Theory of Potentials in statistical Physics

on lattice

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Introduction of a problem

Mathematical Formulation

We study the integrals over a d-dimensional torus:

$$\int_{R^\Lambda} e^{-H(x)}\, dx_\Lambda$$

Application

The Hamiltonian is given by:

$$H(x_\Lambda) = H_g(x_\Lambda) + \sum_{i\in\Lambda} U_i(x_i)$$

where H_g is the ground state energy and U_i represents small petrubations.

Domain

Definition of Λ

The Λ is defined as a d-dimensional torus, denoted by Z^d, where d=2,3...
We presume that Torus Z is very large.

Dimensionality

This is actually very important because of we are on a huge, but finite set.

Quadratic Form on Λ

Definition of Quadratic Form

The quadratic form $H_\beta(X)$ on the torus Λ, or more generally on any graph, is defined as:

$$H_\beta(X) = \sum_{\{i,j\}} a_{ij}(x_i - x_j)^2 + \sum_{i\in\Lambda} \epsilon_i x_i^2$$

where $\{i, j\}$ represents pairs of nodes and ϵ_i are coefficients enhancing stability.

Requirement of Positivity

It is crucial for $H_\beta(X)$ to be positive definite:

Boundary Conditions

Boundary Condition

Define the boundary condition x_{M^c}, where $M^c = \Lambda \setminus M$ for some subset $M \subset \Lambda$. The condition is set as $x_{M^c} = x_M$ implying that the values on the boundary of M^c match those of M.

Objective

The goal is to minimize the value of the Hamiltonian $H_g(x_\Lambda)$ under the above boundary condition. This involves finding the configuration x_M that minimizes H_g across Λ.

Minimizing Configuration

The minimizing configuration, denoted x_M, is the configuration that satisfies:

$$\min_{x \in \Lambda} H_g(x_\Lambda) \text{ subject to } x_{M^c} = x_M$$

Feynman-Kac Lemma

Lemma

Lemma 2.1 (Feynman-Kac) For any $i \in M$,

$$x_i = \sum_P \omega_P X_{e(P)}$$

where the sum is taken over all paths P on Λ starting at i and ending in M^c, and $e(P)$ denotes the endpoints of P.

Path Weight

The weight of each path P, ω_P, is given by:

$$\omega_P = \prod_{\{j,k\}\in P} a_{jk}^{-\frac{1}{2}} \cdot \prod_{l \in P} \frac{1}{\lambda_l}$$

where $\lambda_l = \epsilon_l + \sum_{k\in P} a_{lk}$, and the products are taken over edges of P excluding non-endpoints $l \neq k$ in all steps of P.

Orbits

Orbit Definition

An orbit O is a path starting at some point $b(O)$ in M^c, traveling through M, and ending again at a point $e(O)$ in M^c. The terms $x_{b(O)}$ and $x_{e(O)}$ denote the values at the beginning and end of the orbit, respectively.

Orbit Path Structure

An orbit O consists of the steps:

$$O = (i_1 = b(O), i_2), (i_2, i_3), \ldots, (i_{N-2}, i_{N-1}), (i_{N-1}, i_N = e(O))$$

Here, $b(O)$ and $e(O)$ are the start and end points of the orbit, respectively.

Orbit weight

Weight Calculation

The weight w_O of an orbit O is defined by the product of terms over the path O:

$$w_O = \prod_{(i,j)\in O} a_{ij}^{-1/2} \cdot \prod_{l\in O^o} \frac{1}{\lambda_l}$$

where a_{ij} are the interaction strengths between nodes i and j, and λ_l are coefficients dependent on the properties at each node l along the path.

Interior Ends of O

The second product in the weight calculation extends over all "interior ends of O," i.e., those nodes $l = i_k$ where $k = 2, 3, \ldots, N-1$, that do not include the endpoints of the path.

Orbit theorem

Theorem

The minimal value of the quadratic form, given by

$$H(x_\Lambda) = \sum_{i \in M^c} \lambda_i x_i^2 - \sum_{\text{all } O} w_O x_{b(O)} x_{e(O)}$$

is attained under the condition that x_{M^c} is fixed. Here, λ_i are coefficients corresponding to each point i, and O represents orbits of paths within M.

Orbits

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Thank you

for you attention