LC wave function for nS vector meson states

Miniworkshop on Diffraction and Ultraperipheral Collisions

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1 The scattering amplitude for vector meson production

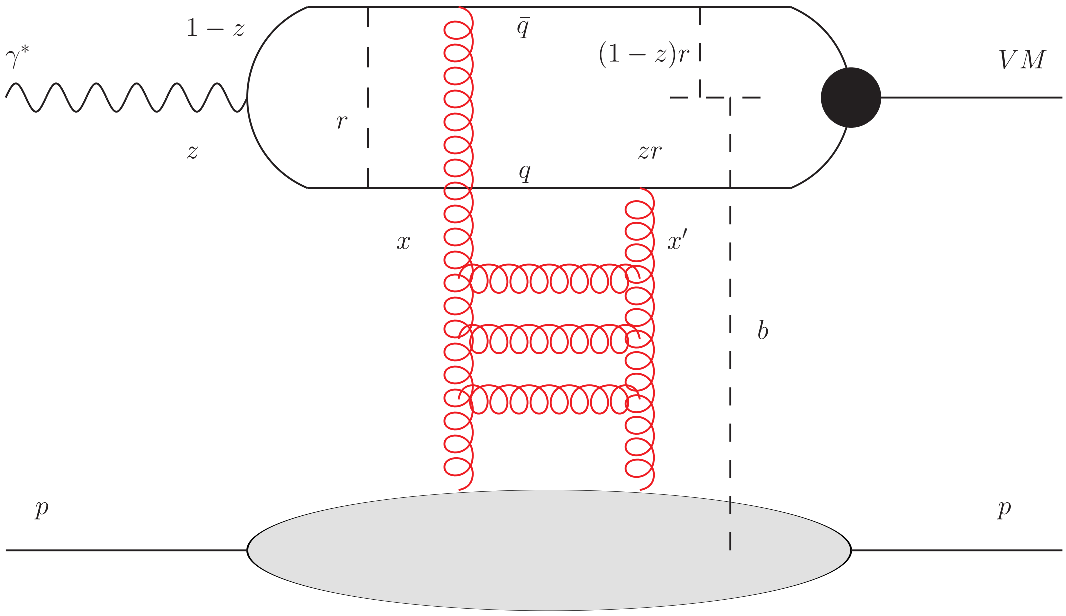

Figure 1: Interaction scheme of the vector meson production

The amplitude for production of a vector meson V is given by [1]

$$\mathcal{A}_{T,L}^{\gamma^* p\to Vp}(x_{Bj},Q,\Delta) = i\int \mathrm{d}^2r\int_0^1\frac{\mathrm{d}z}{4\pi}\int \mathrm{d}^2b\,\Psi_M^*\Psi_{\gamma^*}\Big|_{T,L} e^{-i(\vec{b}-(1-z)\vec{r})\Delta}\frac{\mathrm{d}\sigma_{q\bar{q}}}{\mathrm{d}^2b}, \tag{1}$$

where $\Psi_M^*\Psi_{\gamma^*}|_{T,L}$ is an overlap of a virtual photon and vector meson wave function, $\Delta = \sqrt{-t} \sim p_T^V$ denotes the transverse momentum lost by the outgoing proton, $\vec{r}$ is the transverse dipole size, $\vec{b}$ is the impact parameter of the dipole(transverse distance from the center of the proton to the center of mass of the dipole) and z is a part of photon momenta carried by one of the quarks from the dipole,M_V is the mass of the vector meson, Q is the scale of the incoming photon, W is the energy of a photon and a hadron and Bjorken-x of produced meson is

$$x_{Bj} = \frac{Q^2+M_V^2}{W^2+Q^2}. \tag{2}$$

The elastic diffractive differential cross-section can be written as

$$\frac{\mathrm{d}\sigma_{T,L}^{\gamma^* p\to Vp}}{\mathrm{d}|t|} = \frac{1}{16\pi}\left|\mathcal{A}_{T,L}^{\gamma^* p\to Vp}\right|^2 \tag{3}$$

2 The wave function Gaussian models

The overlap of a virtual photon and vector meson wave function is defined as

$$\Psi_V^*\Psi_{\gamma^*}\Big|_T = \frac{1}{2}\left(\sum_{h,\bar{h}} \Psi^*_{h\bar{h}\lambda=+1}\Big|_V \Psi_{h\bar{h}\lambda=+1}\Big|_{\gamma^*} + \Psi^*_{h\bar{h}\lambda=-1}\Big|_V \Psi_{h\bar{h}\lambda=-1}\Big|_{\gamma^*}\right)$$
$$\Psi_V^*\Psi_{\gamma^*}\Big|_L = \sum_{h,\bar{h}} \Psi^*_{h\bar{h}\lambda=0}\Big|_V \Psi_{h\bar{h}\lambda=0}\Big|_{\gamma^*}. \tag{4}$$

where h and $\bar{h}$ denote helicities of quarks and antiquark and λ is a polarization of incoming photon.

The photon wave function was calculated in perturbative QED[2, 3] from a $\gamma \to f\bar{f}$ vertex.

$$\Psi_{h\bar{h}\lambda=0}\Big|_{\gamma^*}(r,z,Q) = e_f\delta_{f\bar{f}}e\sqrt{N_c}\delta_{h,-\bar{h}}2Qz(1-z)\frac{K_0(\epsilon r)}{2\pi} \tag{5}$$
$$\Psi_{h\bar{h}\lambda=\pm1}\Big|_{\gamma^*}(r,z,Q) = \pm e_f\delta_{f\bar{f}}e\sqrt{2N_c}\left(ie^{\pm i\theta_r}\left(z\delta_{h,\pm1}\delta_{\bar{h},\mp1} - (1-z)\delta_{h,\mp1}\delta_{\bar{h},\pm1}\right)\partial_r + m_f\delta_{h,\pm1}\delta_{\bar{h},\pm1}\right)\frac{K_0(\epsilon r)}{2\pi},$$

where $e = \sqrt{4\pi\alpha_{em}}$, h, θ_r is the azimuthal angle between the vector $\vec{r}$ and the x-axis in the transverse plane, $\epsilon^2 = z(1-z)Q^2 + m_f^2$, $N_c = 3$ is the number of colors, $e_f\delta_{f\bar{f}}$ and m_f are the fractional charge and effective mass of the quark respectively. The partial derivative of the modified Bessel function K_0 with respect to r can be done using the equation $\partial_r K_0(\epsilon r) = -\epsilon K_1(\epsilon r)$.

The vector meson wave function is modelled with the presumption that vector meson is predominantly a quark-antiquark state and the spin and polarization structure is the same as in the photon case [1]

$$\Psi_{h\bar{h}\lambda=0}\Big|_V(r,z,Q) = \sqrt{N_c}\delta_{h,-\bar{h}}\left(M_V + \delta\frac{m_f^2 - \nabla_r^2}{M_V z(1-z)}\right)\Phi_L(r,z) \tag{6}$$
$$\Psi_{h\bar{h}\lambda=\pm1}\Big|_V(r,z,Q) = \pm\sqrt{2N_c}\frac{1}{z(1-z)}\left(ie^{\pm i\theta_r}\left(z\delta_{h,\pm1}\delta_{\bar{h},\mp1} - (1-z)\delta_{h,\mp1}\delta_{\bar{h},\pm1}\right)\partial_r + m_f\delta_{h,\pm1}\delta_{\bar{h},\pm1}\right)\Phi_T(r,z),$$

where $\nabla_r^2 = \frac{1}{r}\partial_r + \partial_r^2$ and δ is a switch enables to include the non-local part of the wave function introduced in [4, 5].

The overlap between photon and vector meson wave function is

$$\Psi_V^*\Psi_{\gamma^*}\Big|_T = e_f\delta_{f\bar{f}}e\frac{N_c}{\pi z(1-z)}\left(m_f^2K_0(\epsilon r)\Phi_T(r,z) - (z^2+(1-z)^2)\epsilon K_1(\epsilon r)\partial_r\Phi_T(r,z)\right)$$
$$\Psi_V^*\Psi_{\gamma^*}\Big|_L = e_f\delta_{f\bar{f}}e\frac{N_c}{\pi}2Qz(1-z)K_0(\epsilon r)\left(M_V\Phi_L(r,z) + \delta\frac{m_f^2-\nabla_r^2}{M_V z(1-z)}\Phi_L(r,z)\right), \tag{7}$$

where $e_f\delta_{f\bar{f}}$ is an effective charge that corresponds to the choice of the vector meson. It can be calculated for arbitrary meson from it's quark wave function by substituting charge for each quark-antiquark pair, e.g.

$$\rho = \frac{u\bar{u} - d\bar{d}}{\sqrt{2}} \Rightarrow e_\rho\delta_{f\bar{f}} = \frac{\frac{2}{3} - (-\frac{1}{3})}{\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad J/\Psi = c\bar{c} \Rightarrow e_{J/\Psi}\delta_{f\bar{f}} = \frac{2}{3}. \tag{8}$$

The scalar part $\Phi_{T,L}$ of the vector meson wave function is model dependent. In the photon case the scalar part is given by modified Bessel functions (kind of a point object), whereas for the vector mesons the hadron at rest is more complicated object (kind of a sphere).

2.1 Gaussian models

These types of parametrization assume that transverse part of the vector meson in the rest frame can be approximated by a Gaussian in the transverse plane. It needs to specify two parameters - a width of a Gaussian R and a normalization of the scalar part N.

Normalization condition for the vector meson wave function comes from the fact, that the vector meson is composed solely of the quark-antiquark pair from the dipole. Therefore, no contribution from gluons and sea quarks is considered and

$$1 = \sum_{h,\bar{h}} \int \mathrm{d}^2 r \int_0^1 \frac{\mathrm{d}z}{4\pi} |\Psi^V_{h\bar{h}\lambda}(r, z, Q)|^2 \tag{9}$$

This can be transformed into the normalization of the scalar parts [1]

$$\begin{aligned} 1 &= \frac{N_c}{2\pi} \int_0^1 \frac{\mathrm{d}z}{z^2(1-z)^2} \int \mathrm{d}^2 r \left(m_f^2 \Phi_T^2(r, z) + (z^2 + (1-z)^2)(\partial_r \Phi_T(r, z))^2 \right) \\ 1 &= \frac{N_c}{2\pi} \int_0^1 \mathrm{d}z \int \mathrm{d}^2 r \left(M_V \Phi_L(r, z) + \delta \frac{m_f^2 - \nabla_r^2}{M_V z(1-z)} \Phi_L(r, z) \right)^2 . \end{aligned} \tag{10}$$

Another constraint on the vector meson wave functions is obtained from the decay width. It is assumed that the perturbative decay width of $\bar{q}q \to \gamma^* \to l^+ l^-$ is factorized from the vector meson wave function. The electromagnetic current is then

$$\begin{aligned} f_{V,T} &= e_f \frac{N_c}{2\pi M_V} \int_0^1 \frac{\mathrm{d}z}{z^2(1-z)^2} \left(m_f^2 - (z^2 + (1-z)^2) \nabla_r^2 \right) \Phi_T(r=0, z) \\ f_{V,L} &= e_f \frac{N_c}{\pi} \int_0^1 \mathrm{d}z \left(M_V + \delta \frac{m_f^2 - \nabla_r^2}{M_V z(1-z)} \right) \Phi_L(r=0, z). \end{aligned} \tag{11}$$

The coupling of the electromagnetic current to the vector meson is obtained from the measured electronic decay width as

$$\Gamma_{M \to e^+ e^-} = \frac{4\pi \alpha_{em}^2 f_V^2}{3 M_V}, \tag{12}$$

where the value of f_V comes from the measurement. It is presumed that $f_V = f_{V,L} = f_{V,T}$, however, some models predict different values of $f_{V,T}$ and $f_{V,L}$. In this case, the longitudinal part is preferably set to be equal to measured value [1].

For excited states, additional parameters mean that we need additional normalization conditions. It is, therefore, anticipated that 2S wave function is orthogonal to 1S one and 3S wave function is orthogonal to both 1S and 2S ones

$$\begin{aligned} 0 &= \frac{N_c}{2\pi} \int_0^1 \frac{\mathrm{d}z}{z^2(1-z)^2} \int \mathrm{d}^2 r \left(m_f^2 \Phi_T^{(n-1)S}(r, z) \Phi_T^{nS}(r, z) + (z^2 + (1-z)^2) \partial_r \Phi_T^{(n-1)S}(r, z) \partial_r \Phi_T^{nS}(r, z) \right) \\ 0 &= \frac{N_c}{2\pi} \int_0^1 \frac{\mathrm{d}z}{z^2(1-z)^2} \int \mathrm{d}^2 r \left(m_f^2 \Phi_T^{(n-2)S}(r, z) \Phi_T^{nS}(r, z) + (z^2 + (1-z)^2) \partial_r \Phi_T^{(n-2)S}(r, z) \partial_r \Phi_T^{nS}(r, z) \right). \end{aligned} \tag{13}$$

2.1.1 Boosted Gaussian model

The boosted Gaussian model [4, 5, 6] assumes that the boost from the rest frame to the LC frame can be done by changing the three-momentum $\vec{p}$ in the rest frame of the meson to a boost-invariant form $p^2 = (k^2 + m_f^2)/(4z(1-z)) - m_f^2$. Therefore, supplying a scalar spatial part in the rest frame allows to boost it to proper frame. This model has proper short distance limit and assumes $\delta = 1$. If we use the Gaussian form of the rest frame meson wave function (quadratic potential in Schrodinger equation) we get

$$\Phi^{1S}_{T,L}(r,z) = N_{T,L} z(1-z) e^{-\frac{m_f^2 R^2}{8z(1-z)} - \frac{2z(1-z)r^2}{R^2} + \frac{m_f^2 R^2}{2}} \tag{14}$$

Note, that the original paper on boosted Gaussian model [4] included also Coulomb-like part of the scalar wave function. However, it has not since been used in any other paper. It may influence only kinematical regions with contribution from large dipoles, since it enhances the tail in r.

Meson	$\hat{e}_f$	M_V[GeV]	m_f[GeV]	N_T	N_L	R^2[GeV^{-2}]
J/Ψ	2/3	3.09692	1.4	0.582	0.578	2.24
ϕ	1/3	1.019461	0.14	0.918	0.823	11.3
ρ	$1/\sqrt{2}$	0.77526	0.14	0.909	0.853	12.95
ω	$1/3\sqrt{2}$	0.78265	0.14	0.885	0.835	16.6
$\Upsilon(1S)$	1/3	9.4603	4.2	0.478	0.478	0.585

Table 1: Table of boosted Gaussian scalar part parameters from [1, 7]

The formula for nS state is [8]

$$\Phi^{nS}_{T,L}(r,z) = \left[\sum_{k=0}^{n-1} \alpha_k R^{2k} \tilde{D}^{2k}\right] \Phi^{1S}_{T,L}(r,z) \qquad \tilde{D}^2 = \frac{m_f^2 - \nabla_r^2}{4z(1-z)} - m_f^2 \qquad \alpha_0 = 1 \tag{15}$$

For the first and the second excited states, one gets

$$\Phi^{2S}_{T,L}(r,z) = \Phi^{1S}_{T,L}(r,z)\,(1 + \alpha_1 g(r,z)) \qquad g(r,z) = 2 + \frac{m_f^2 R^2}{4z(1-z)} - \frac{4z(1-z)r^2}{R^2} - m_f^2 R^2 \tag{16}$$

$$\Phi^{3S}_{T,L}(r,z) = \Phi^{1S}_{T,L}(r,z)\left(1 + \alpha_1 g(r,z) + \alpha_2\left(g(r,z)^2 + 4\left(1 - \frac{4z(1-z)r^2}{R^2}\right)\right)\right) \tag{17}$$

Meson	$\hat{e}_f$	M_V[GeV]	m_f[GeV]	N_T	N_L	R^2[GeV^{-2}]	α_1[1]	α_2[1]
$\Psi(2S)$	2/3	3.686109	1.4	0.666	0.658	3.705	−0.6225	0
$\Upsilon(2S)$	1/3	10.02326	4.2	0.614	0.610	0.831	−0.568	0
$\Upsilon(3S)$	1/3	10.3552	4.2	0.668	0.668	1.028	−1.219	0.217

Table 2: Table of boosted Gaussian scalar part parameters [9, 8]

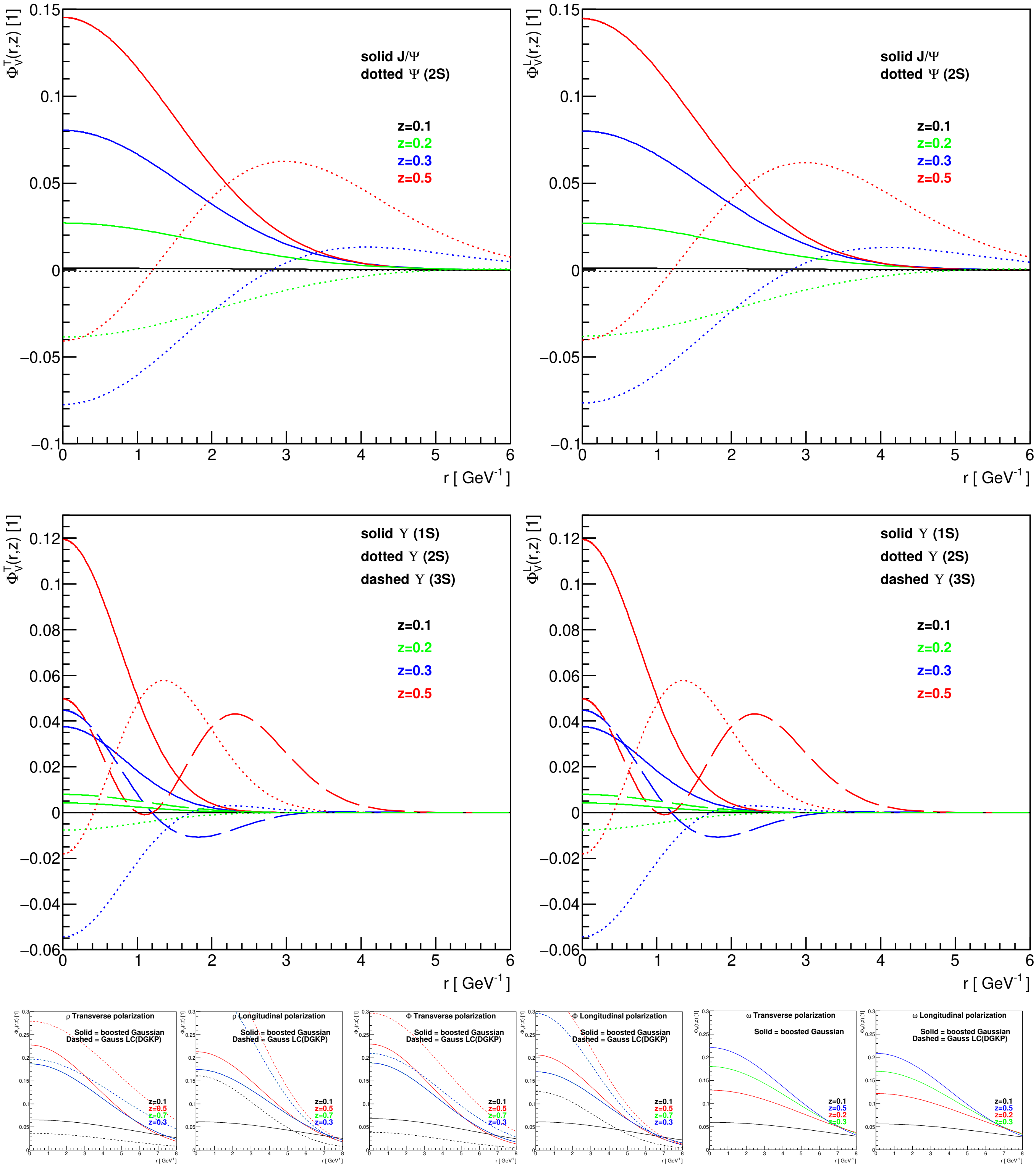

Figure 2: Scalar part of nS states (J/Ψ,Ψ(2S), $\Upsilon(nS)$, ρ,ω and Φ meson) wave function.

3 The wave function non-Gaussian models

If we do not want to guess the shape of the rest frame wave function one has to calculate it using the Schodinger equation. For that we re-formulate the overlap between the photon and the vector meson part to [10]

$$\begin{aligned}\Psi^*_V\Psi_{\gamma^*}\Big|_T &= e_f\frac{\sqrt{N_c\alpha_{em}}}{2\pi\sqrt{2}}2m_fK_0(\epsilon r)\psi(r,z)\\ \Psi^*_V\Psi_{\gamma^*}\Big|_L &= e_f\frac{\sqrt{N_c\alpha_{em}}}{2\pi\sqrt{2}}4Qz(1-z)K_0(\epsilon r)\psi(r,z),\end{aligned}\tag{18}$$

3.1 non-Gaussian model

The boosted non-Gaussian model (also called NNPZ) [4, 5, 11, 10] assumes as in Gaussian version that the boost into the LC frame is done in a boost-invariant form. But rather than guessing the rest frame form of a wave function, it supplies the solution of the Schrodinger equation for realistic potential. The spatial part of the $q\bar{q}$ wave function satisfies the Schrodinger equation (non-relativistic!) [10]

$$\left(-\frac{\Delta}{2\mu}+V(r)\right)\Psi_{nlm}(\vec{r}) = E_{nl}\Psi_{nlm}(\vec{r}) \qquad \mu = \frac{m_q m_{\bar{q}}}{m_q+m_{\bar{q}}} = \frac{m_q}{2},\tag{19}$$

where μ is the reduced mass of the $q\bar{q}$ pair and

$$\Delta = \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.\tag{20}$$

If we use the standard factorized form of the wave function to radial and angular part

$$\Psi_{nlm}(\vec{r}) = \psi_{nl}(r)\times Y_{lm}(\theta,\varphi)\tag{21}$$

the Schrodinger equation can be separated to two equations

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right)+m_q(E-V(r))r^2\psi(r) &= l(l+1)\psi(r)\\ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\varphi)}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 Y(\theta,\varphi)}{\partial\varphi^2} &= -l(l+1)Y(\theta,\varphi)\end{aligned}\tag{22}$$

with $l=0$ for 1S,2S states and $l=1$ for 1P,2P etc. If there is no spin rotation, the solution of angular equation provides constant factor, that is absorbed into the normalization. Now, we can use the formula

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi(r)}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r))\tag{23}$$

to rewrite the radial Schrodinger equation to

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi(r))+m_q(E-V(r))\psi(r) = \frac{l(l+1)}{r^2}\psi(r).\tag{24}$$

If we use a substitution $u(r)=r\psi(r)$ we have

$$\frac{\partial^2 u(r)}{\partial r^2}+m_q(E-V(r))u(r) = \frac{l(l+1)}{r^2}u(r).\tag{25}$$

This can be written in the final form

$$\frac{\partial^2 u(r)}{\partial r^2} = (V_{eff}(r) - \epsilon)u(r) \qquad V_{eff}(r) = m_q V(r) + \frac{l(l+1)}{r^2} \qquad \epsilon = m_q E. \tag{26}$$

This equation can be solved using e.g. Runge-Kutta method with initial conditions[12] $r = r_{min}, u(r_{min}) = r_{min}^{l+1}, u'(r_{min}) = (l+1)r_{min}^{l}$ or Numerov method with initial conditions $r = r_{min}, u(r_{min}) = r_{min}^{l+1}, u(r_{min+1}) = r_{min+1}^{l+1}$. The normalization of the solution is that the following relation holds

$$\int_{0}^{+\infty} |u(r)|^2 \mathrm{d}r = 1 \tag{27}$$

and so, $u(r)$ has a dimension $\mathrm{GeV}^{\frac{1}{2}}$ and $\psi_{nl}(r)$ has a dimension $\mathrm{GeV}^{\frac{3}{2}}$. Since the solution of the Schrodinger equation $u(r)$ is normalized the actual wave function $\psi(r)$ is then normalized as follows

$$\psi(r) = \frac{u(r)}{r}\,[\mathrm{GeV}^{\frac{3}{2}}] \qquad \int_{0}^{+\infty} |u(r)|^2 \mathrm{d}r = 1 \Rightarrow \int |\psi(r)|^2 \mathrm{d}^3 r = 4\pi. \tag{28}$$

In order to obtain normalized wave function, we have to re-normalize it according to previous formula with

$$\psi(r) = \frac{1}{\sqrt{4\pi}}\psi(r). \tag{29}$$

This normalized solution has to be now Fourier transformed (notation according to [13]) to momentum space using following definition of Fourier transformation

$$\psi(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{i\vec{x}\vec{y}} \psi(x) \mathrm{d}^n x \qquad \psi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\vec{x}\vec{y}} \psi(y) \mathrm{d}^n y \tag{30}$$

yielding

$$\psi(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3 r\, e^{i\vec{p}\vec{r}} \psi(r) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{0}^{+\infty} r^2 \mathrm{d}r \psi(r) \int_{0}^{2\pi} \mathrm{d}\phi \int_{0}^{\pi} \mathrm{d}\theta e^{ipr\cos\theta} \sin\theta. \tag{31}$$

We can use the integral definition of the Bessel function

$$J_{\frac{1}{2}}(t) = \frac{\sqrt{t}}{\sqrt{2\pi}} \int_{0}^{\pi} e^{it\cos\theta} \sin\theta \mathrm{d}\theta = \sqrt{\frac{2}{\pi t}} \sin(t) \tag{32}$$

to write

$$\psi(p) = \frac{2\pi}{(2\pi)^{\frac{3}{2}}} \int_{0}^{+\infty} r^2 \mathrm{d}r \psi(r) J_{\frac{1}{2}}(pr) \frac{\sqrt{2\pi}}{\sqrt{pr}} \tag{33}$$

and, consequently,

$$\psi(p) = \frac{2}{\sqrt{2\pi}p} \int_{0}^{+\infty} r \mathrm{d}r \psi(r) \sin(pr) \quad [\mathrm{GeV}^{-\frac{3}{2}}]. \tag{34}$$

The normalization is such that the integral in the Fourier transformation has to leave the normalization intact and following the fact that we started with the wave function normalized to 1 we have

$$\int |\psi(p)|^2 \mathrm{d}^3 p = 1. \tag{35}$$

Now, the wave function in momentum space has to be boosted to proper frame. We use the fact that invariant mass of the quark pair is the same in all frames and we can write it in terms of light-cone variables ([4, 10, 13]) as

$$M_{q\bar{q}} = (p_q + p_{\bar{q}})^2 = \frac{p_T^2 + m_q^2}{z(1-z)} \tag{36}$$

assuming quarks are on shell and the same can be done in the rest frame of the quark pair as

$$M_{q\bar{q}} = (p_q + p_{\bar{q}})^2 = (2E)^2 = 4(p^2 + m_q^2) \tag{37}$$

since $\vec{p}_q = -\vec{p}_{\bar{q}}$. And so, following relations hold

$$4(p^2 + m_q^2) = \frac{p_T^2 + m_q^2}{z(1-z)} \qquad p^2 = \frac{p_T^2 + (1-2z)^2 m_q^2}{4z(1-z)} \qquad p_L^2 = \frac{(p_T^2 + m_q^2)(1-2z)^2}{4z(1-z)}. \tag{38}$$

The identification between wave functions in both frames is done using probability conservation per unit phase space

$$\mathrm{d}^3p|\psi(p)|^2 = \mathrm{d}^2p_T\mathrm{d}z|\psi(p_T, z)|^2. \tag{39}$$

We need to express d^3p in terms of light-cone variables. This can be done using

$$\mathrm{d}^3p = \mathrm{d}p_L\mathrm{d}^2p_T, \tag{40}$$

where

$$\mathrm{d}p_L = \frac{\sqrt{p_T^2 + m_q^2}}{4\sqrt{(z(1-z))^3}}\mathrm{d}z \tag{41}$$

and from the correspondence equation we have

$$\frac{\sqrt{p_T^2 + m_q^2}}{4\sqrt{(z(1-z))^3}}\mathrm{d}z\mathrm{d}^2p_T|\psi(p)|^2 = \mathrm{d}^2p_T\mathrm{d}z|\psi(p_T, z)|^2 \tag{42}$$

and we can write the formula for the boosted wave function in $(z, \vec{p}_T)$ space as

$$\psi(p_T, z) = \psi\left(p = \sqrt{\frac{p_T^2 + (1-2z)^2 m_q^2}{4z(1-z)}}\right)\left(\frac{p_T^2 + m_q^2}{16(z(1-z))^3}\right)^{\frac{1}{4}} \qquad [\mathrm{GeV}^{-1}] \tag{43}$$

or

$$\psi(p_T, z) = \psi\left(p = \sqrt{\frac{p_T^2 + (1-2z)^2 m_q^2}{4z(1-z)}}\right)\sqrt{2}\left(\frac{(p^2 + m_q^2)^{\frac{3}{4}}}{(p_T^2 + m_q^2)^{\frac{1}{2}}}\right) \qquad [\mathrm{GeV}^{-1}]. \tag{44}$$

Following previous definition of the normalization and the Fourier transformation, this wave function has to be normalized according to

$$1 = \int |\psi(p)|^2\mathrm{d}^3p = \int |\psi(p_T, z)|^2\mathrm{d}^2p_T\mathrm{d}z = \int |\psi(p_T, z)|^2 2\pi p_T\mathrm{d}p_T\mathrm{d}z. \tag{45}$$

The final form of the wave function is done by transforming the wave function from (p_T, z) space to (r, z) space using 2D Fourier transformation

$$\psi(r, z) = \int \frac{\mathrm{d}^2p_T}{2\pi}e^{-i\vec{p}_T\vec{r}}\psi(p_T, z) = \int_0^{+\infty} \frac{p_T\mathrm{d}p_T}{2\pi}\psi(p_T, z)\int_0^{2\pi}\mathrm{d}\theta e^{-ipr\cos\theta} = \int_0^{+\infty} p_T\mathrm{d}p_T\psi(p_T, z)J_0(p_T r). \tag{46}$$

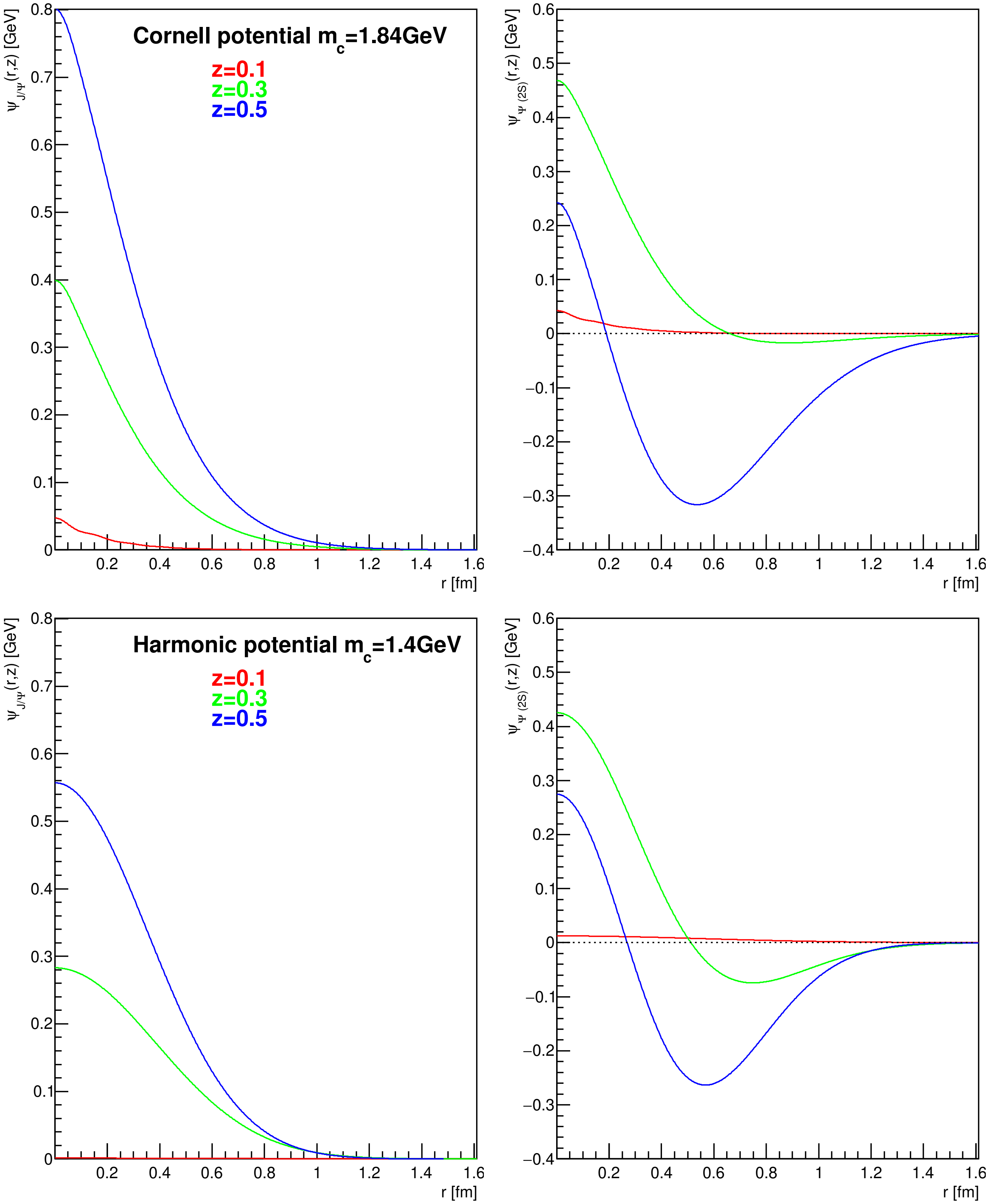

Figure 3: Backward Fourier transformed boosted Ψ wave function for fixed values of z.

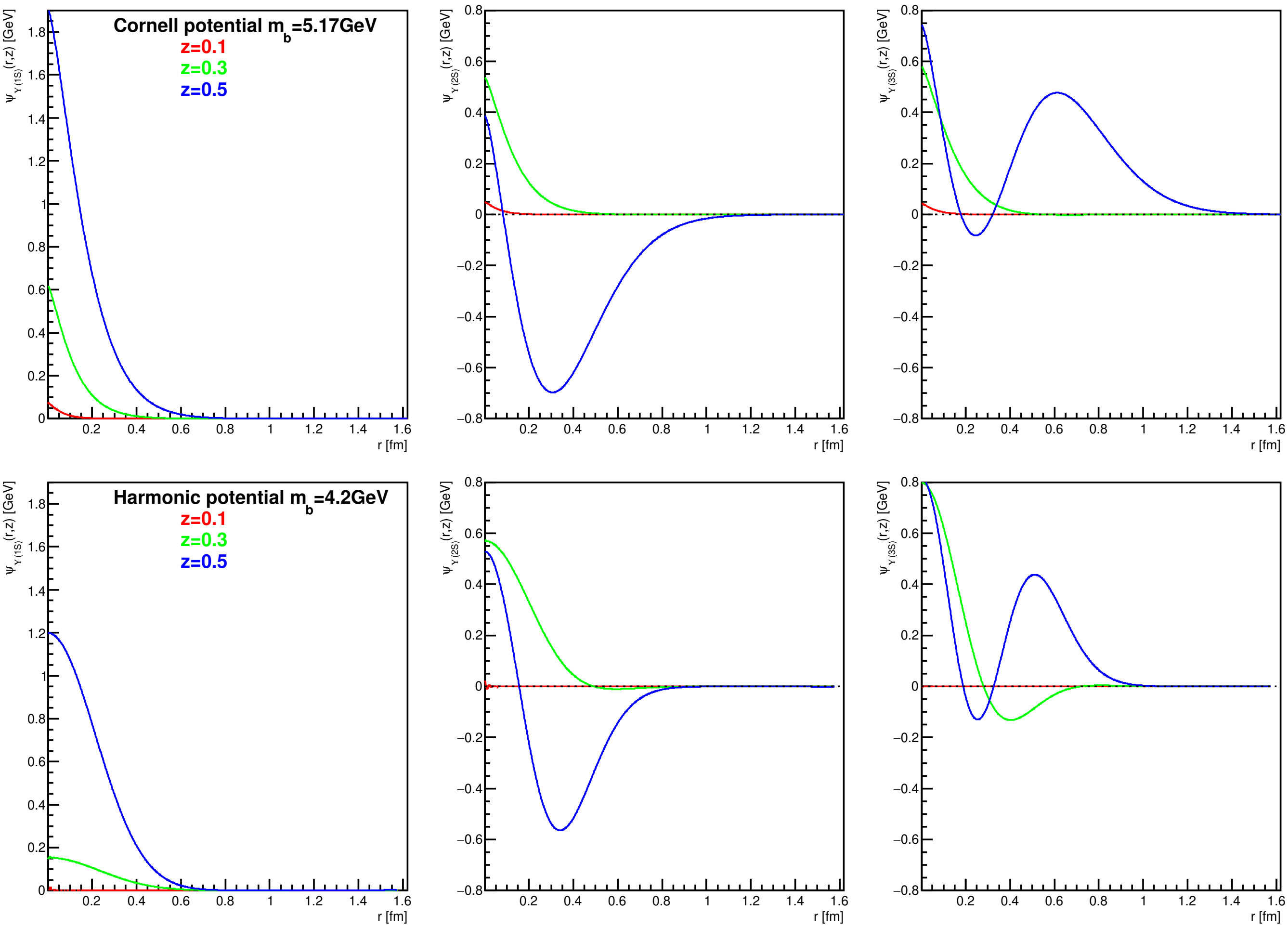

Figure 4: Backward Fourier transformed boosted Υ wave function for fixed values of z.

The resulting wave function is then in GeV.
Following previous definitions of the normalization and the Fourier transformation, this wave function has to be normalized according to

$$1 = \int |\psi(p_T, z)|^2 \mathrm{d}^2 p_T \mathrm{d}z = \int |\psi(r, z)|^2 \mathrm{d}^2 r \mathrm{d}z = 2\pi \int |\psi(r, z)|^2 r \mathrm{d}r \mathrm{d}z. \tag{47}$$

3.1.1 Quarkonia potentials

In order to calculate the radial part of the wave function of vector mesons, one needs to specify a potential between these two quarks.

Harmonic oscillator potential

Most common choice of the potential leads to the gaussian shape of the wave function. It is used in Gaussian models.

$$V(r) = \frac{1}{2} m_q \omega^2 r^2, \tag{48}$$

where

$$\omega = \frac{1}{2}(M_{2S} - M_{1S}) \tag{49}$$

which is 0.3GeV for J/Ψ and 0.28GeV for Υ. The mass of the quark is taken $m_c = 1.4$ GeV and $m_b = 4.2$GeV. Schrodinger equation with this potential has analytic solution in the form

$$u(r) = e^{-\frac{1}{4}m_q\omega r^2}. \tag{50}$$

Cornell potential

The Cornell potential was published in [14, 15] and also used in [13, 10]

$$V(r) = -\frac{k}{r} + \frac{r}{a^2} \quad k = 0.52 \quad a = 2.34\text{GeV}^{-1} \tag{51}$$

with $m_c = 1.84$ GeV and $m_b = 5.17$ GeV.

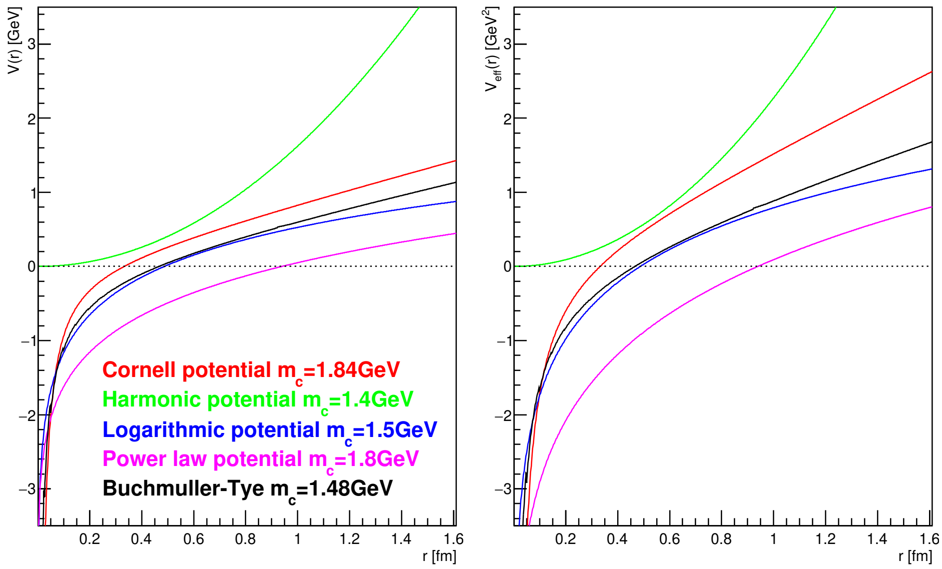

Figure 5: Left: Various potentials used for the calculation. Right: Effective potentials V_{eff} for $L = 0$ and c-quark.

4 Comparing both

When one compared the two approaches, one has to let 7 be equal to 18 and this gives us the condition for the scalar parts in both scenarios

$$\psi(r,z)^{nG} = \frac{\sqrt{N_c}}{\sqrt{2\pi}} \left(M_V + \delta \frac{m_f^2 - \nabla_r^2}{M_V z(1-z)} \right) \Phi_L^{BG}(r,z) \tag{52}$$

$$m_f K_0(\epsilon r)\psi(r,z)^{nG} = \frac{\sqrt{N_c}}{\sqrt{2\pi} z(1-z)} \left(m_f^2 K_0(\epsilon r) \Phi_T^{BG}(r,z) - (z^2 + (1-z)^2)\epsilon K_1(\epsilon r)\partial_r \Phi_T^{BG}(r,z) \right) \tag{53}$$

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