Quantum-like model of uncertainty for dynamic decision making

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In previous episodes:

- 2022

About use of Everett's interpretation of quantum mechanics in decision making

ver. 0.1

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Conclusions

Quantum FPD

- is a new approach with DM inspired by physical formalism
- provides better modelling and reasoning at micro and macro level
- considers human-like judgement (e.g. zooming) and cognition
- opens a way for general human AI
- may bring interesting mathematics to those who are interested in

Thank you for your attention!

- 2023

Does quantum version of DM make sense?

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Intermediate conclusions

- Current status: Learning... learning... learning
- Partially got inspired by Everett's theory and FPD
- Aim: to get promising theoretical approach that could provide mathematical reasons why qDM describes human behaviour
- The most expected answer: "Is qDM indeed needed?"
- being formalized and prepared for review by experts in QM

- SPMS 2024?

Thank you for your attention!

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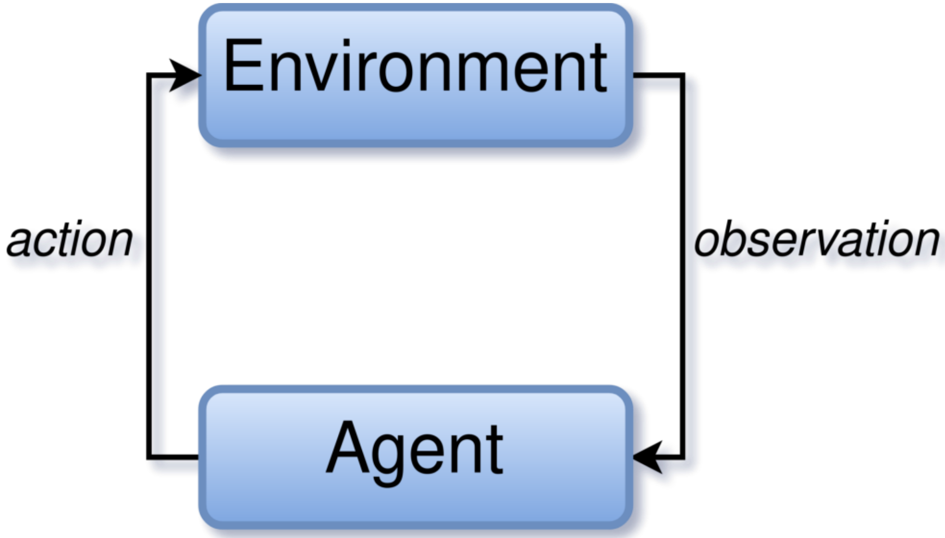
Motivation

Discrepancy: traditional theories vs observed human judgements:

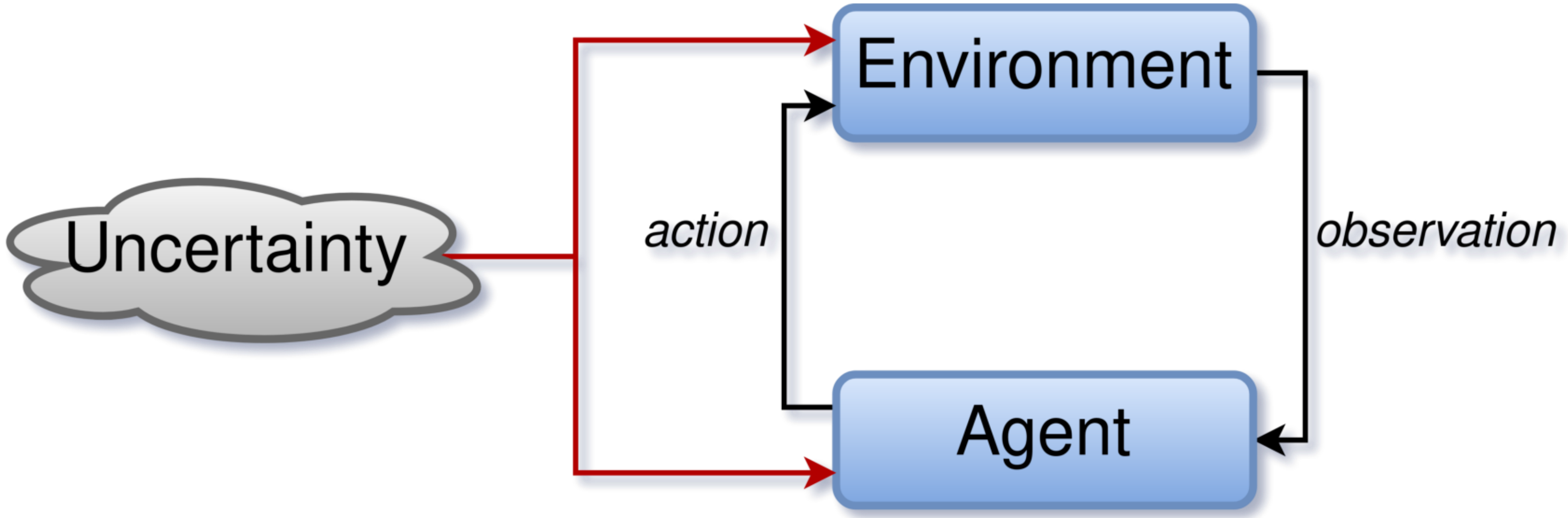
- Judgements are based on *indefinite states* (aka superposition of all definite states).
- Judgements interfere and introduce uncertainty => ***order effect***.
- Judgements *do not obey Boolean logic* => classic probability theory is not enough
- Humans' "***irrationality***" , i.e. deviation from rational decision making
- Human's sensitivity to a decision-making ***context***
- Paradoxes (Ellsberg, Allais) connected with ***uncertainty and risk***
- …

Closed decision loop



- **Agent:**
  - *influences/learns* the environment behavior
  - *influences/improves* own knowledge about the environment (model, loss function, …)
- **Examples:** control; automation; reinforcement learning; prediction; filtering.
- **Applications:** autonomous cars; IoT; (smart) robotics; forecasting; Deep Learning; non-invasive examinations (e.g. medical one); language and image processing, etc.

Closed decision loop



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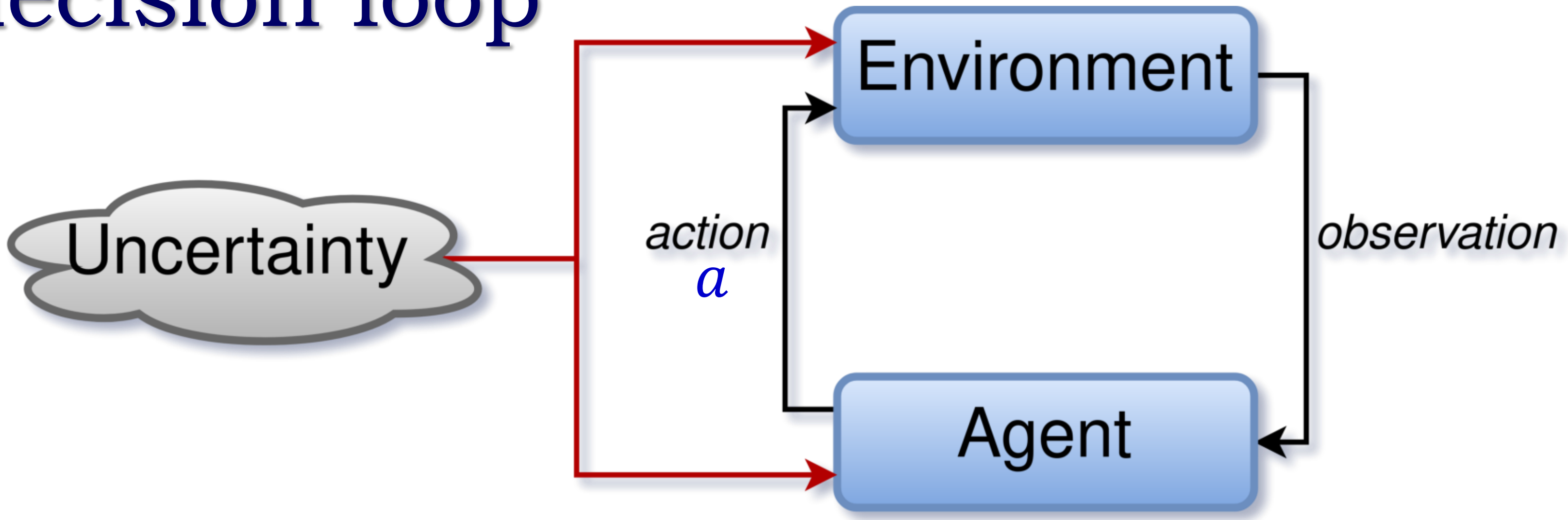
Main idea

To analyse general formulation of decision making (DM) problem

Key aspects of the proposed solution:

- DM problem is formalised via behaviours ordered by a loss function.
- Uncertainty is independent on decision strategy (no hidden feedback).
- Use SVD of matrix of losses
  - ⇒ atomic events are vectors; random events form subspaces of a Hilbert space.
- Schrödinger equation describes time evolution of uncertainty
  - ⇒ need of non-Kolmogorov's probability

Closed decision loop



Notions:

- behaviour $b = (g, a, k)$
- g – ignorance (uncertainty is a *part* of g)
- k – knowledge (e.g. data up to time t at which a is chosen)
- $r : \mathbf{k} \rightarrow \mathbf{a}$ – decision rule
- $S = (r_1, r_2, \dots)$ – strategy = a sequence of decision rules

Main steps

- $\exists$ order of behaviours $\preccurlyeq_{\mathbf{b}}$ that
  - is complete
  - preserves transitivity
- $\exists$ loss function $z : \mathbf{b} \mapsto \mathbf{R}$ that ranks order $\preccurlyeq_{\mathbf{b}}$ and
  - is isotone with $\preccurlyeq_{\mathbf{b}}$
  - preserves equivalence on $\mathbf{b}$
- S and U are pointers to strategy/uncertainty, resp.
- $S, U \to \mathbf{b}$ and existence of $z \quad \Rightarrow \quad \exists$ matrix of losses $\mathbb{L}_{S,U}$, with $\mathbf{S}, \mathbf{U}$ countable
- Apply SVD to split $\mathbb{L}$ into $U -$ dependent and $S -$ dependent parts

$$\mathbb{L} = \underbrace{\mathbb{V} \cdot \mathbb{D}}_{\mathbb{S}} \cdot \mathbb{U}^* = \mathbb{S} \cdot \mathbb{U}^* = \mathbb{S} \cdot \underbrace{\mathbb{P} \cdot \mathbb{P}^*}_{=\mathbb{I}} \cdot \mathbb{U}^*$$

Quantification of order on a set of strategies

- Define a set of functions $\mathbf{\Lambda} \coloneqq \{l_S: \mathbf{U} \to \mathbf{R} \mid \exists\, S \in \mathbf{S}\colon\ l_S(U) = \mathbb{L}_{S,U}\}$.
- Functional $\Upsilon(l_S) = \sum_j \mathcal{K}\left(l_S(U_j), U_j\right) p(U_j)$ quantifies order of strategies ($\mathcal{K}(\cdot)$ is increasing):

$$S_1 \preccurlyeq_{\mathbf{S}} S_2 \iff l_{S_1} \preccurlyeq_{\mathbf{\Lambda}} l_{S_2} \iff \Upsilon(l_{S_1}) \leq \Upsilon(l_{S_2}),$$

- Model uncertainty by atomic random events U_j and consider them as elements in a $\mathcal{H}$ space.
- For each U_j, define a unique unit vector $|\eta_j\rangle$ in $\mathcal{H}$, $\dim(\mathcal{H}) = \mathrm{card}(\mathbf{U})$.
- Using Gleason theorem, a measure on $\mathcal{H}$ can be written as $\mu(\eta_j) = \langle \eta_j | \hat{T}_\mu | \eta_j \rangle$ with

  non-negative kernel $\hat{T}_\mu = \mathrm{diag}(p(U_1), p(U_2), \dots),\ \mathrm{tr}(\hat{T}_\mu) = 1$.
- The functional in quantum probability reads: $\Upsilon(l_S) = \sum_j \mathcal{K}\left(l_S(U_j), U_j\right) \mu(\eta_j)$ and optimal strategy is

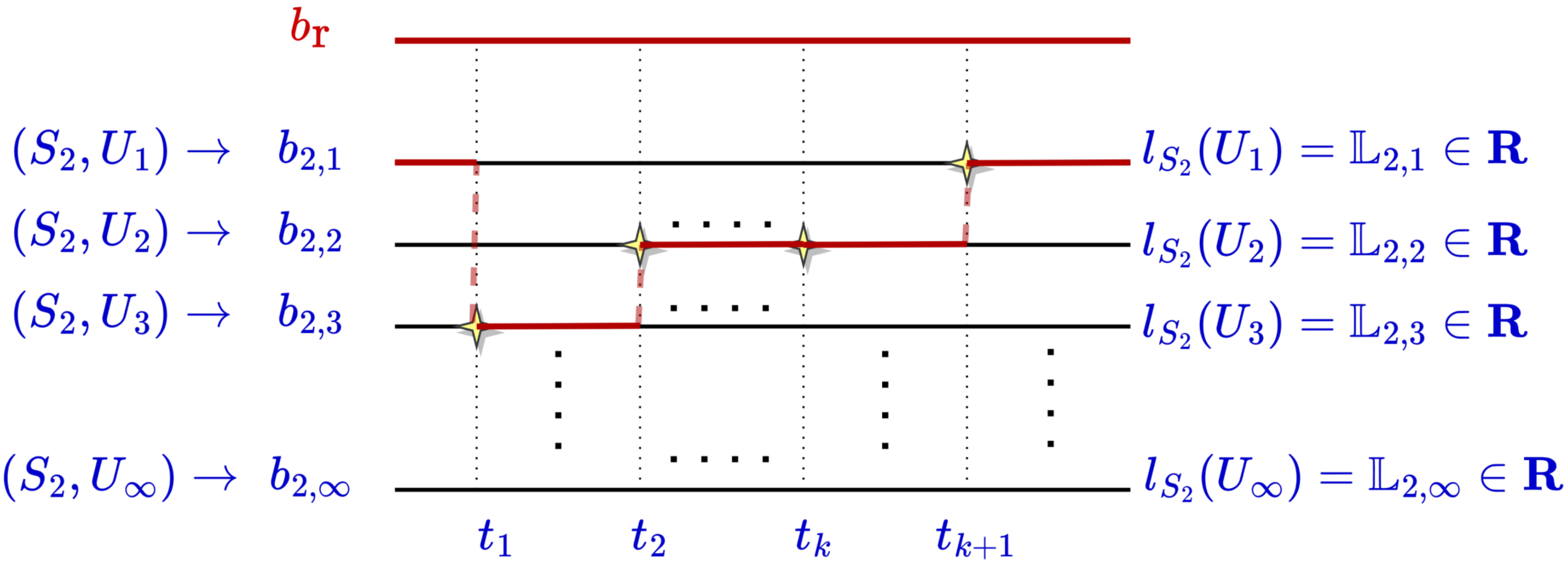
$$S^{opt} = arg \min_{S \in \mathbf{S}} \Upsilon(l_S).$$

In one-shot (static) case: the solution coincides with classical Kolmogorov's theory.

Dynamic case: two time-scales formalism

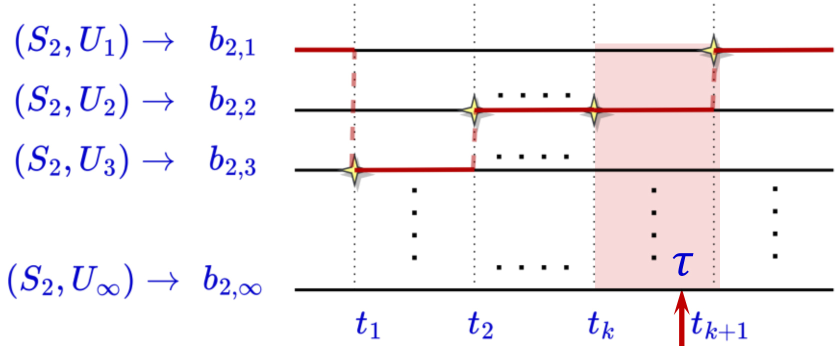
strategy $\mathbf{S} = \{S_1, S_2\}$

uncertainty $\mathbf{U} = \{U_1, U_2, \ldots, \}$



$$\mathbb{L} = \begin{pmatrix} \mathbb{L}_{1,1} & \mathbb{L}_{1,2} & \cdots & \mathbb{L}_{1,\infty} \\ \mathbb{L}_{2,1} & \mathbb{L}_{2,2} & \cdots & \mathbb{L}_{2,\infty} \end{pmatrix}$$

Dynamic case



Distinguish: discrete times (actions applied) and continuous time when uncertainty evolves

- At $\tau \in (t_k, t_{k+1})$, $\mathbb{U}$ changes and we express this by introducing matrix $\mathbb{M}_{t_k,\tau}$ that

$$\mathbb{U}_{t_k+\tau} = \mathbb{M}_{t_k,\tau} \cdot \mathbb{U}_{t_k}$$ (as $\mathbb{U}_{t_k}$ and $\mathbb{U}_{t_k+\tau}$ are unitary, $\mathbb{M}_{t_k,\tau}$ is unitary too)

- By introducing $\mathbb{M}_{t_k,\tau} = exp(i\tau\mathbb{H}_t)$, $\mathbb{H}_t$ - positive semidefinite matrix

$$\mathbb{U}_{t_k+\tau} = exp(i\tau\mathbb{H}_t) \cdot \mathbb{U}_{t_k}$$ - looks similar to ansatz

- By deriving $\frac{\partial}{\partial\tau}$ we got the Schrödinger's equation

$$\frac{\partial}{\partial\tau}\mathbb{U}_{t_k+\tau} = i\mathbb{H}_t \cdot \mathbb{U}_{t_k+\tau} \qquad (*)$$

with $\mathbb{H}_t$ interpreted as energy.

Random events are subspaces of $\mathcal{H}$ and (*) transforms uncertainty ($\mathbb{M}_{t,\tau}$ rotates vector $|\eta_{j,t}\rangle$ in $\mathcal{H}$):

$$\mu(\eta_{j,t+\tau}) = \langle\eta_{j,t}|\mathbb{M}_{t,\tau}\mathbb{T}_{\mu,t}\mathbb{M}_{t,\tau}|\eta_{j,t}\rangle.$$

$$\mathbb{L} = \begin{pmatrix} \mathbb{L}_{1,1} & \mathbb{L}_{1,2} & \dots & \mathbb{L}_{1,\infty} \\ \mathbb{L}_{2,1} & \mathbb{L}_{2,2} & \dots & \mathbb{L}_{2,\infty} \end{pmatrix}$$

Summary

Aim: to show *uncertainty dynamics* in DM

Used: general formulation of DM problem and realistic assumptions

Findings:

- Explanation why quantum modelling is inevitable
- *Dynamic case:* Schrödinger eqn describes time evolution of uncertainty => quantum modelling
- *Static case:* the solution coincide with classical DM theory

Possible future work:

- Solve partially observable Markov decision process in quantum setting
- Compare the with DM based on Open Quantum System approach
- More deep analysis of interrelations with physics (e.g. commutativity aspects, etc.)
- …

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Q & A

Operators H_t used in the exponential representation of the unitary operators: it seems that one should proceed under condition of their commutativity? i.e., that $[H_t,H_s] \equiv H_tH_s - H_sH_t = 0$.

- Time-evolution of uncertainty is “valid” within one decision epoch. At the end the agent takes a new action and changes the situation.
- Generally, commutativity of Hamiltonians from two different decision epochs, say (t_k, t_{k+1}) and (t_{k+1}, t_{k+2}) might have some meaning.
- A detailed analysis whether (and under what conditions) the commutativity matters is to be studied in future.

Q & A

Your basic equation is the Schrödinger equation (3.31). In the finite dimensional case its solution would generally fluctuate forever. Is it not an obstacle for getting a decision in finite time?

- No, its not. Equation (3.31) models evolution of uncertainties *between* decision epochs $t_1, .. t_k, t_{k+1}, \ldots$ (chosen by the agent) in which it selects the actions. Thus, whenever the solution (3.31) be, the agent selects a new action irrespectively.