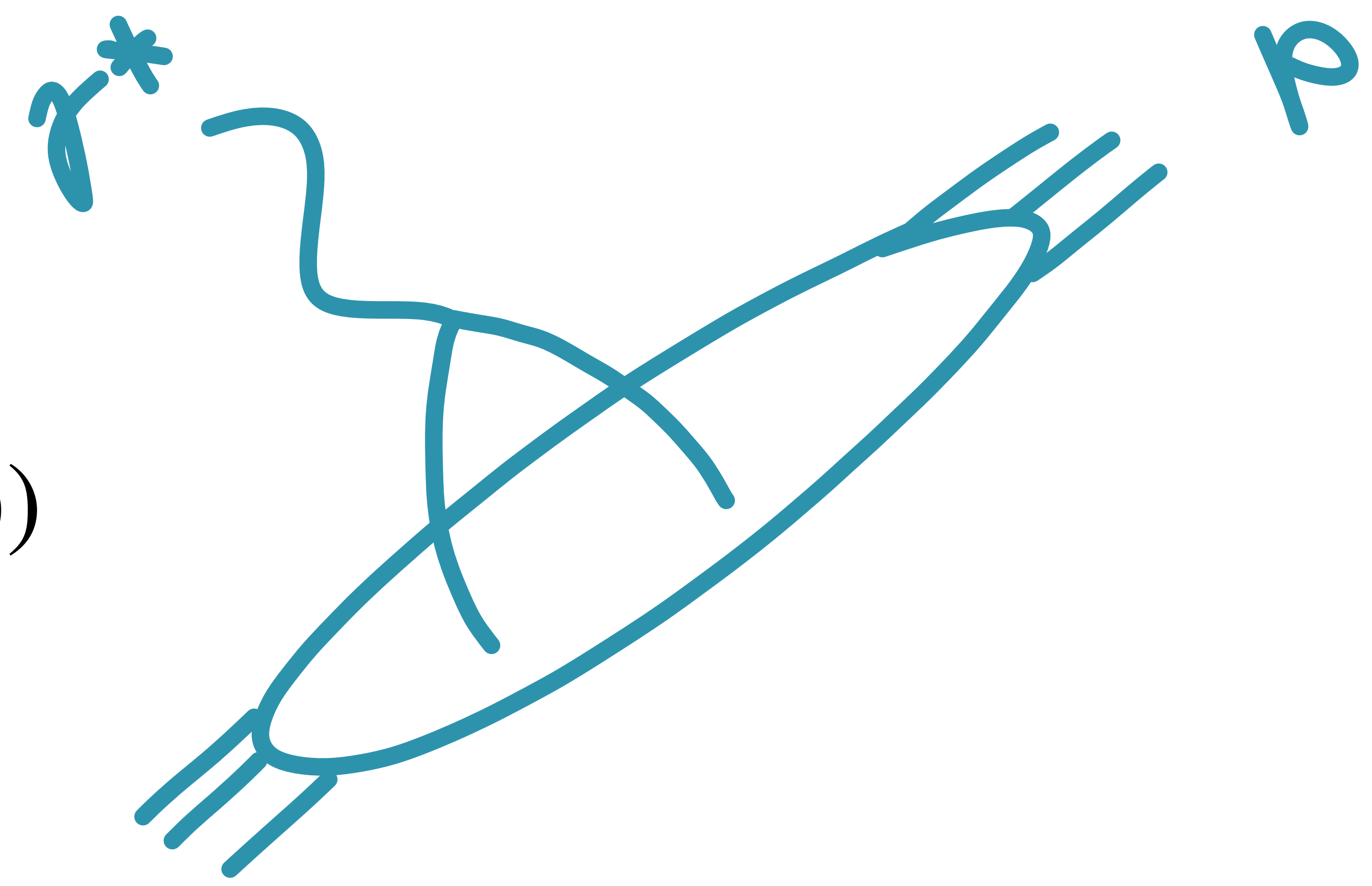
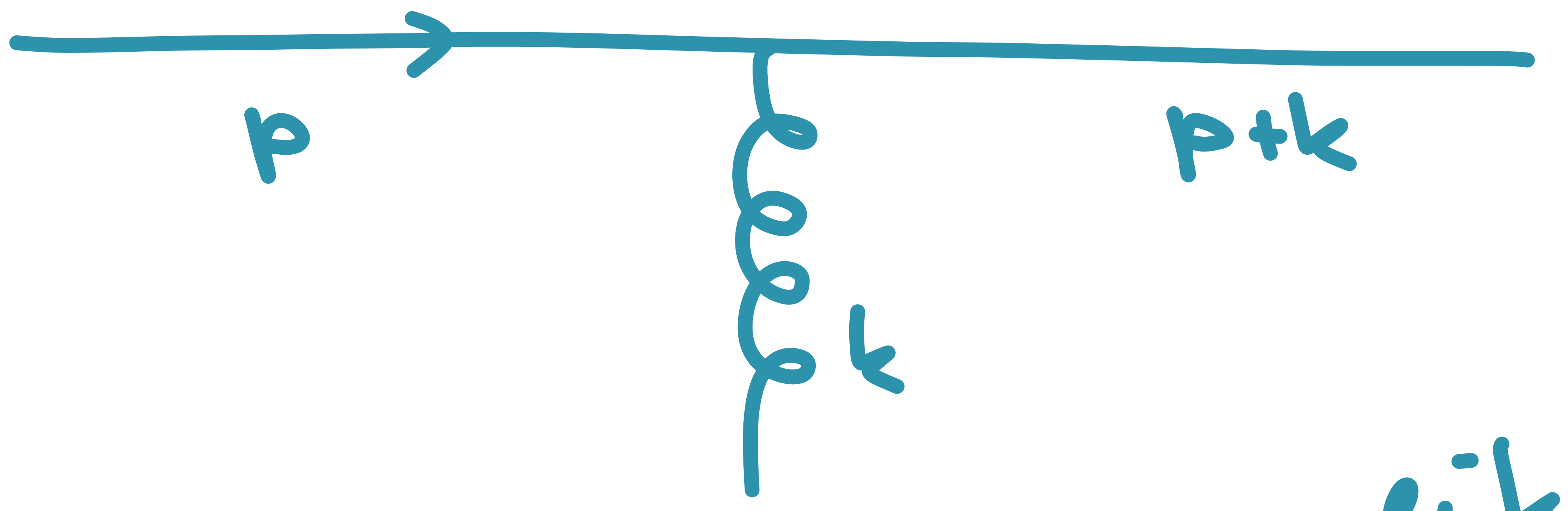


the colour dipole model

$$\sigma_{L,T}^{\gamma^* p}(x, Q^2) = \sum_f \int \mathrm{d}^2\vec{r} \int_0^1 \mathrm{d}z \, |\psi_{T,L}^{(f)}(\vec{r}, Q^2, z)|^2 \, 2 \int \mathrm{d}^2\vec{b} N(\vec{r}, \vec{b}, \tilde{x}_f(x))$$



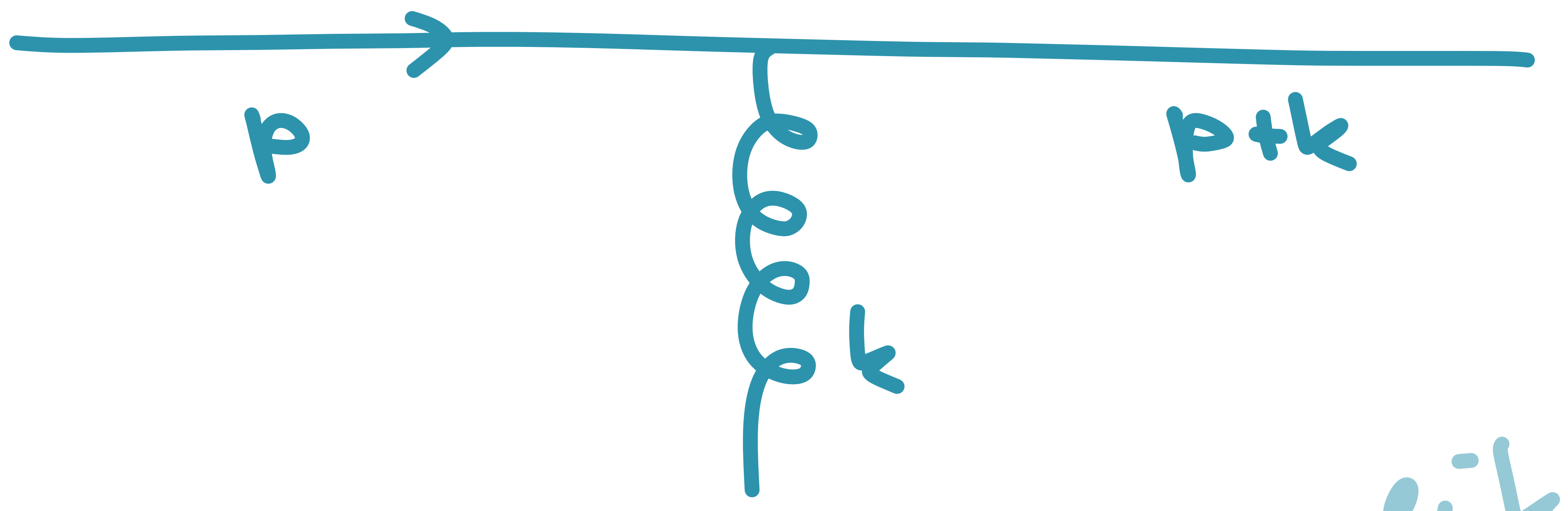
dipole-hadron scattering



eikonal ~ $p^- \gg k^-$

$$\frac{1}{\sqrt{2p^-}}\frac{1}{\sqrt{2(p^-+k^-)}}\bar{u}(p+k)\; ig\gamma_\mu A^\mu(k)\; u(p) \;\approx\; igA^+(k)$$

dipole-hadron scattering



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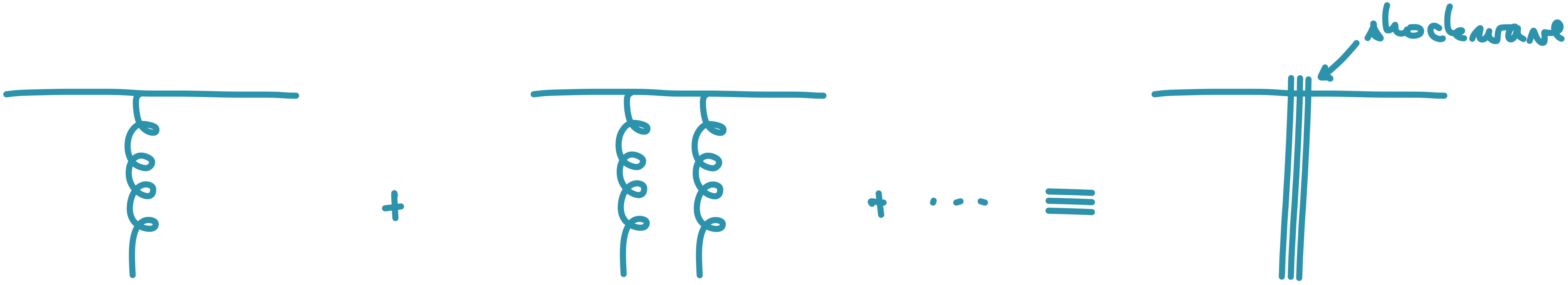
$$\to \int \mathrm{d}x^- igA^+(\vec{x})$$

dipole-hadron scattering



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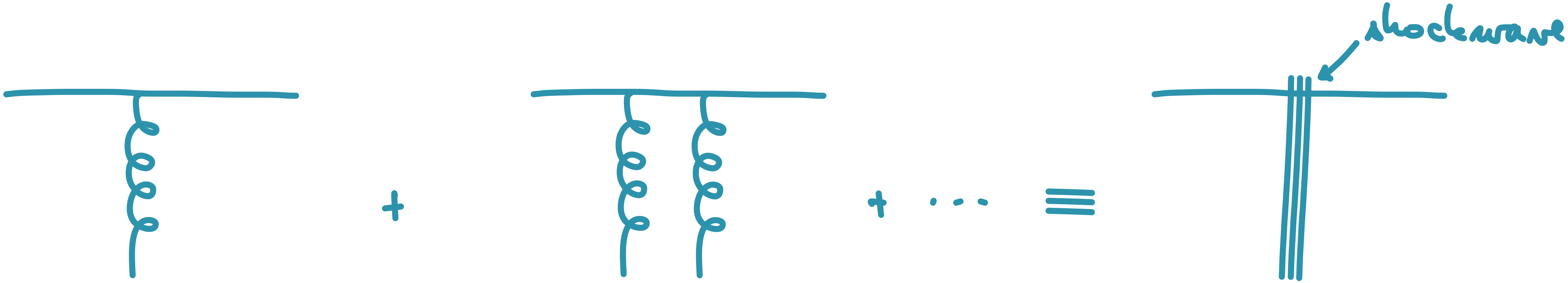
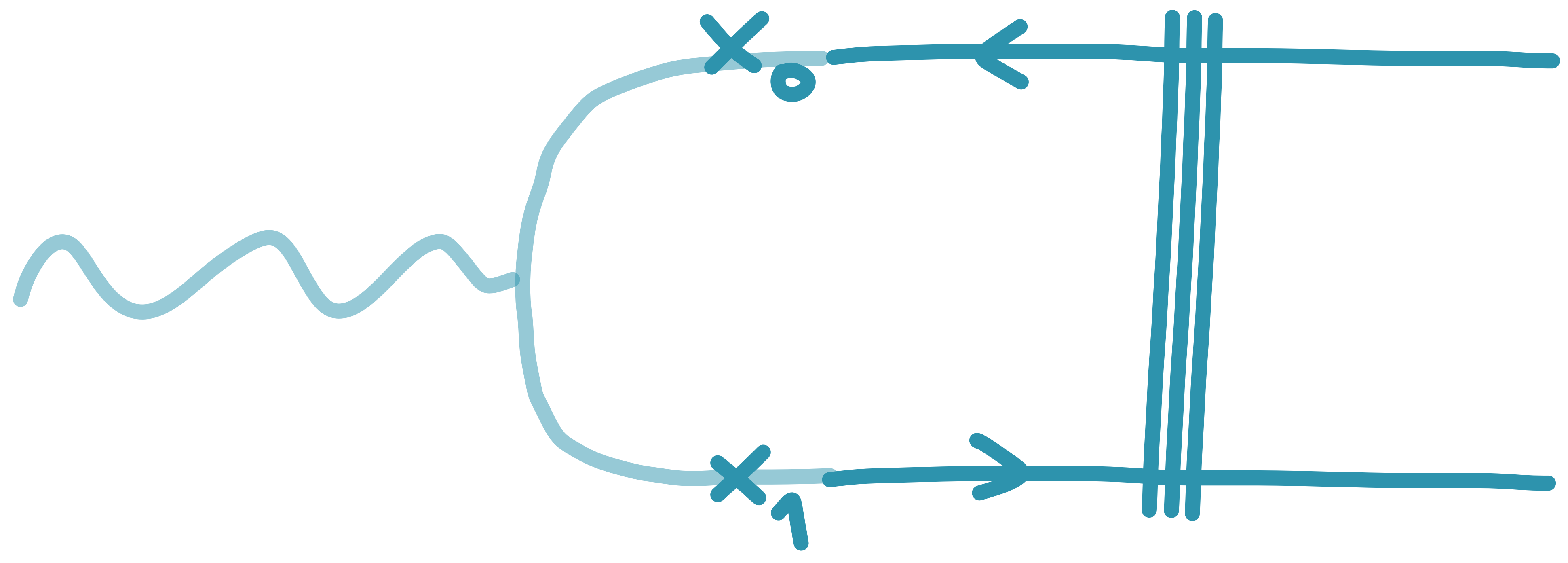


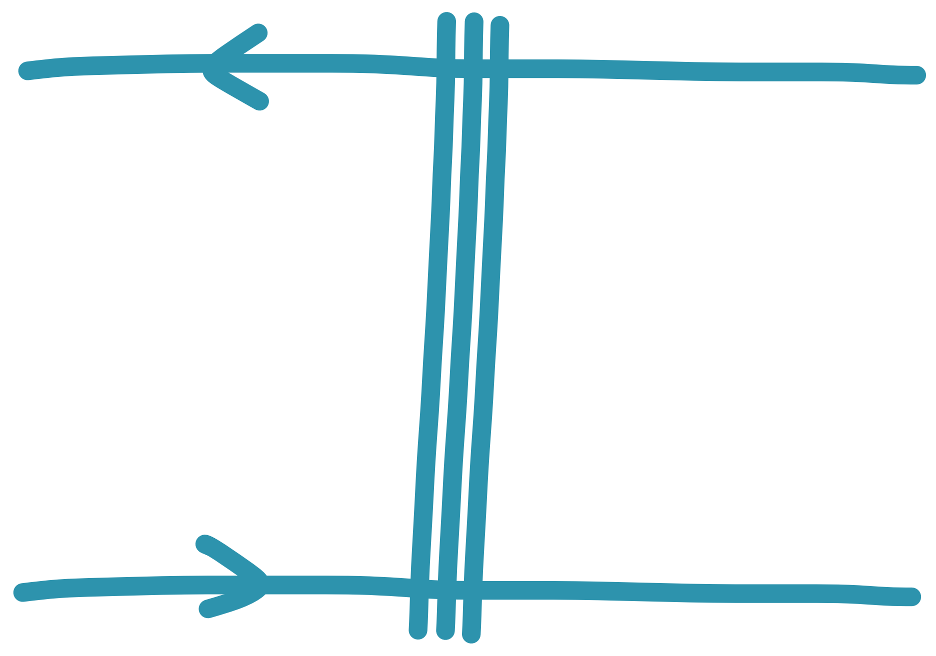
$$ig\int_{-\infty}^{+\infty} \mathrm{d}x_1^- A^+(\vec{x}_1) + (ig)^2 \int_{-\infty}^{+\infty} \mathrm{d}x_1^- \int_{x_1^-}^{+\infty} \mathrm{d}x_2^- A^+(\vec{x}_2) A^+(\vec{x}_1) \equiv \underbrace{P\exp\left\{ig\int_{-\infty}^{+\infty} \mathrm{d}x^- A^+(\vec{x})\right\}}_{V_{x_\perp}} - 1$$

dipole-hadron scattering

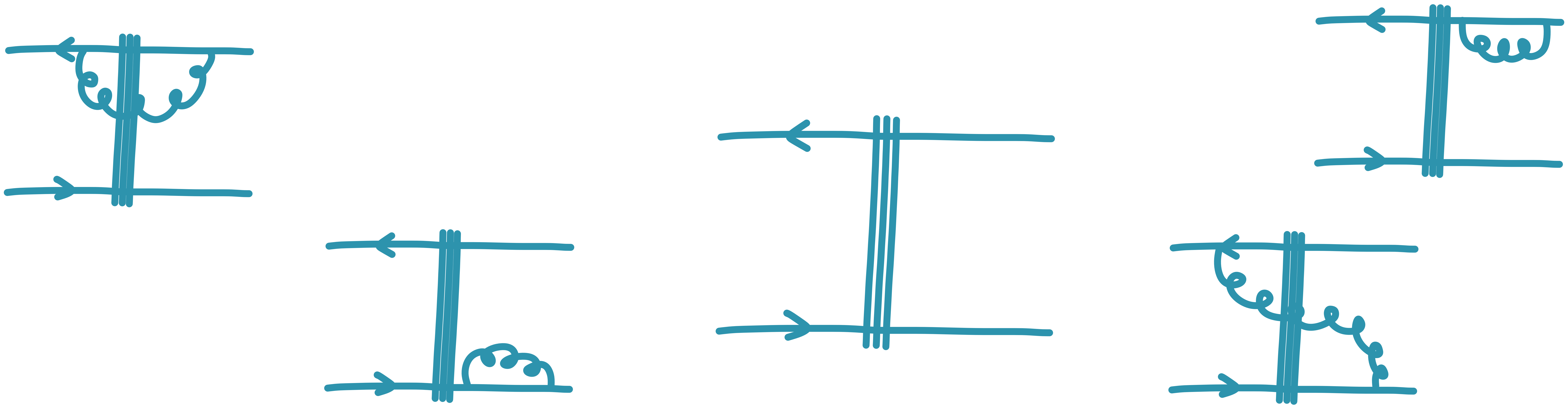


$$N(\underline{x}_0, \underline{x}_1) = 1 - \left\langle \frac{1}{N_C} \mathrm{tr}\left[V_1 V_0^\dagger\right] \right\rangle$$

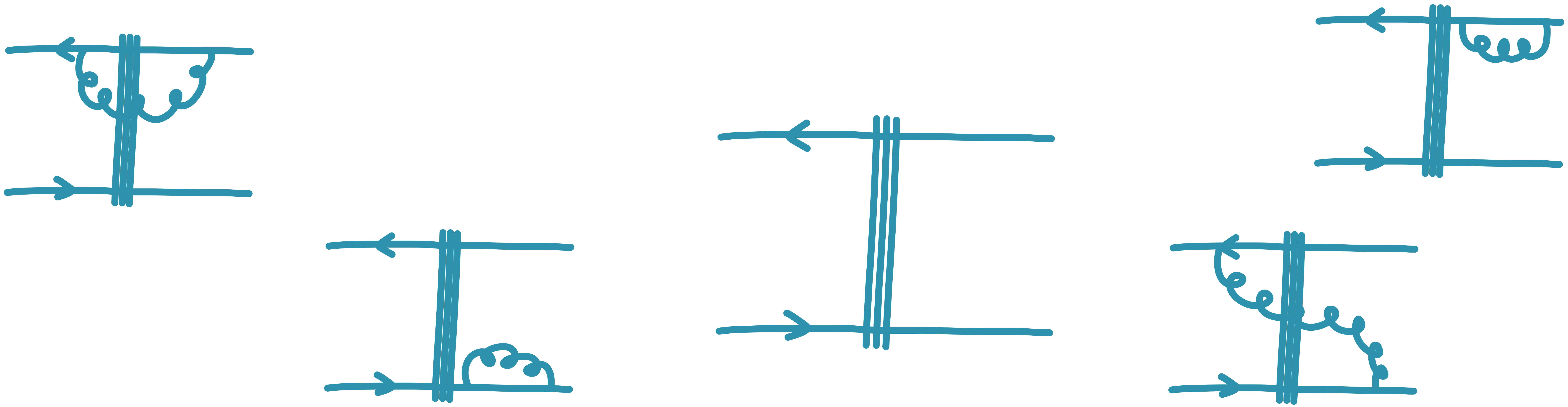
dipole-hadron scattering



dipole-hadron scattering



dipole-hadron scattering



$$\frac{\partial N(\underline{r},\underline{b},\eta)}{\partial \eta} = \int \mathrm{d}^2\underline{r}_1 K(\underline{r},\underline{r}_1,\underline{r}_2) \left[N(\underline{r}_1,\underline{b}_1,\eta_1) + N(\underline{r}_2,\underline{b}_2,\eta_2) - N(\underline{r},b,\eta) - N(\underline{r}_1,\underline{b}_1,\eta_1)N(\underline{r}_2,\underline{b}_2,\eta_2)\right]$$

$\eta = \ln \frac{x}{x_0}$

dipole-hadron scattering

$$K_{lo} = \frac{\alpha_s N_C}{2\pi} \frac{r^2}{r_1^2 r_2^2}$$

$$K_{ci} = \frac{\bar{\alpha}_s}{2\pi} \frac{r^2}{r_1^2 r_2^2} \left[\frac{r^2}{\min\{r_1^2, r_2^2\}} \right]^{\pm \bar{\alpha}_s A_1}$$

$$\frac{\partial N(\underline{r}, \underline{b}, \eta)}{\partial \eta} = \int \mathrm{d}^2 \underline{r}_1 K(\underline{r}, \underline{r}_1, \underline{r}_2) \left[N(\underline{r}_1, \underline{b}_1, \eta_1) + N(\underline{r}_2, \underline{b}_2, \eta_2) - N(\underline{r}, b, \eta) - N(\underline{r}_1, \underline{b}_1, \eta_1) N(\underline{r}_2, \underline{b}_2, \eta_2) \right]$$

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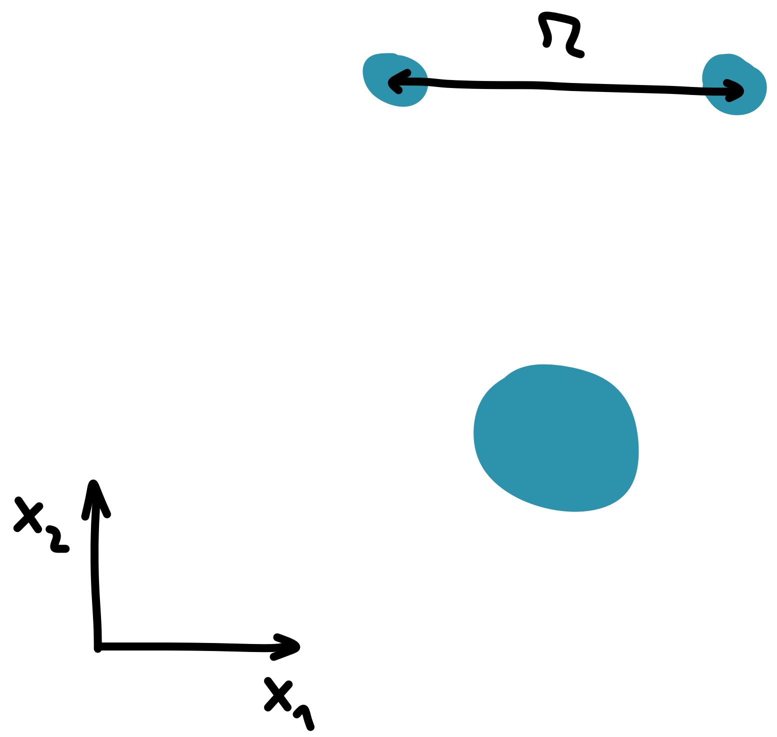
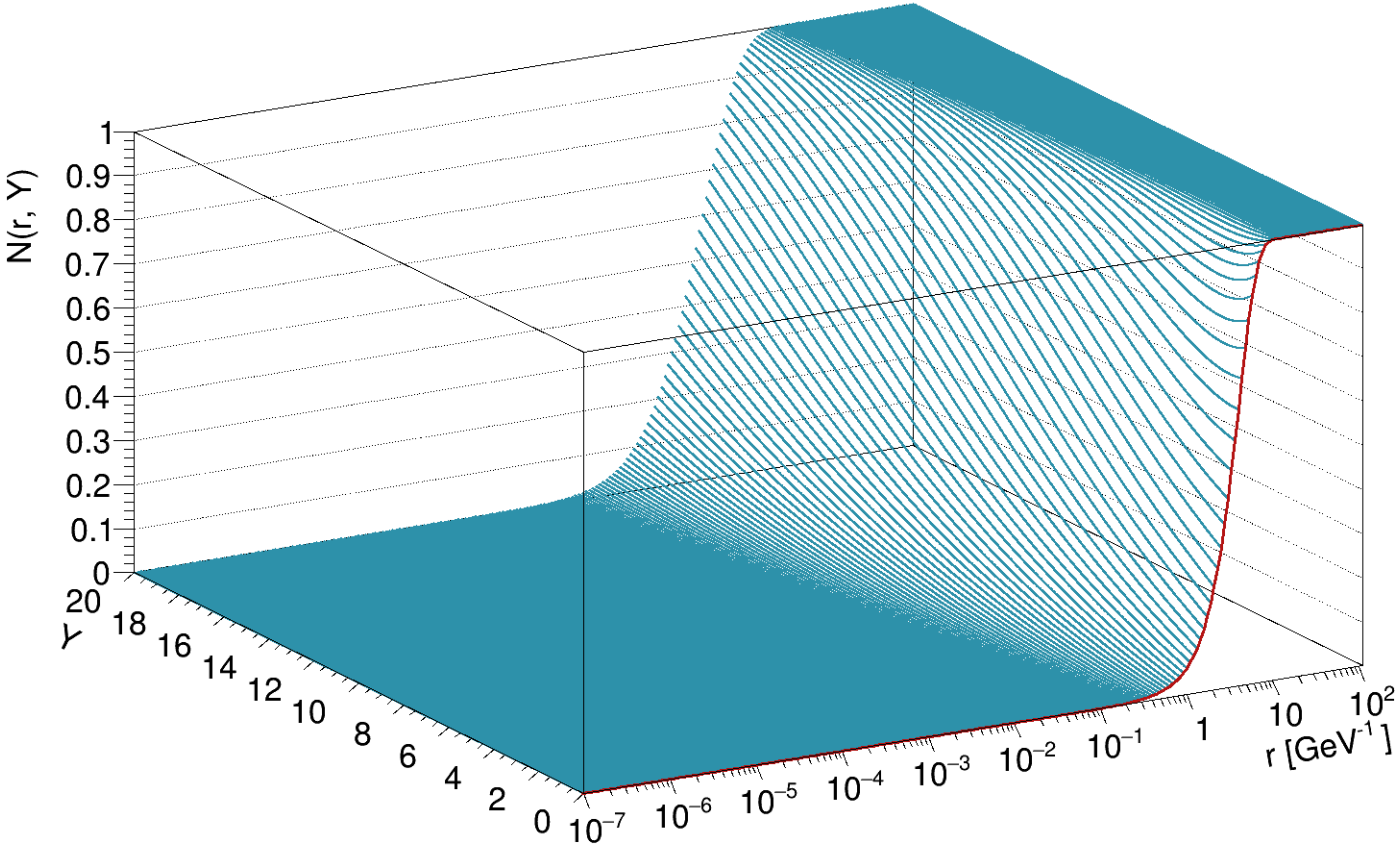
numerical solutions - 1D

$$2\int \mathrm{d}\underline{b}\, N(\underline{r},\underline{b},Y) \approx \sigma_0\, N(r,Y)$$

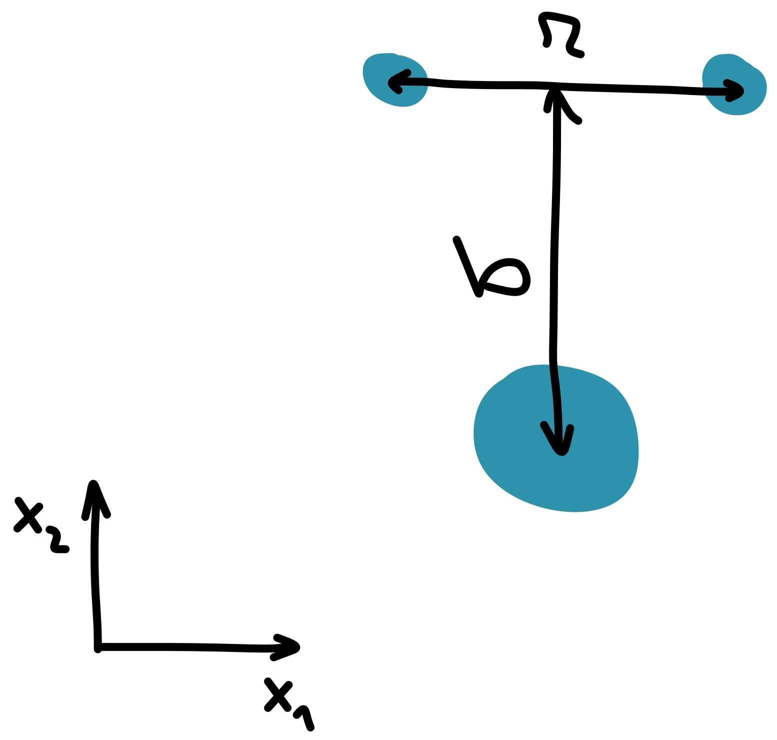


numerical solutions - 1D



numerical solutions - 2D

$$2\int \mathrm{d}\underline{b}N(\underline{r},\underline{b},Y) \approx 4\pi\int \mathrm{d}bN(r,b,Y)$$



numerical solutions - 2D

detour: *Y* vs *η*

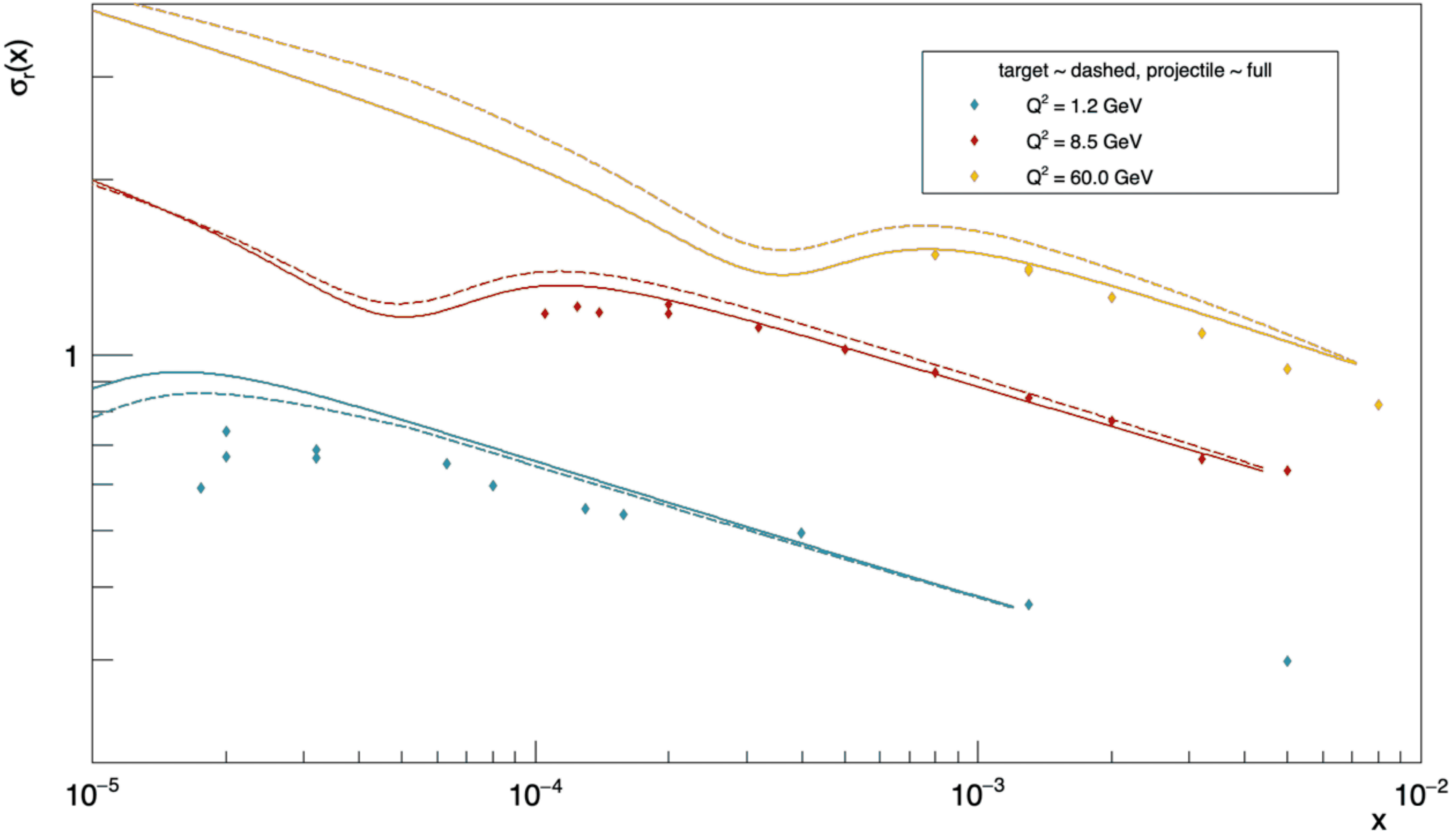
$$Y = \eta + \ln \frac{Q^2}{Q_s^2}$$

$$\frac{\partial N(\underline{r}, \underline{b}, Y)}{\partial Y} = \int \mathrm{d}^2 \underline{r}_1 K(\underline{r}, \underline{r}_1, \underline{r}_2) \left[N(\underline{r}_1, \underline{b}_1, Y) + N(\underline{r}_2, \underline{b}_2, Y) - N(\underline{r}, b, Y) - N(\underline{r}_1, \underline{b}_1, Y) N(\underline{r}_2, \underline{b}_2, Y) \right]$$

$$\frac{\partial N(\underline{r}, \underline{b}, \eta)}{\partial \eta} = \int \mathrm{d}^2 \underline{r}_1 K(\underline{r}, \underline{r}_1, \underline{r}_2) \left[N(\underline{r}_1, \underline{b}_1, \underline{\eta_1}) + N(\underline{r}_2, \underline{b}_2, \underline{\eta_2}) - N(\underline{r}, b, \eta) - N(\underline{r}_1, \underline{b}_1, \underline{\eta_1}) N(\underline{r}_2, \underline{b}_2, \underline{\eta_2}) \right]$$

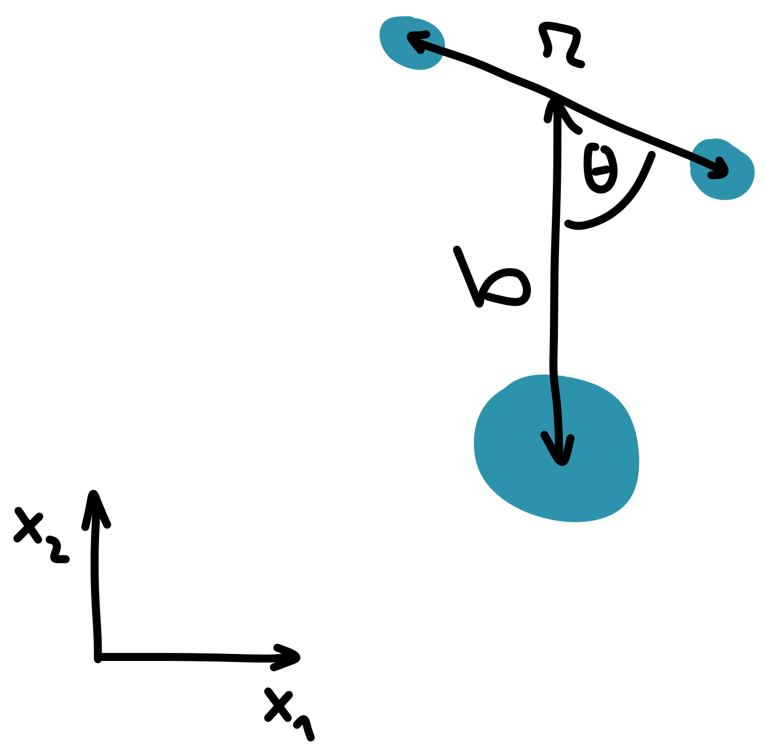
$$\eta_j = \eta - \max\left\{0, 2 \ln \frac{r}{r_j}\right\}$$

numerical solutions - 2D

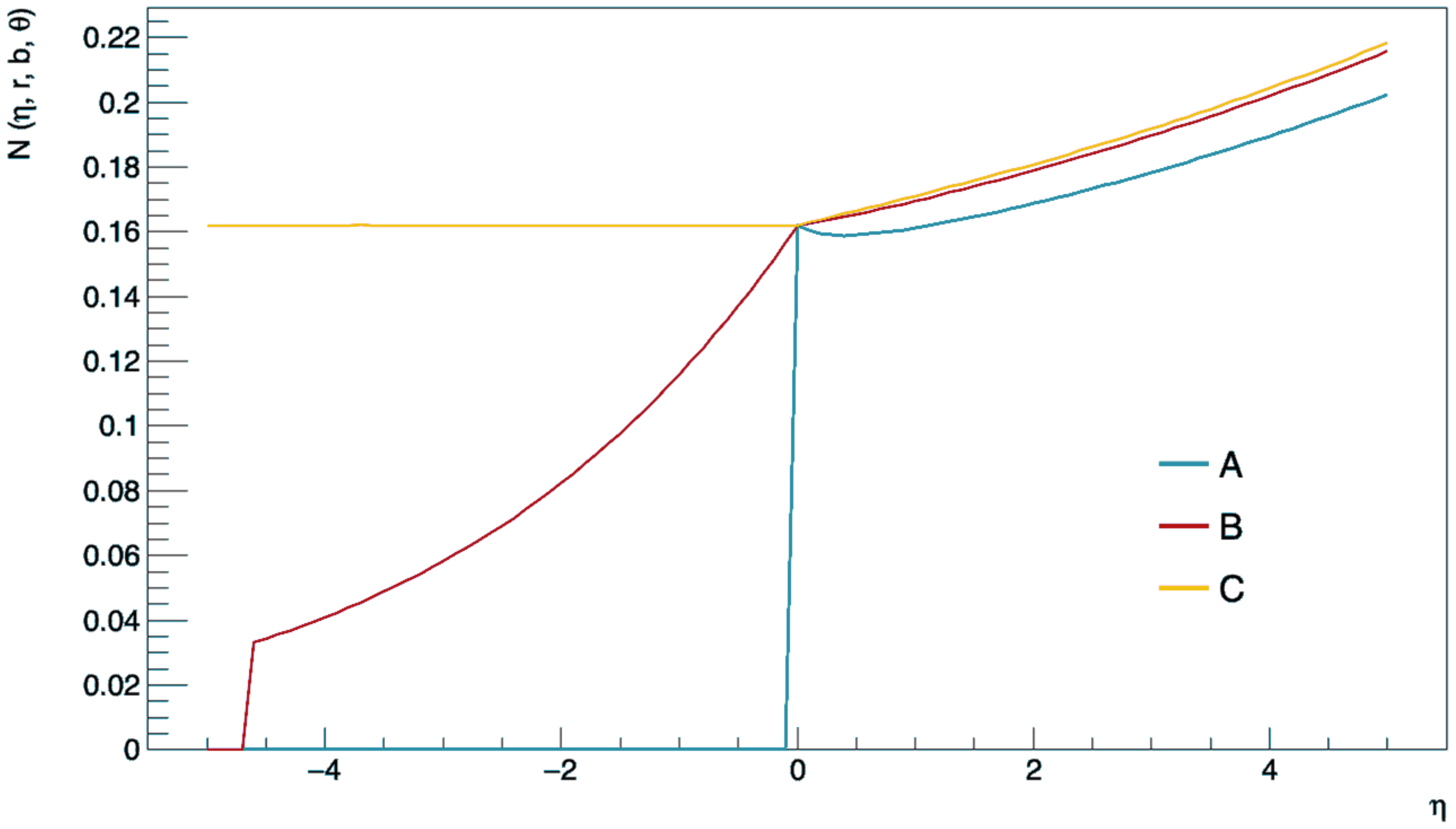


numerical solutions - 3D

$$2\int \mathrm{d}\underline{b}N(\underline{r},\underline{b},\eta) \approx 2\int \mathrm{d}bN(r,b,\theta,\eta)$$



numerical solutions - 3D



further steps

- 4D solution
- observables sensitive to 3D and 4D
    - dijet production
- analytical prescription
- nuclear target calculations
- higher order corrections

thank you

backup

$$N(r,b,\theta,x) = 1 - e^{-(\frac{x_0}{x})^{\lambda} Q_{s0}^2 \frac{r^2}{4} T(b,r)\left(1 + c\cos(2\theta)\right)}$$

$$N(r,b,\theta) = \left(1 - e^{-Q_{s0}^2 \frac{r^2}{4}}\right) T(b,r)\left(1 + (1 - e^{-\left(\frac{rb}{2B}\right)^2})\cos(2\theta)\right)$$

$$N(r,b,\theta) = 1 - e^{-Q_{s0}^2 \frac{r^2}{4} T(b,r)\left(1 + \frac{c}{2}\cos(2\theta)\right)}$$

$$N(r,b,\theta) = 1 - e^{-Q_{s0}^2 \frac{r^2}{4}\left[\ln\left(\frac{1}{r^2m^2} + e\right) + \frac{b^2}{6m^2R^4}\cos(2\theta)\right]T(b,r)}$$

$$N(r,b,\theta) = 1 - e^{-Q_{s0}^2 \frac{r^2}{4} T(b,r)\left(1 + c\cos(2\theta)\right)} \qquad T(r,b) = e^{-\frac{b^2 + (r/2)^2}{2B}}$$