Modelling slow dynamics and hysteresis using non-equilibrium strain theory

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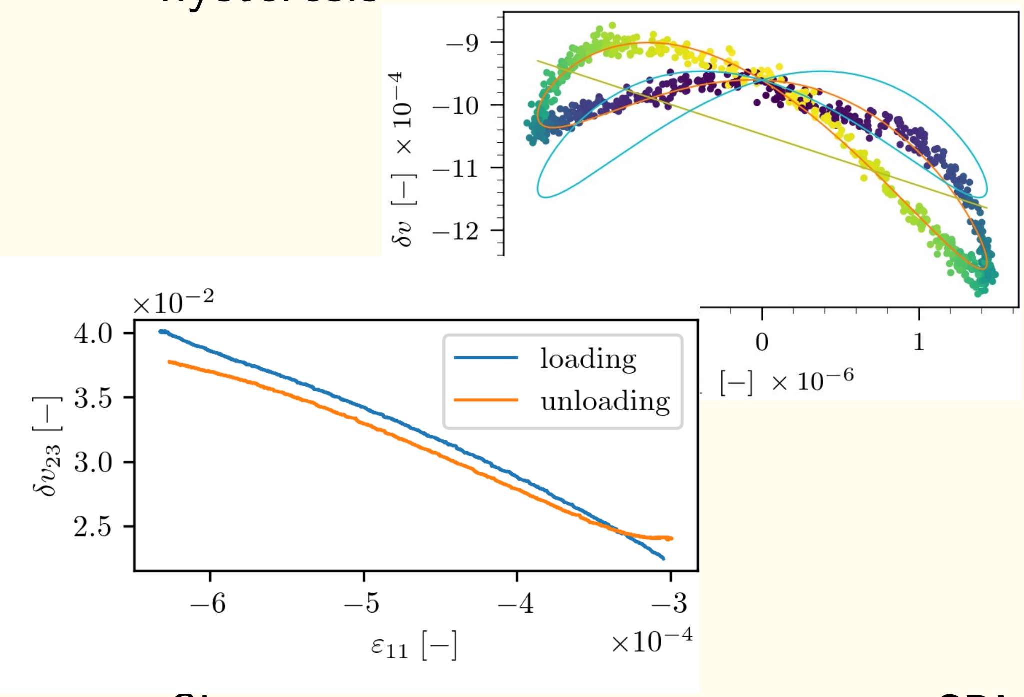
Nonlinear behavior of consolidated granular materials

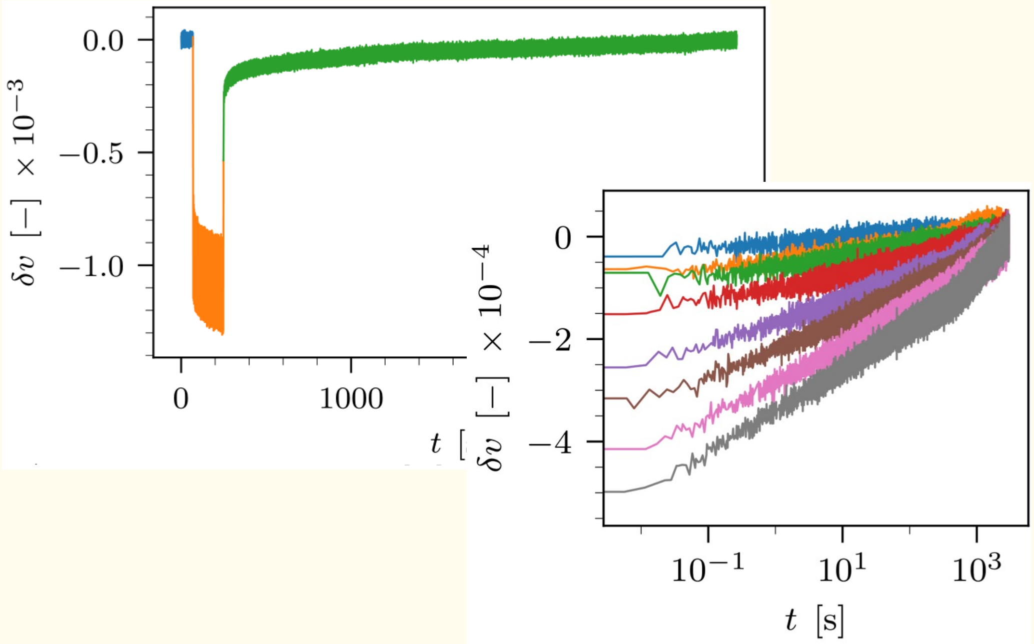
Fast processes

- classical nonlinearity
- hysteresis



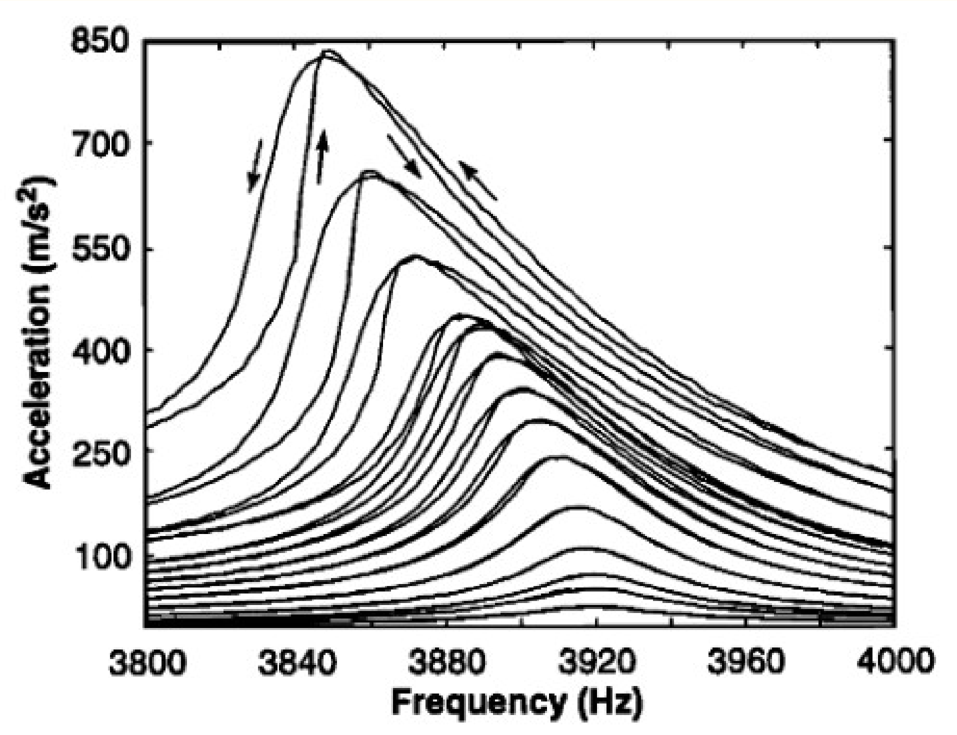
Slow processes

- slow dynamics
- creep



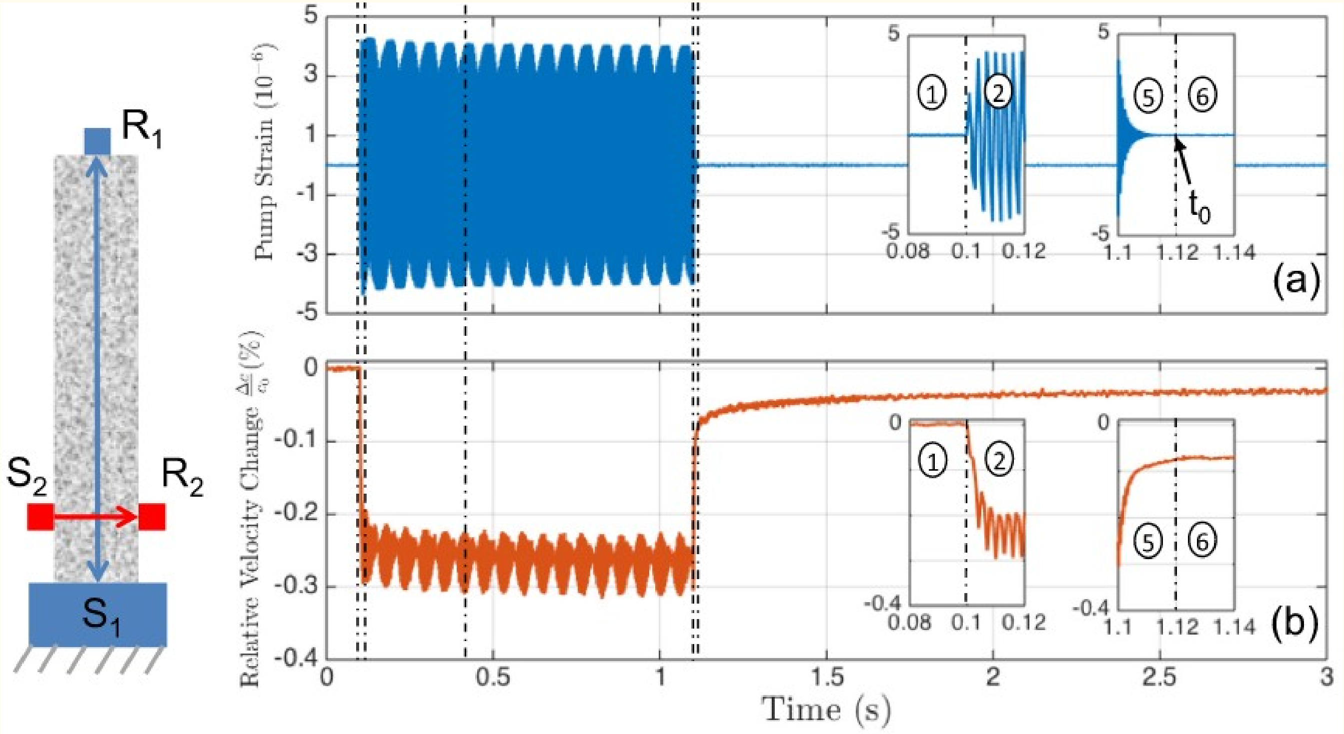
Bar resonance measurements

- sandstone bar
- depend on amplitude:
- modulus (frequency) decreases,
- classical nonlinearity
- nonlinear resonance ultrasonic spectroscopy (NRUS)
- depend on sweep direction, timing
- memory effects



J. A. TenCate and T. J. Shankland, “Slow dynamics in the nonlinear elastic response of Berea sandstone,” Geophys. Res. Lett. 1996.

Dynamic acoustoelastic testing



P. Shokouhi, J. Rivière, R. A. Guyer, and P. A. Johnson, "Slow dynamics of consolidated granular systems: Multi-scale relaxation," Appl. Phys. Lett., 2017.

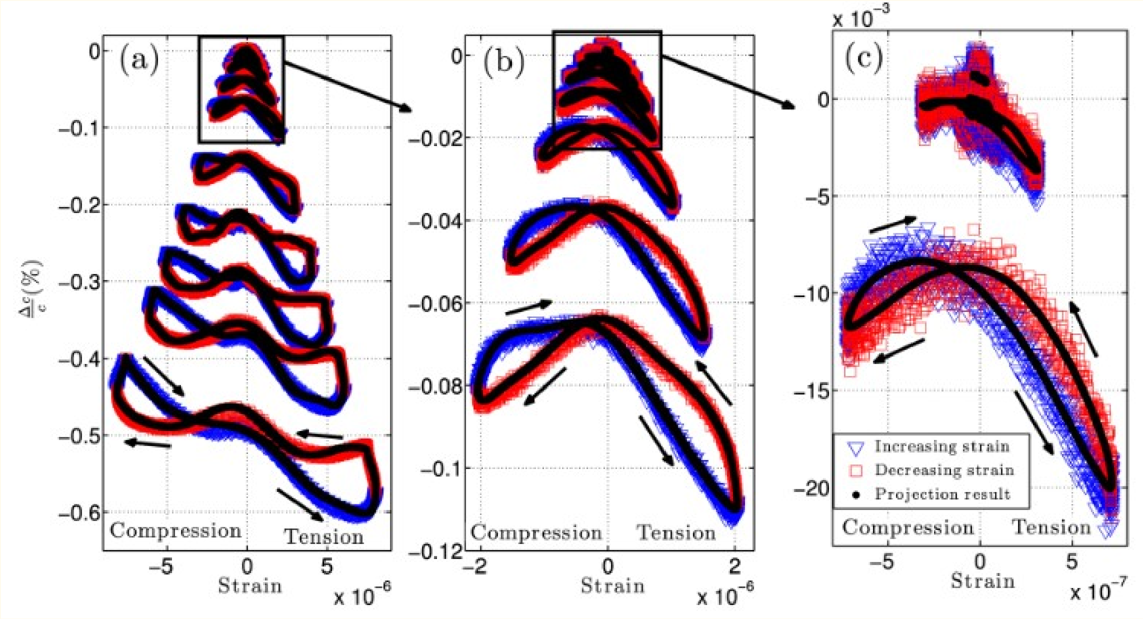
- Conditioning: high strain excitation, softening / decrease in modulus, new equilibrium is reached
- Relaxation:
  no excitation, log-time recovery
- Fully reversible

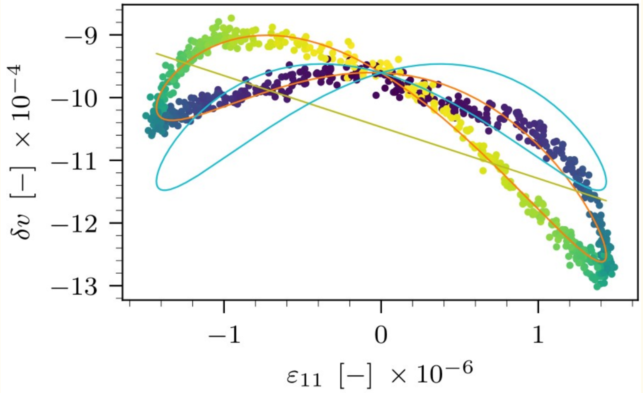
Dynamic acoustoelastic testing

Classical nonlinearity and hysteresis:

- strain and strain rate dependent modulus / velocity

$$\delta v\left(\varepsilon\left(t\right)\right) = \beta\varepsilon + \delta\varepsilon^2 + \alpha\left(\Delta\varepsilon + \varepsilon\mathrm{sign}\dot{\varepsilon}\right)$$





J. Rivière, G. Renaud, R. A. Guyer, and P. A. Johnson, “Pump and probe waves in dynamic acousto-elasticity: Comprehensive description and comparison with nonlinear elastic theories,” Journal of Applied Physics, 2013.

Multirelaxation process

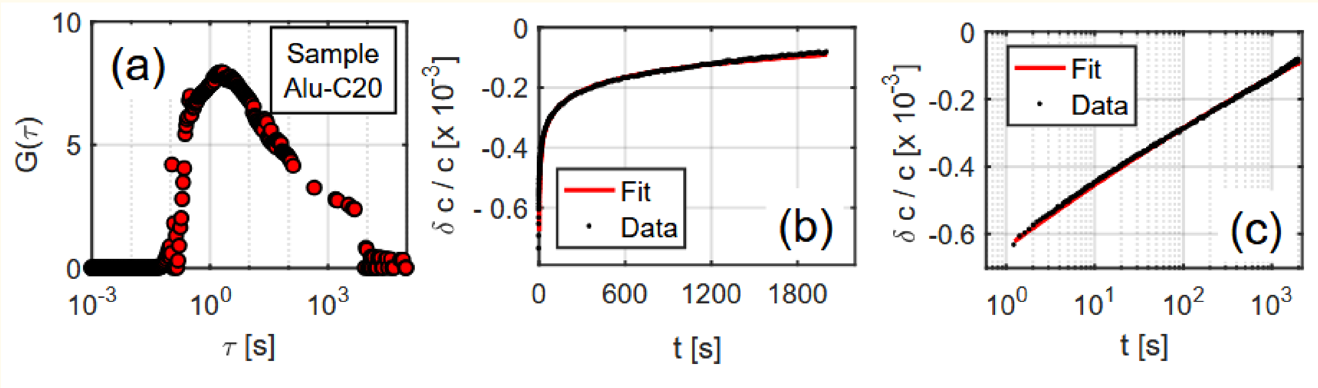
$$\delta v = \lambda \int_0^{+\infty} F(\tau) \exp\left(-\frac{t}{\tau}\right) \mathrm{d}\tau$$

distribution of relaxation times *F(τ)*

- material property
- challenging to derive from experiments
  - discrete relaxation times
  - continuous distribution – generalized Weibull

$$F(\tau) \sim \left(\frac{\tau}{a}\right)^b \exp\left[-\left(\frac{\tau}{a}\right)^c\right]$$

$$a > 0, b, c < 0$$



J. Kober, A. Kruisova, and M. Scalerandi, "Elastic Slow Dynamics in Polycrystalline Metal Alloys," Applied Sciences, 2021.

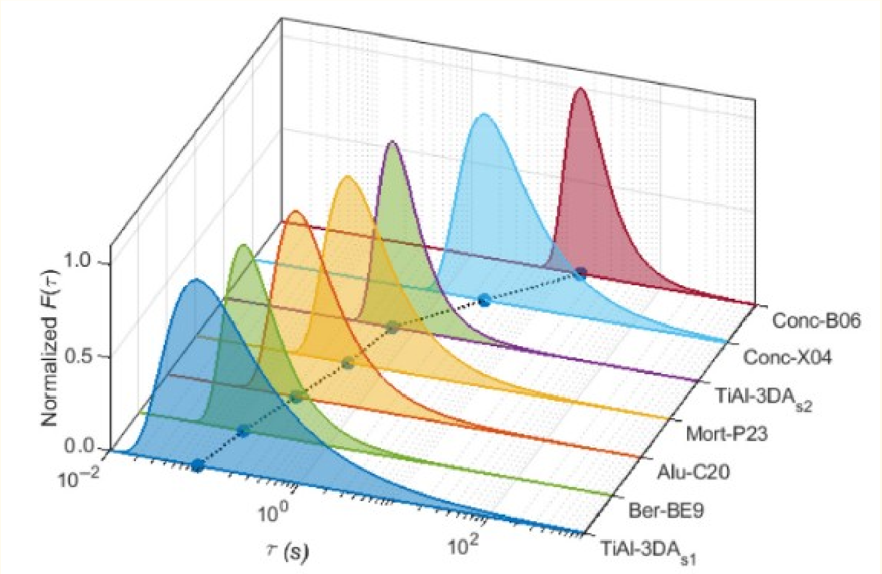
Multirelaxation process

Materials exhibiting SD

- consolidated granular materials (sandstone, limestone, concrete, mortar) metal alloys, unconsolidated granular materials

distribution of relaxation times $F(\tau)$

- related to microstructure
- relaxation times increasing with grain size
- defects manifested as additional relaxation times



J. Kober, A. S. Gliozzi, M. Scalerandi, and M. Tortello, “Material Grain Size Determines Relaxation-Time Distributions in Slow-Dynamics Experiments,” Phys. Rev. Applied, 2022.

Modeling nonlinear elasticity

- classical nonlinearity:
  stress-strain relations, strain-dependent modulus, hysteresis

$$\delta v\left(\varepsilon\left(t\right)\right)=\beta\varepsilon+\delta\varepsilon^{2}+\alpha\left(\Delta\varepsilon+\varepsilon\text{sign}\dot{\varepsilon}\right)$$

K. E.-A. Van Den Abeele, A. Sutin, J. Carmeliet, and P. A. Johnson, “Micro-damage diagnostics using nonlinear elastic wave spectroscopy (NEWS),” NDT & E International, 2001.

- slow dynamics
  - physical origin: unknown
    - redistribution of fluids, adhesion and friction, rearrangement of dislocations
  - restoration of microscopic contacts inhibited by a smooth energy barriers spectrum

J. A. TenCate and T. J. Shankland, “Slow dynamics in the nonlinear elastic response of Berea sandstone,” Geophys. Res. Lett. 1996.

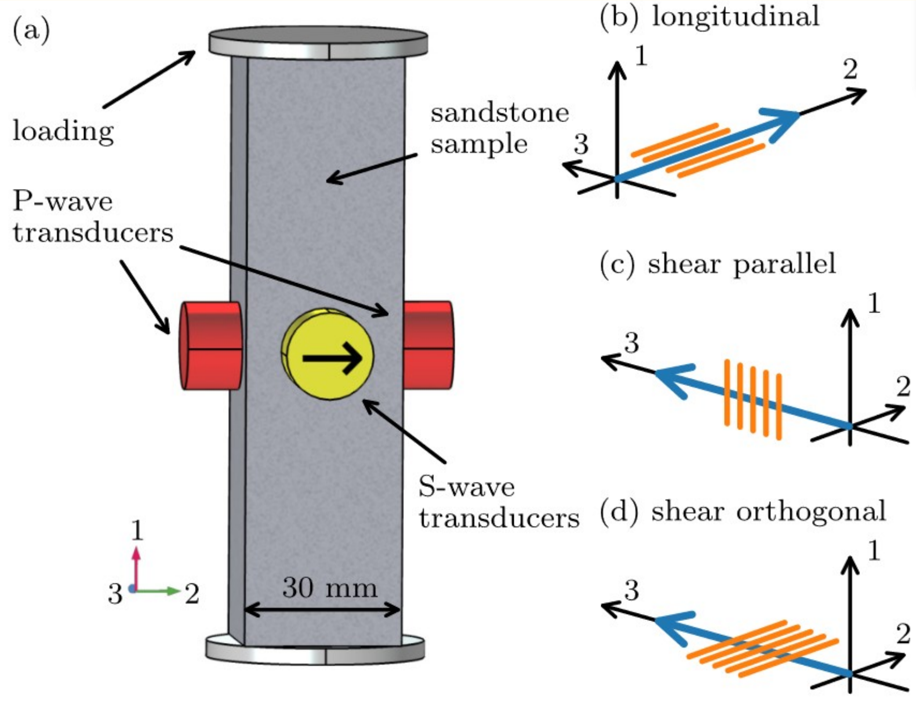
  - evolution of defects, exponential models
  - modulus proportional to concentration on defects
- the aim: universal viscoelastic model

Non-equilibrium strain

- same anisotropy in static uniaxial loading, conditioning, relaxation
  - stronger effect in the direction of loading
- non-equilibrium strain introduces conditioning into the classical non-linearity

$$\delta v_{ij} = \beta_{ij} \left(\varepsilon_{11} + \varepsilon_{\text{neq}} \right)$$

- accumulates during loading (conditioning), depends on strain amplitude
- relaxes to zero when applied stress is removed
- describes hysteresis and slow dynamics



J. Kober et al., Appl. Phys. Lett., 2023.

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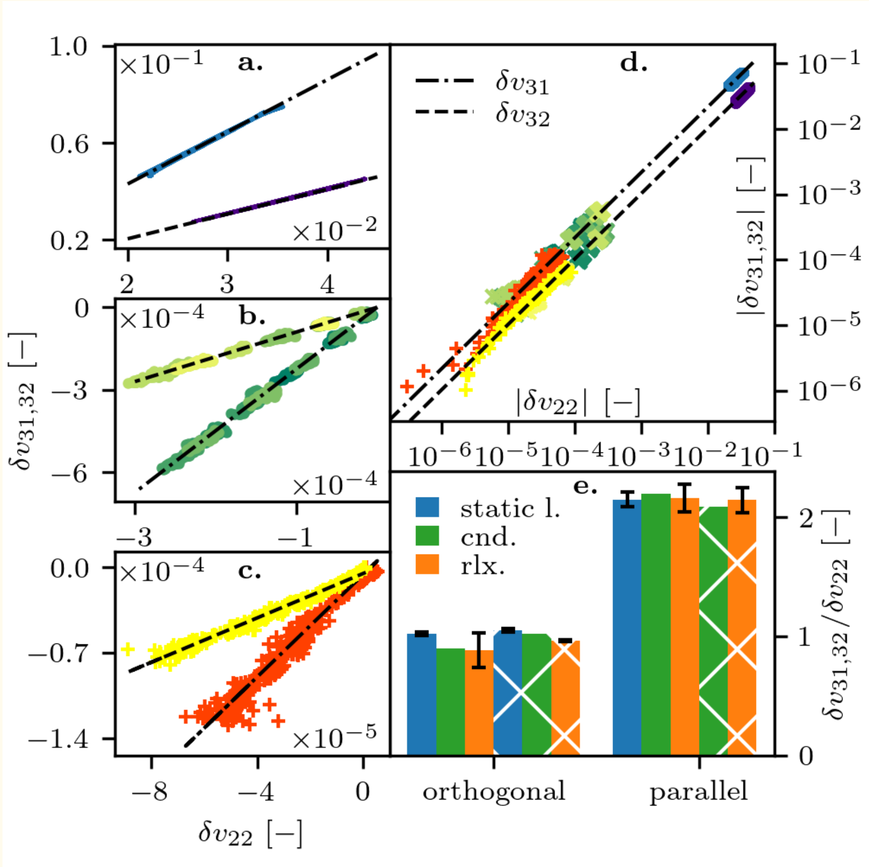


J. Kober et al., Appl. Phys. Lett., 2023.

Model of non-equilibrium strain

- superposition of components with relaxation time τ

$$\varepsilon_{\mathrm{neq}} = \int_0^{+\infty} \Psi\left(\tau\right) \varepsilon_{\mathrm{neq}}^{\tau} \mathrm{d}\tau$$

- independent evolution of components

$$\dot{\varepsilon}_{\mathrm{neq}}^{\tau} = \left(\varepsilon + \tau\dot{\varepsilon}\right)^2 - \frac{1}{\tau}\varepsilon_{\mathrm{neq}}^{\tau}$$

  - conditioning term: accumulation, hysteresis loops
  - relaxation term: exponential evolution

Model of non-equilibrium strain

conditioning term: $\dot{\varepsilon}^{\tau}_{\mathrm{neq}} = \underline{(\varepsilon + \tau\dot{\varepsilon})^2} - \frac{1}{\tau}\varepsilon^{\tau}_{\mathrm{neq}}$

- non-negative, even function of strain or strain rate

  ⇒ accumulation

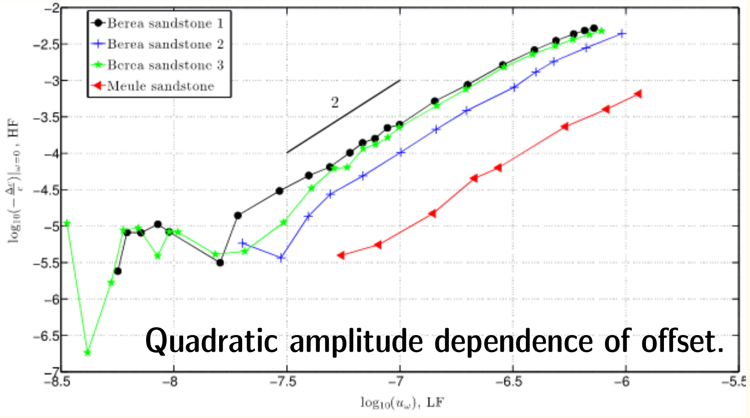
  ⇒ second harmonic oscillations (loops)

- phase shift
  - τ-dependent combination

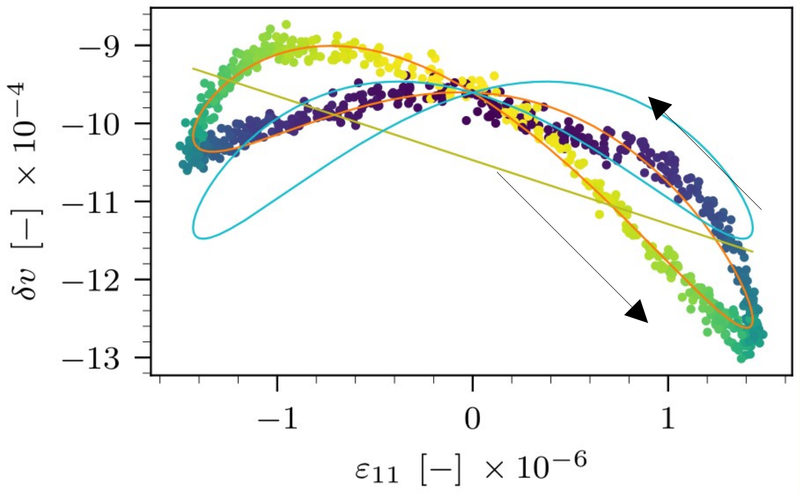
    of strain and strain rate

    ⇒ correct shape and orientation

    of the loops for all components



Rivière et al., Journal of Applied Physics, 2013.



Velocity variation as function of strain

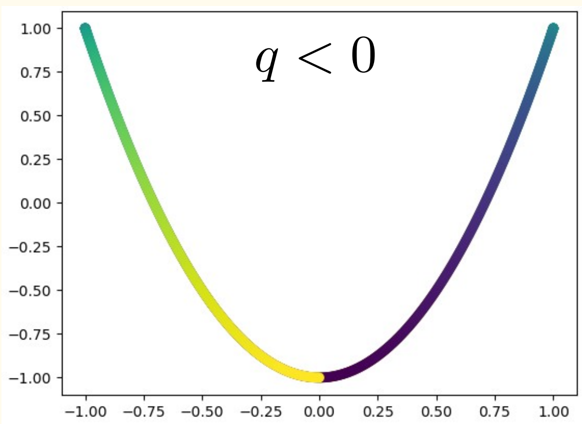
Solution for components

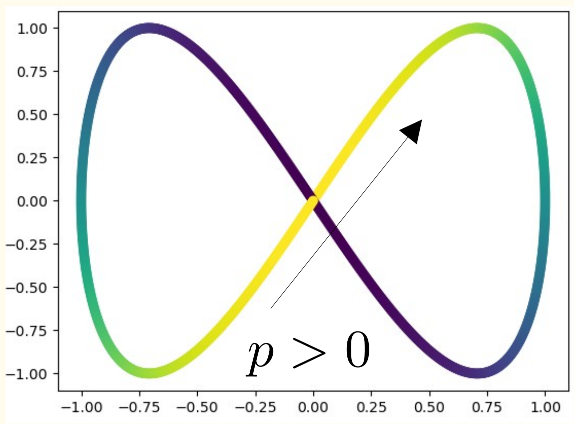
- independent evolution of components

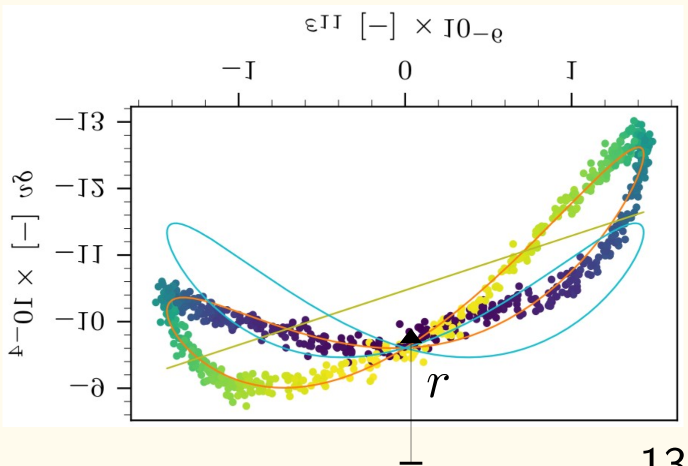
$$\dot{\varepsilon}^{\tau}_{\text{neq}} = \left(\varepsilon + \tau\dot{\varepsilon}\right)^2 - \frac{1}{\tau}\varepsilon^{\tau}_{\text{neq}}$$

- analytical solution for harmonic loading $\varepsilon\left(t\right) = A\sin\omega t$

$$\varepsilon^{\tau}_{\text{neq}}\left(t\right) = r\left[1 - \exp\left(-\frac{t}{\tau}\right)\right] + q\left(\cos 2\omega t - 1\right) + p\sin 2\omega t$$







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Distribution Ψ(τ)

- superposition of components with relaxation time τ

$$\varepsilon_{\mathrm{neq}} = \int_0^{+\infty} \Psi\left(\tau\right) \varepsilon_{\mathrm{neq}}^{\tau} \mathrm{d}\tau$$

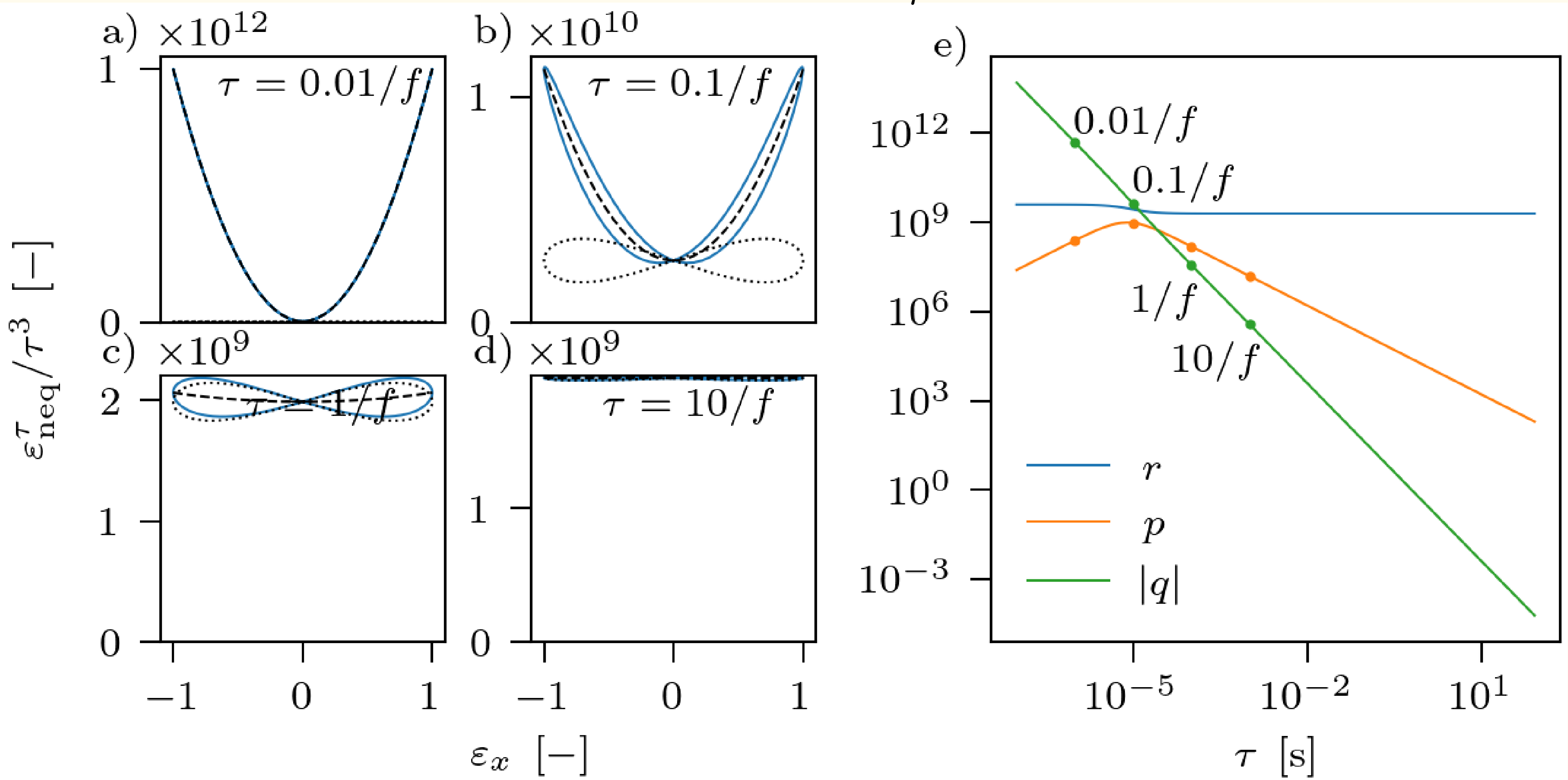
- distribution

$$\Psi\left(\tau\right) = k\tau^{-4}, \quad \tau \in \left(\tau_{\mathrm{min}}, \tau_{\mathrm{max}}\right)$$

  - same contribution of different timescales, log(t) process
  - $1/\tau^3$ normalization of components $\left\langle \varepsilon_{\mathrm{neq}}^{\tau} \right\rangle \sim \tau^3$
  - 1/τ underlying distribution (uniform distribution of log(τ))
  - different from relaxation distribution F(τ)

Contribution of time scales

$$\dot{\varepsilon}^{\tau}_{\mathrm{neq}} = (\varepsilon + \tau\dot{\varepsilon})^2 - \frac{1}{\tau}\varepsilon^{\tau}_{\mathrm{neq}}$$



Contribution of time scales

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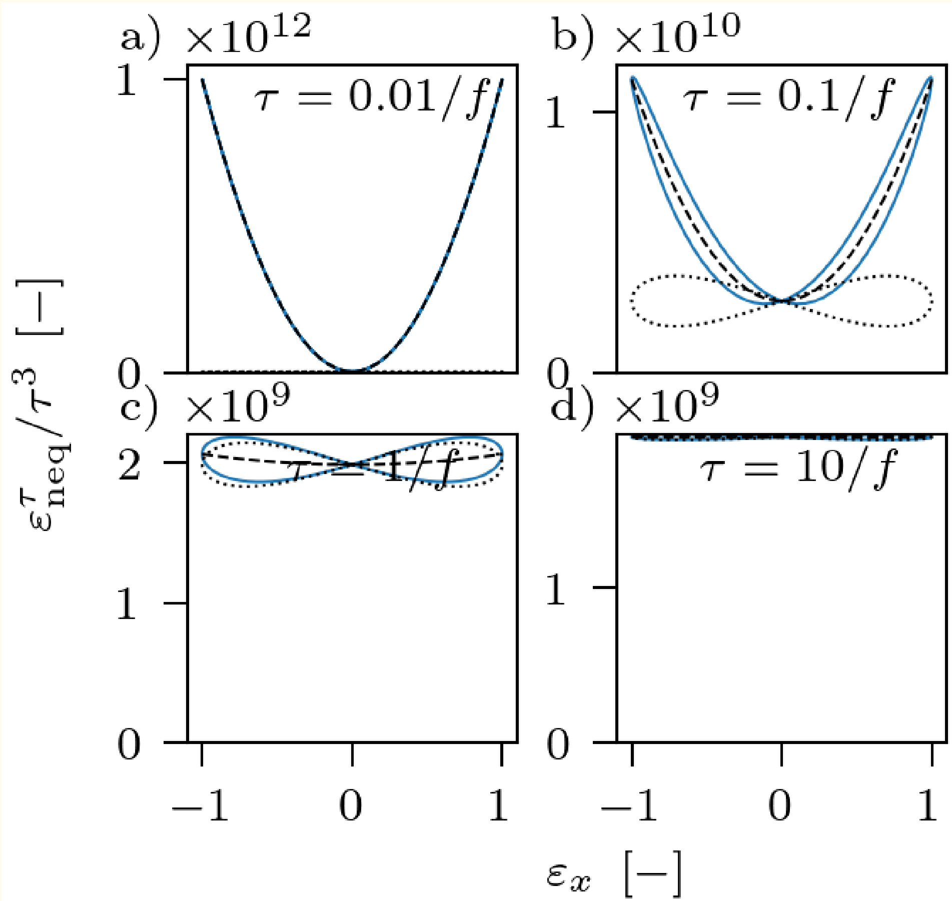


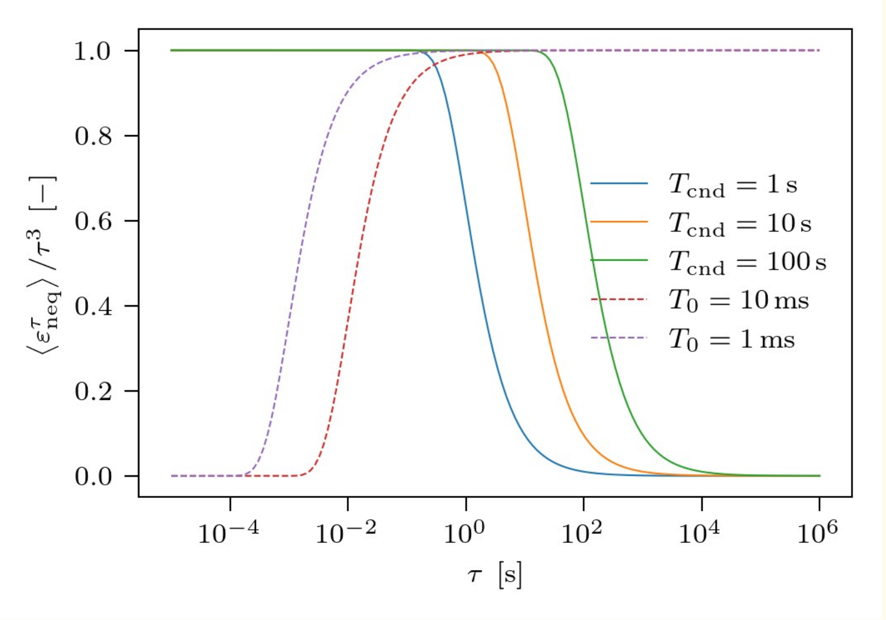
- fast τ:
  - following ε^2, low phase shift, quadratic loop
- moderate τ:
  - driven by strain rate, higher phase shift, hysteretic loop
- slow τ:
  - only conditioning offset

Numerical results

- **Finite conditioning induces limited spectrum**
  - S_∞ full spectrum from τ_{min} to τ_{max}
  - conditioning duration T_{cnd}

$$S_{T_{\text{cnd}}}(\tau) = S_\infty(\tau)\left(1 - \exp\left(-\frac{T_{\text{cnd}}}{\tau}\right)\right)$$

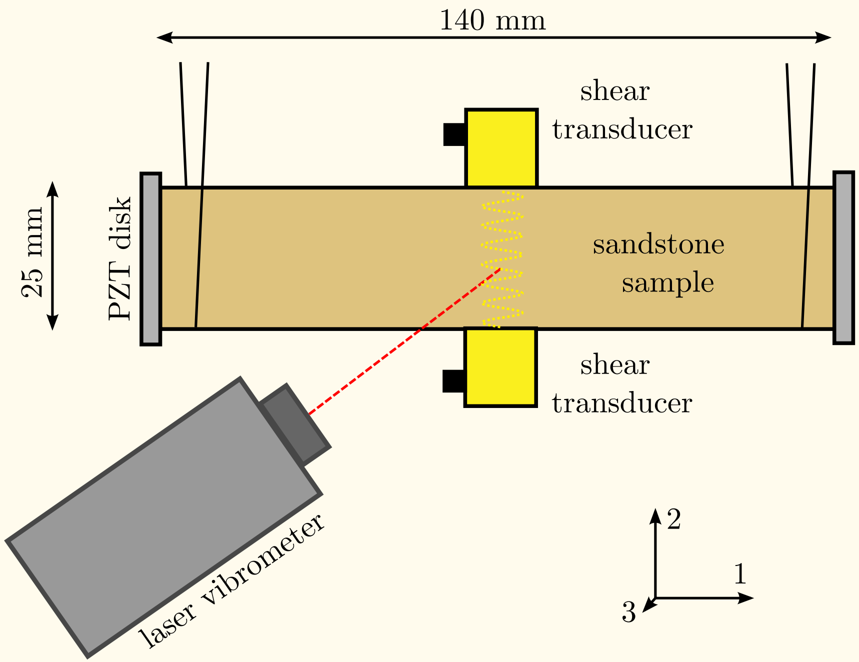
- **Fast relaxation times not measurable**
  - time gap before relaxation probing T_0, fast rlx times gone $S_{T_{\text{cnd}}}(\tau)\exp\left(-\frac{T_0}{\tau}\right)$
  - leads to generalized Weibull
  - ringdown, definition of t_0
- Relaxation spectrum seen through window function $F(\tau) = S_\infty(\tau)\,W(\tau)$



Normalized spectrum of non-eq. strain

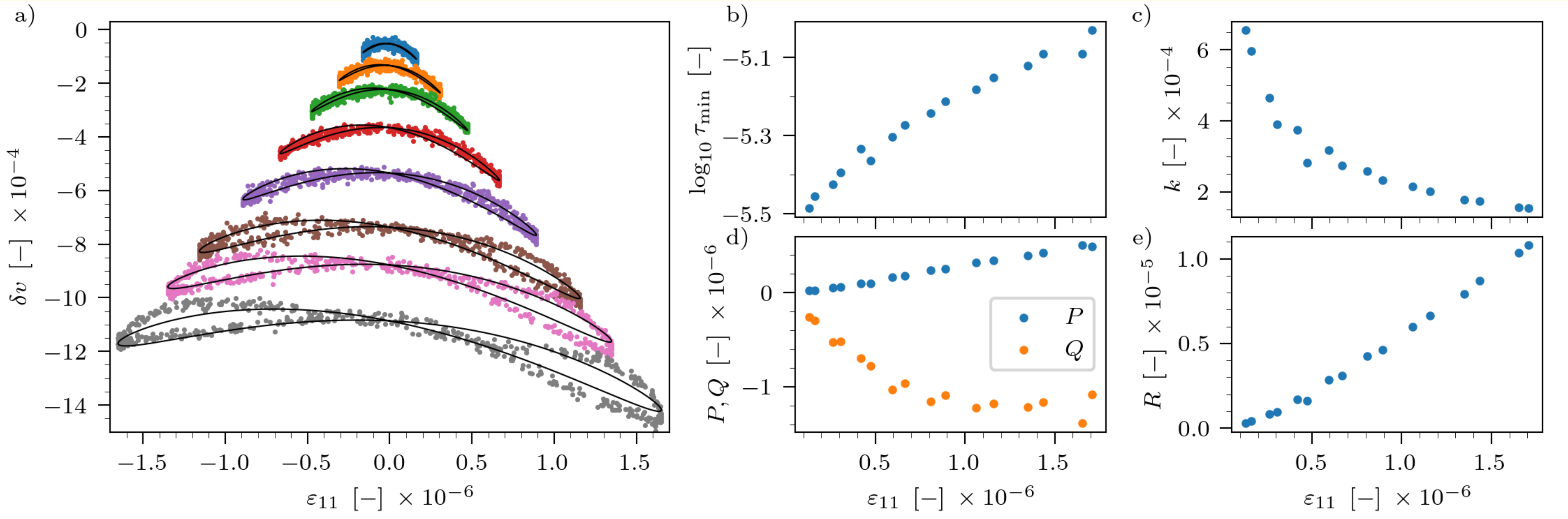
DAET experiment

- low frequency PZT disk
  - sinusoidal loading, first longitudinal mode 8.4 kHz
  - output amplitudes 0.4 – 1.8 V, 0.2 V step, power amplifier
- shear transducers
  - 1 MHz pulses, 1 period, low amplitude
  - 100 Hz repetition rate
- laser vibrometer: strain estimation using normal surface velocity
- 60 s preconditioning, 180 s conditioning, 3000 s relaxation



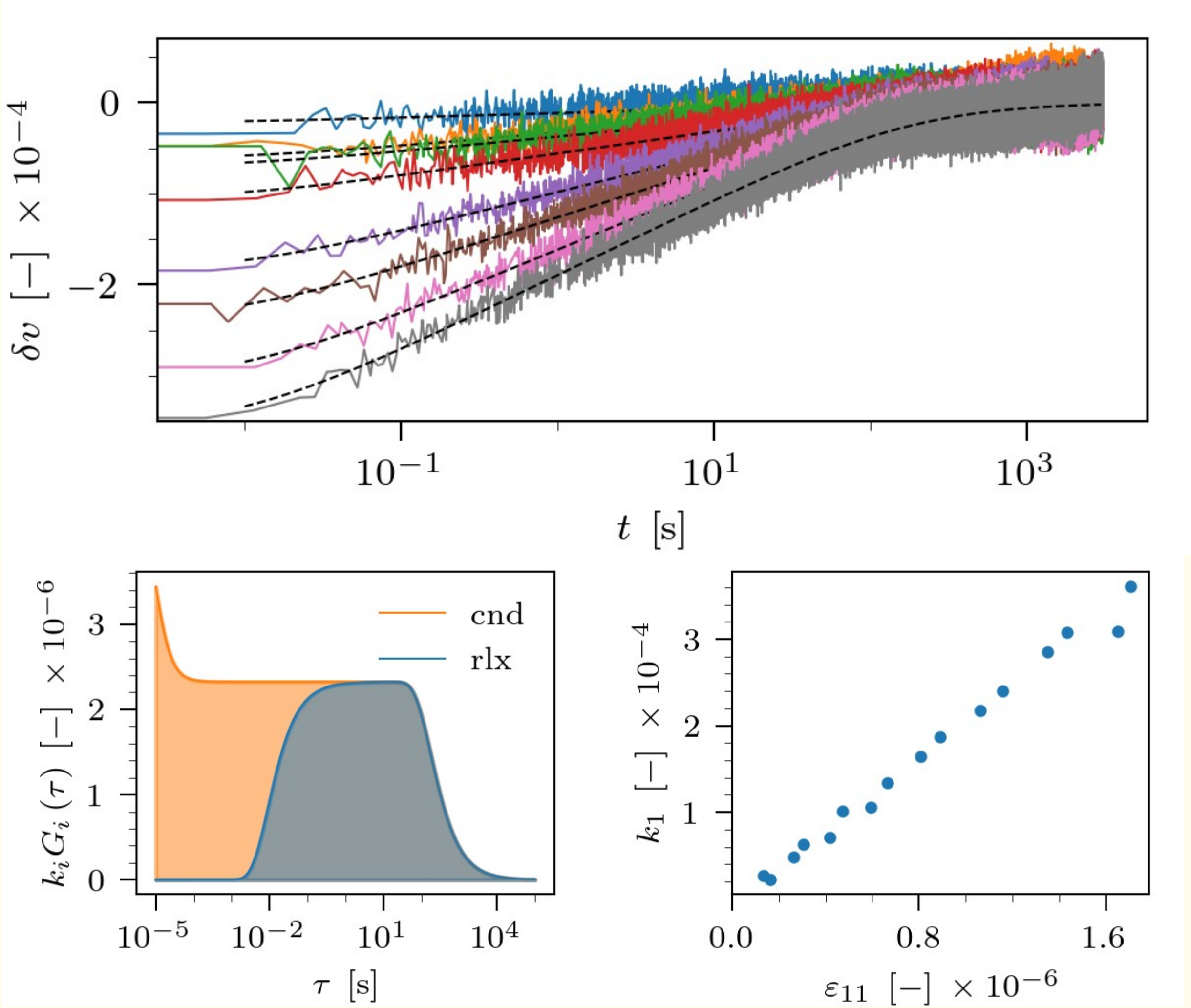
DAET experiment

- fit distribution parameters: $\tau_{\min}$, k (scaling)
- $\tau_{\max}$ irrelevant, upper bound given by conditioning duration
- amplitude-dependent, $\tau_{\min} \approx 10^{-5}$ s



DAET experiment

- relaxation spectrum given
  by conditioning duration and
  ringdown,
  same for all amplitudes
  - exponential ringdown, 10 ms
  - linear amplitude dependence

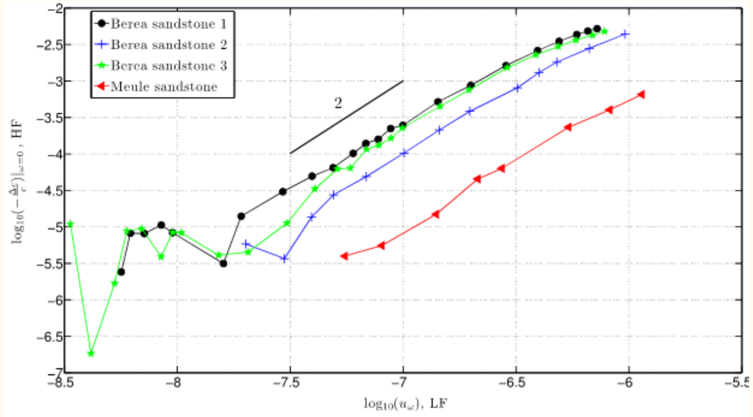


Remarks, discussion, discrepancies

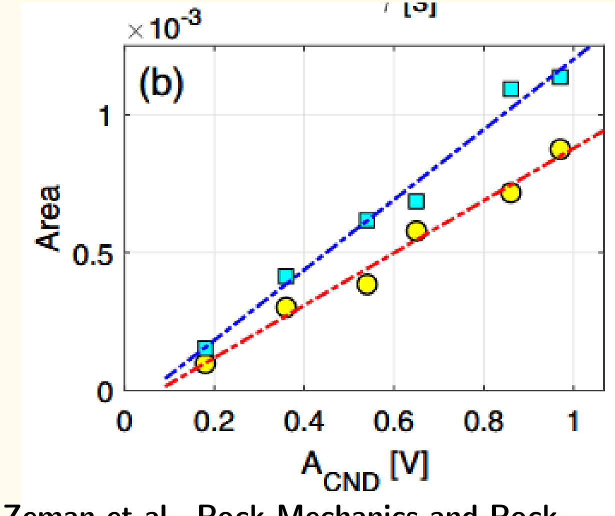
Amplitude dependence

- linear, quadratic, more complicated
- range of amplitudes
- strain estimation: input voltage / linear approximation

- different conditioning function?
    - conclusions on relaxation times remain valid



Rivière et al., Journal of Applied Physics, 2013.

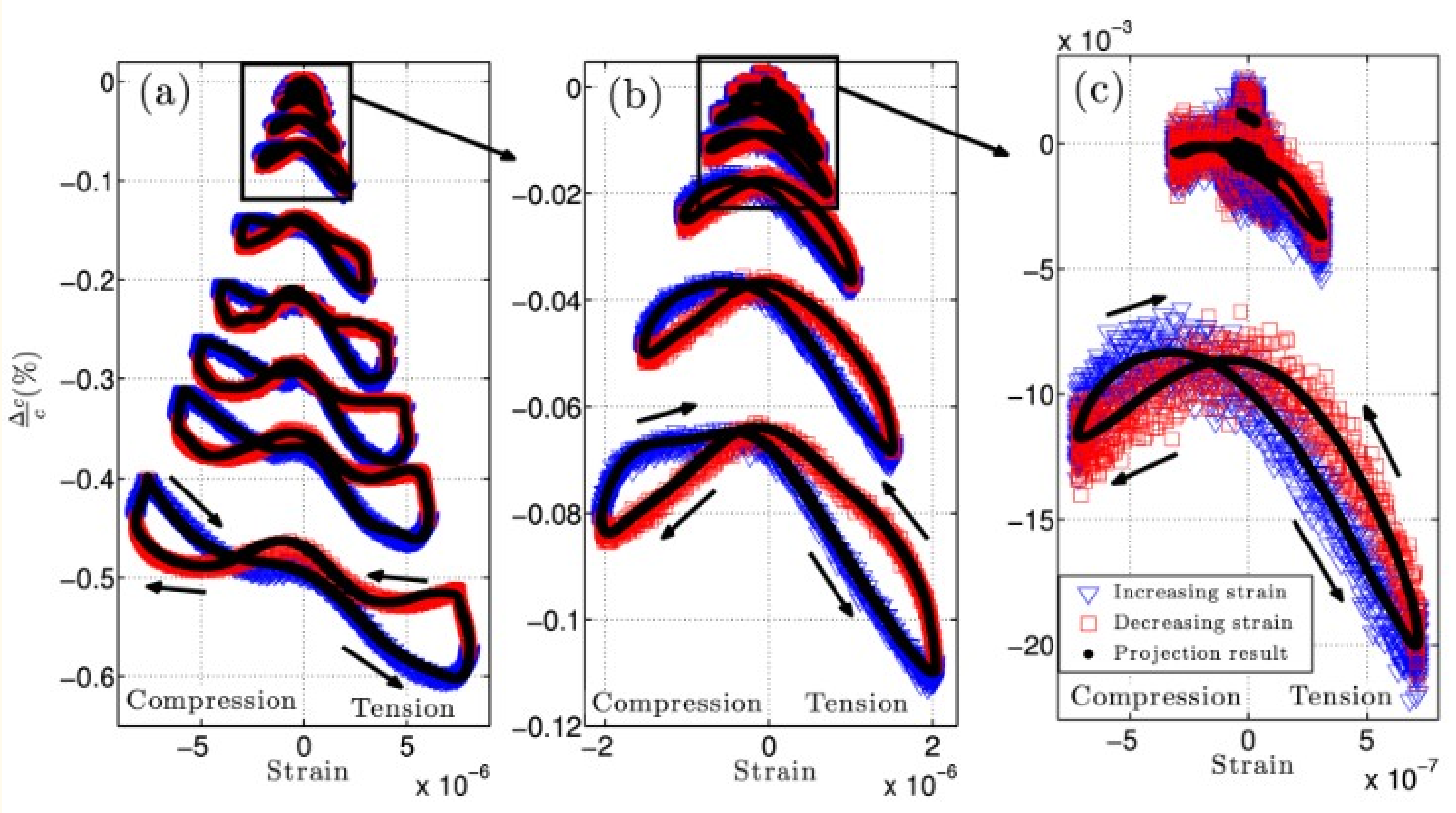


Zeman et al., Rock Mechanics and Rock Engineering, 2024.

Remarks, discussion, discrepancies

Amplitude dependence

- conditioning loops



Rivière et al., Journal of Applied Physics, 2013.

Remarks, discussion, discrepancies

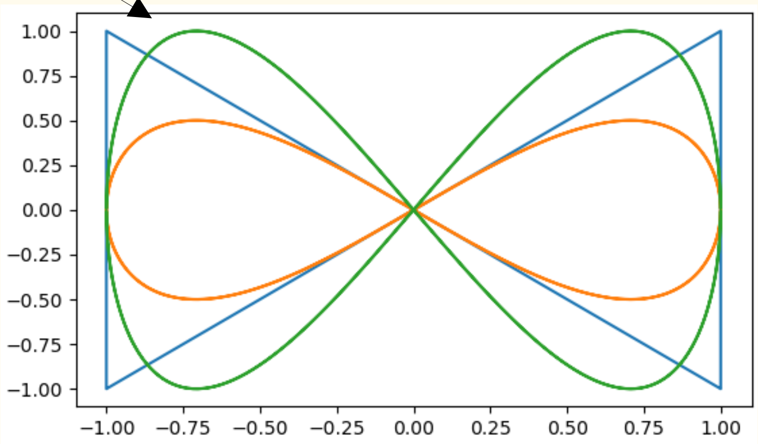
Link with strain-dependent modulus

$$\delta v = \beta\varepsilon + \delta\varepsilon^2 + \alpha\left(\Delta\varepsilon + \varepsilon \operatorname{sign}\dot{\varepsilon}\right) + o \qquad P, Q, R = \int_{\tau_{\min}}^{\tau_{\max}} \Psi\left(\tau\right) p, q, r\left(\tau\right)$$

$$\delta v = \beta\left(\varepsilon + \varepsilon_{\text{neq}}\right) = \beta\varepsilon + \beta R\left(t\right) + \beta Q\left(\cos 2\omega t - 1\right) + \beta P \sin 2\omega t$$

$$= \beta\varepsilon + \beta R\left(t\right) - 2\beta Q/A^2\varepsilon^2 \pm \beta P/A^2\varepsilon\sqrt{A^2 - \varepsilon^2}$$

- quadratic term: $\delta \sim Q$
- hysteretic term: $\alpha \sim P$
- offset: $o + \alpha\Delta\varepsilon \sim R$
- P, Q (ratio) determined by τ_{min}
- offset + loop size given by k



Remarks, discussion, discrepancies

- relaxation spectrum determined by experimental protocol
- conditioning spectrum includes wider range of relaxation times
- fast relaxation times needed for conditioning loops

Thank you