

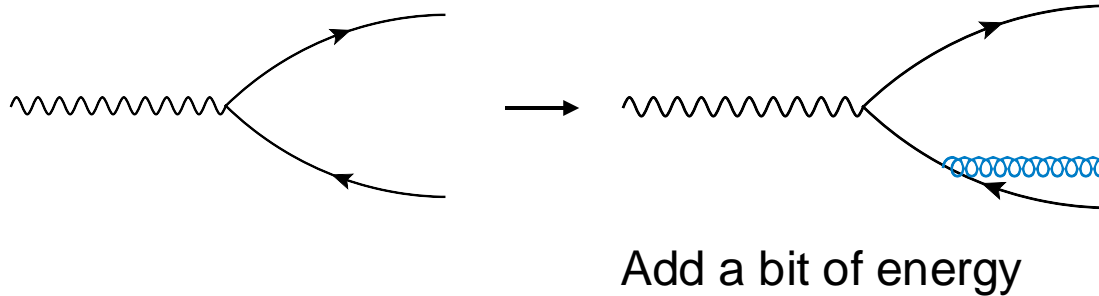
# NLO BK

Marek Matas  
Miniworkshop difrakce a ultraperiferních srážek  
ČVUT Děčín

# THE BK EQUATION

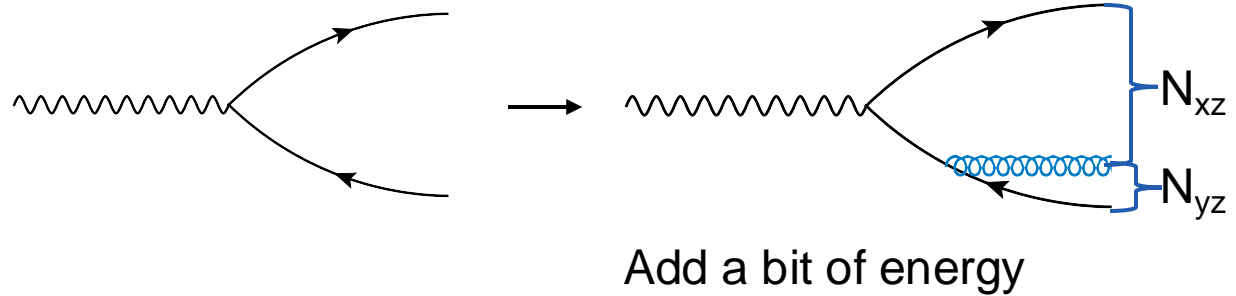
# NLO BK IN PICTURES

$$\frac{\partial N(r_{xy}, Y)}{\partial Y} = \int d\vec{r}_{xz} K(r_{xy}, r_{xz}, r_{zy}) [N(r_{xz}, Y) + N(r_{zy}, Y) - N(r_{xy}, Y) - N(r_{xz}, Y)N(r_{zy}, Y)]$$



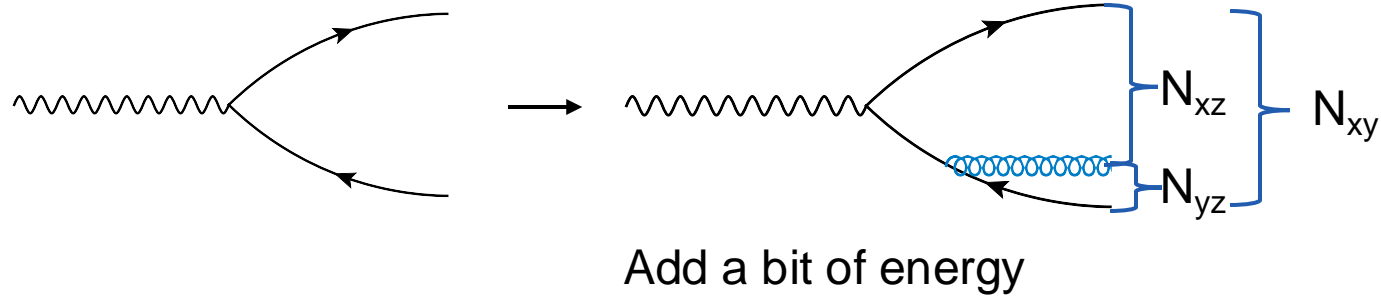
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$$\frac{\partial N(r_{xy}, Y)}{\partial Y} = \int d\vec{r}_{xz} K(r_{xy}, r_{xz}, r_{zy}) [N(r_{xz}, Y) + N(r_{zy}, Y) - N(r_{xy}, Y) - N(r_{xz}, Y)N(r_{zy}, Y)]$$



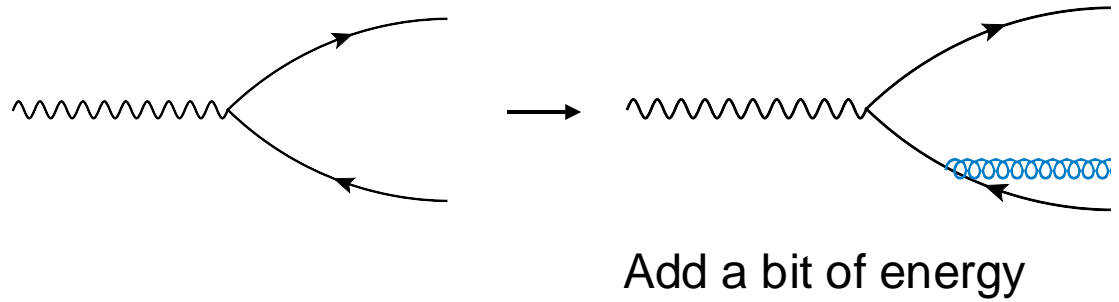
# NLO BK IN PICTURES

$$\frac{\partial N(r_{xy}, Y)}{\partial Y} = \int d\vec{r}_{xz} K(r_{xy}, r_{xz}, r_{zy}) [N(r_{xz}, Y) + N(r_{zy}, Y) - N(r_{xy}, Y) - N(r_{xz}, Y)N(r_{zy}, Y)]$$

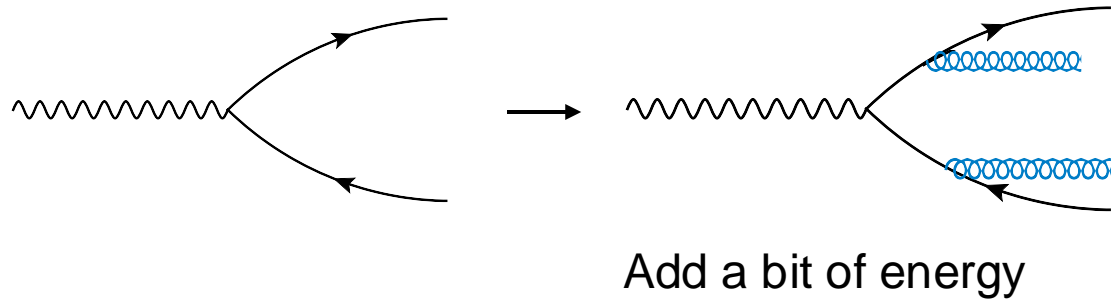


# NLO

# NLO BK IN PICTURES

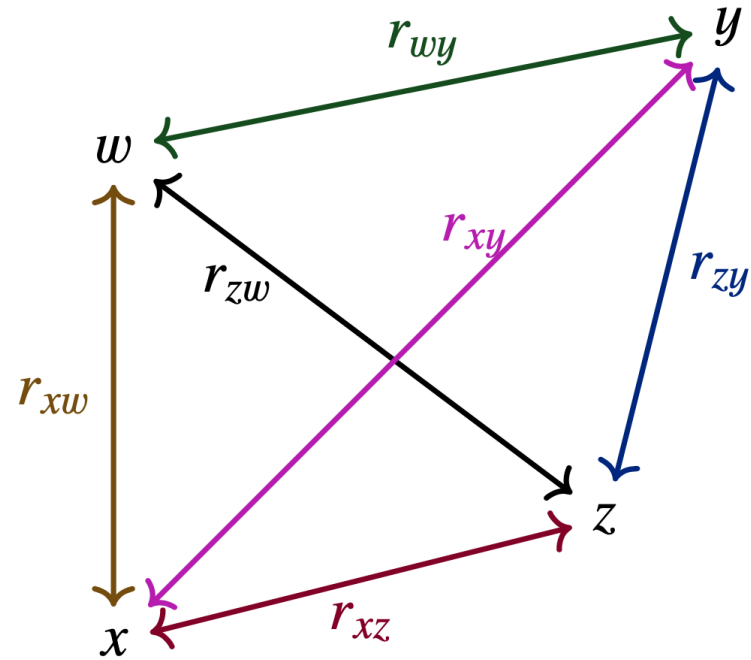
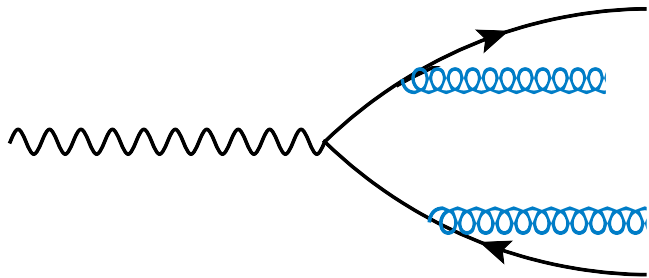


# NLO BK IN PICTURES



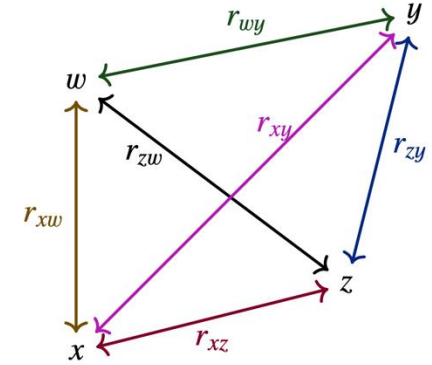


# NLO BK IN PICTURES



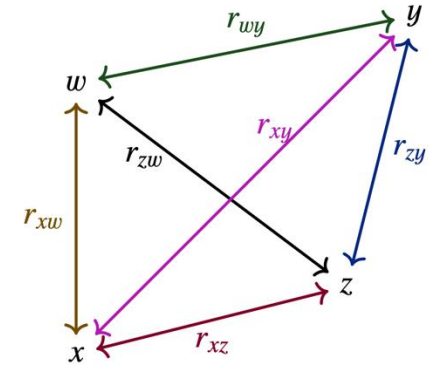
# THE MATH

# NLO BK IN WRITING



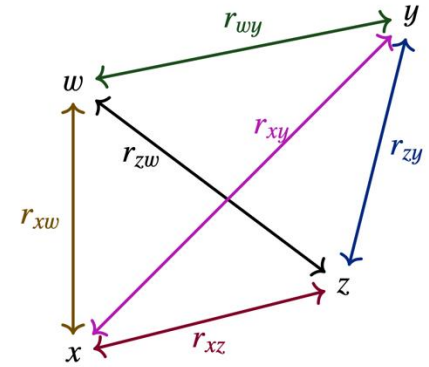
$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
 &+ \int d^2z d^2w K_b \left[ N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \right. \\
 &\quad \left. - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \right] \\
 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

# NLO BK IN WRITING



$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
 &+ \int d^2z d^2v K_b \left[ N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \right. \\
 &\quad \left. - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \right] \\
 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

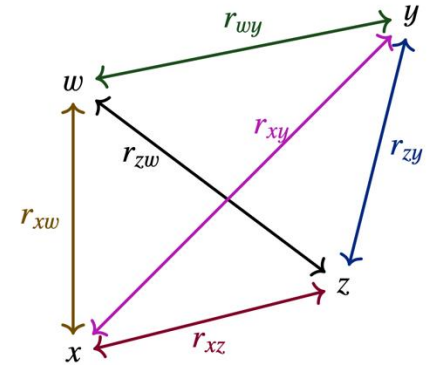
# NLO BK IN WRITING



$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
 &+ \int d^2z d^2w K_b \left[ N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \right. \\
 &\quad \left. - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \right] \\
 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

$$K_f = \frac{\alpha_S^2 n_f N_C^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$

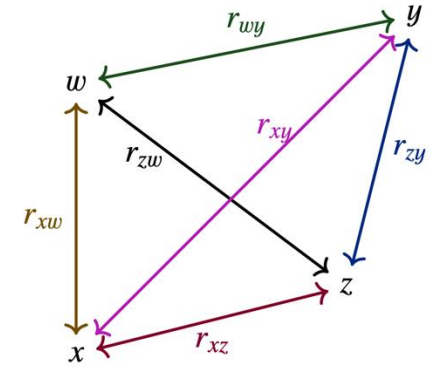
# NLO BK IN WRITING



$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
 &+ \int d^2z d^2v K_b \left[ N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \right. \\
 &\quad \left. - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \right] \\
 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

$$K_b = \frac{\alpha_S^2 N_c^2}{8\pi^4} \left( -\frac{2}{r_{zw}^4} + \left[ \frac{r_{xz}^2 r_{wy}^2 + r_{xw}^2 r_{zy}^2 - 4r^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} + \frac{r_{xy}^4}{r_{xz}^2 r_{wy}^2 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} + \frac{r_{xy}^2}{r_{xz}^2 r_{wy}^2 r_{zw}^2} \right] \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$

# NLO BK IN WRITING

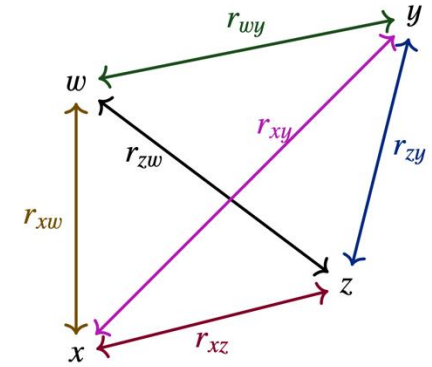


$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
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 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{rc} K_{STL} K_{DLA} - K_{sub} + K_{fin}$$

# NLO BK IN WRITING

$$K_{\text{rc}}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi} \left[ \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} + \frac{1}{r_{xz}^2} \left( \frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1 \right) + \frac{1}{r_{zy}^2} \left( \frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1 \right) \right]$$



$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{\text{rc}} K_{\text{STL}} K_{\text{DLA}} - K_{\text{sub}} + K_{\text{fin}}$$



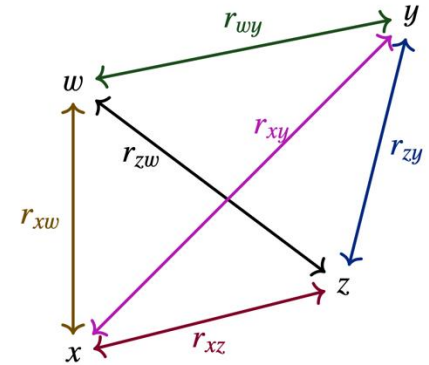


# NLO BK IN WRITING

$$K_{rc}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi} \left[ \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} + \frac{1}{r_{xz}^2} \left( \frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1 \right) + \frac{1}{r_{zy}^2} \left( \frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1 \right) \right]$$

$$K_{STL} = \exp \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{sub} r_{xy}}{\min \{r_{xz}, r_{zy}\}} \right) \right| \right]$$

$$K_{DLA}(\rho) = \frac{J_1(2\sqrt{\bar{\alpha}_S \rho^2})}{\sqrt{\bar{\alpha}_S \rho^2}} \quad \rho = \sqrt{L_{r_{xz}r_{xy}} L_{r_{zy}r_{xy}}}; \quad L_{r_i r_{xy}} = \ln \left( \frac{r_i^2}{r_{xy}^2} \right)$$



$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{rc} K_{STL} K_{DLA} - K_{sub} + K_{fin}$$

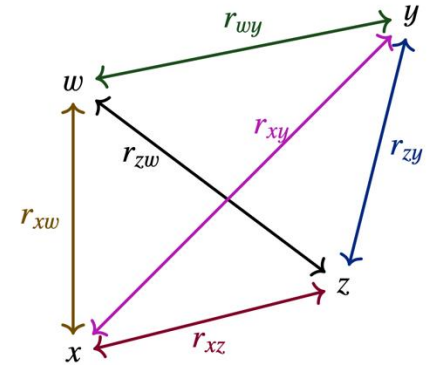
# NLO BK IN WRITING

$$K_{rc}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi} \left[ \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} + \frac{1}{r_{xz}^2} \left( \frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1 \right) + \frac{1}{r_{zy}^2} \left( \frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1 \right) \right]$$

$$K_{STL} = \exp \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{sub} r_{xy}}{\min \{r_{xz}, r_{zy}\}} \right) \right| \right]$$

$$K_{DLA}(\rho) = \frac{J_1(2\sqrt{\bar{\alpha}_S \rho^2})}{\sqrt{\bar{\alpha}_S \rho^2}} \quad \rho = \sqrt{L_{r_{xz}r_{xy}} L_{r_{zy}r_{xy}}}; \quad L_{r_i r_{xy}} = \ln \left( \frac{r_i^2}{r_{xy}^2} \right)$$

$$K_{sub} = \frac{\bar{\alpha}_S}{2\pi} \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{sub} r_{xy}}{\min \{r_{xz}^2, r_{zy}^2\}} \right) \right| \right] \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2}$$



$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{rc} K_{STL} K_{DLA} - K_{sub} + K_{fin}$$

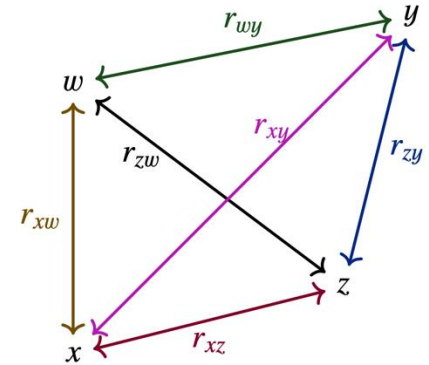
# NLO BK IN WRITING

$$K_{\text{rc}}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi} \left[ \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} + \frac{1}{r_{xz}^2} \left( \frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1 \right) + \frac{1}{r_{zy}^2} \left( \frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1 \right) \right]$$

$$K_{\text{STL}} = \exp \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{\text{sub}} r_{xy}}{\min \{r_{xz}, r_{zy}\}} \right) \right| \right]$$

$$K_{\text{DLA}}(\rho) = \frac{J_1(2\sqrt{\bar{\alpha}_S \rho^2})}{\sqrt{\bar{\alpha}_S \rho^2}} \quad \rho = \sqrt{L_{r_{xz}r_{xy}} L_{r_{zy}r_{xy}}}; \quad L_{r_i r_{xy}} = \ln \left( \frac{r_i^2}{r_{xy}^2} \right)$$

$$K_{\text{sub}} = \frac{\bar{\alpha}_S}{2\pi} \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{\text{sub}} r_{xy}}{\min \{r_{xz}^2, r_{zy}^2\}} \right) \right| \right] \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \quad K_{\text{fin}} = \frac{\bar{\alpha}_S^2}{8\pi} \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \left[ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{n_f}{N_c} \right]$$



$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{\text{rc}} K_{\text{STL}} K_{\text{DLA}} - K_{\text{sub}} + K_{\text{fin}}$$

# NLO BK IN WRITING

$$\begin{aligned} \partial_Y N(r_{xy}) &= \int d^2z K_a [N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy})] \\ &+ \int d^2z d^2w K_b [N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \\ &\quad - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy})] \\ &+ \int d^2z d^2w K_f [N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy})]. \end{aligned}$$

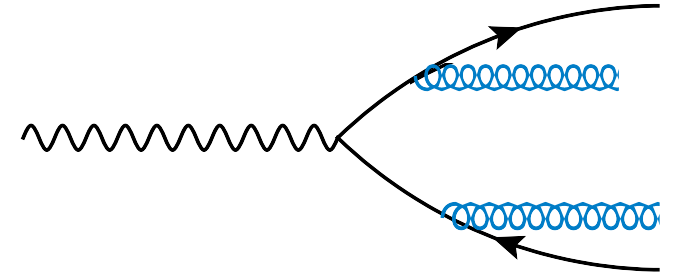
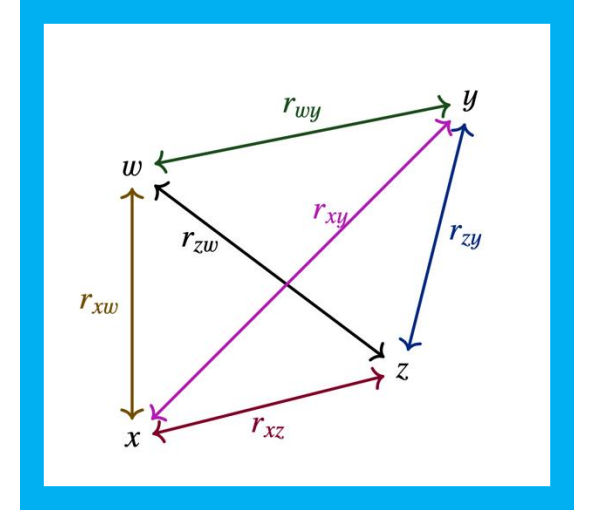
$$K_{\text{STL}} = \exp \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{\text{sub}} r_{xy}}{\min\{r_{xz}, r_{zy}\}} \right) \right| \right] \quad K_{\text{DLA}}(\rho) = \frac{J_1(2\sqrt{\bar{\alpha}_S \rho^2})}{\sqrt{\bar{\alpha}_S \rho^2}} \quad \rho = \sqrt{L_{r_{xz}r_{xy}} L_{r_{zy}r_{xy}}}; \quad L_{r_i r_{xy}} = \ln \left( \frac{r_i^2}{r_{xy}^2} \right)$$

$$K_{\text{rc}}(r_{xy}, r_{xz}, r_{zy}) = \frac{\bar{\alpha}_S(r_{xy})}{2\pi} \left[ \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} + \frac{1}{r_{xz}^2} \left( \frac{\alpha_S(r_{xz})}{\alpha_S(r_{zy})} - 1 \right) + \frac{1}{r_{zy}^2} \left( \frac{\alpha_S(r_{zy})}{\alpha_S(r_{xz})} - 1 \right) \right] \quad K_{\text{fin}} = \frac{\bar{\alpha}_S^2}{8\pi} \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \left[ \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9} \frac{n_f}{N_c} \right]$$

$$K_{\text{sub}} = \frac{\bar{\alpha}_S}{2\pi} \left[ -\bar{\alpha}_S A_1 \left| \ln \left( \frac{C_{\text{sub}} r_{xy}}{\min\{r_{xz}^2, r_{zy}^2\}} \right) \right| \right] \frac{r_{xy}^2}{r_{xz}^2 r_{zy}^2} \quad K_f = \frac{\alpha_S^2 n_f N_c^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$

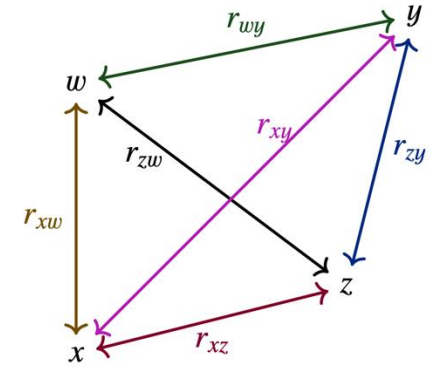
$$K_b = \frac{\alpha_S^2 N_c^2}{8\pi^4} \left( -\frac{2}{r_{zw}^4} + \left[ \frac{r_{xz}^2 r_{wy}^2 + r_{xw}^2 r_{zy}^2 - 4r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} + \frac{r_{xy}^4}{r_{xz}^2 r_{wy}^2 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} + \frac{r_{xy}^2}{r_{xz}^2 r_{wy}^2 r_{zw}^2} \right] \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$

$$K_1(r_{xy}, r_{xz}, r_{zy}) = K_{\text{rc}} K_{\text{STL}} K_{\text{DLA}} - K_{\text{sub}} + K_{\text{fin}}$$



# THE PROBLEMS

# NLO BK IN WRITING

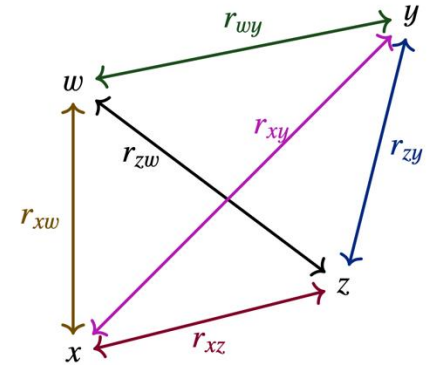


$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a \left[ N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy}) \right] \\
 &+ \int d^2z d^2w K_b \left[ N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \right. \\
 &\quad \left. - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy}) \right] \\
 &+ \int d^2z d^2w K_f \left[ N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy}) \right].
 \end{aligned}$$

# NLO BK IN WRITING

This takes ages to calculate

$$\begin{aligned}
 \partial_Y N(r_{xy}) &= \int d^2z K_a [N(r_{xz}) + N(r_{zy}) - N(r_{xy}) - N(r_{xz})N(r_{zy})] \\
 &+ \int d^2z d^2w K_b [N(r_{wy}) + N(r_{zw}) - N(r_{zy}) - N(r_{xz})N(r_{zw}) - N(r_{xz})N(r_{wy}) - \\
 &\quad - N(r_{zw})N(r_{wy}) + N(r_{xz})N(r_{zy}) + N(r_{xz})N(r_{zw})N(r_{wy})] \\
 &+ \int d^2z d^2w K_f [N(r_{xw}) - N(r_{xz}) - N(r_{zy})N(r_{xw}) + N(r_{xz})N(r_{zy})].
 \end{aligned}$$

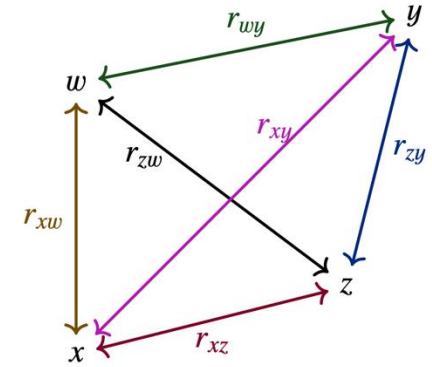


nsteps in r \* nsteps in angle ~ 4000x slower



# NLO BK IN WRITING

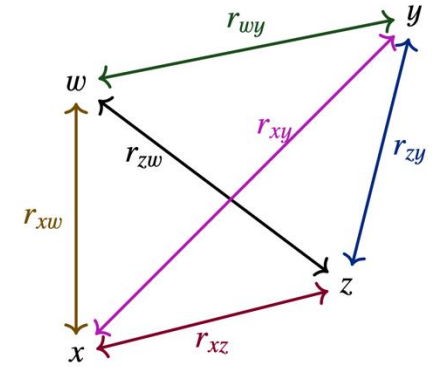
$$K_f = \frac{\alpha_S^2 n_f N_C^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$



This is OK!

# NLO BK IN WRITING

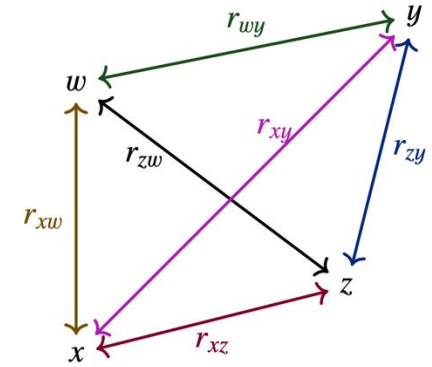
$$K_f = \frac{\alpha_S^2 n_f N_C^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} \right) \mathbb{1} \left[ \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right]$$



This is not

# NLO BK IN WRITING

$$K_f = \frac{\alpha_S^2 n_f N_C^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} \ln \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$



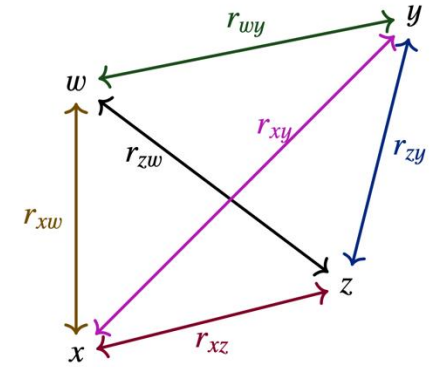
This also not

# NLO BK IN WRITING

$$K_f = \frac{\alpha_S^2 n_f N_C^2}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zw}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} - 1 + \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2} \right)$$

$$r_{xw}^2 r_{zy}^2 = r_{xz}^2 r_{wy}^2$$

$$r_{xw}^2 r_{zy}^2$$



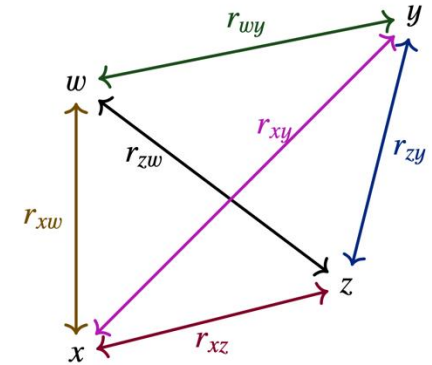
# NLO BK IN WRITING

This does not work 😊

$$\frac{J^2 V_C}{8\pi^4} \left( \frac{2}{r_{zw}^4} - \frac{r_{xw}^2 r_{zy}^2 + r_{wy}^2 r_{xz}^2 - r_{xy}^2 r_{zw}^2}{r_{zy}^4 (r_{xz}^2 r_{wy}^2 - r_{xw}^2 r_{zy}^2)} - 1 \right) \frac{r_{xz}^2 r_{wy}^2}{r_{xw}^2 r_{zy}^2}$$

$$r_{xw}^2 r_{zy}^2 = r_{xz}^2 r_{wy}^2$$

$$r_{xw}^2 r_{zy}^2$$



So we cut out the problematic part of phase space and check for stability

# THE RESULTS

# RESULTS



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# RESULTS



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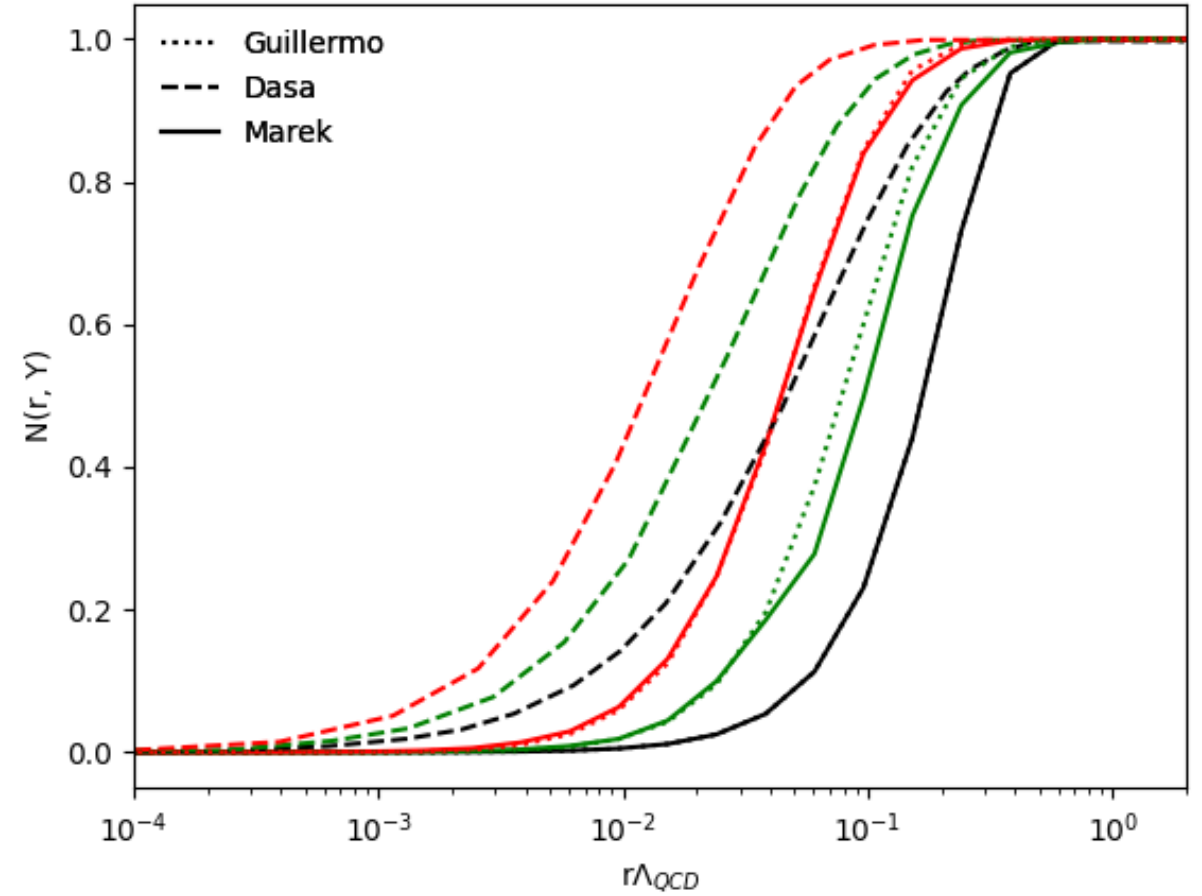
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**STAY TUNED FOR  
OBSERVABLES**

**!**



*That's all Folks!*