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Translation and rotation invariant method of Renyi dimension estimation

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ABSTRACT

A fractal dimension is a non-integer characteristic that measures the space filling of an arbitrary set. The conventional methods usually provide a biased estimation of the fractal dimension, and therefore it is necessary to develop more complex methods for its estimation. A new characteristic based on the Parzen estimate formula is presented, and for the analysis of correlation dimension, a novel approach that employs the log-linear dependence of a modified Renyi entropy is used. The new formula for the Renyi entropy has been investigated both theoretically and experimentally on selected fractal sets.

1. Introduction

A fractal dimension is a non-integer characteristic that allows to measure the space-filling property of any set in Euclidean space. There are several definitions of dimension. The most general definition is the Hausdorff dimension [1] based on the Hausdorff measure and covering of the set with smaller sets with different radii. The similarity dimension [2] is based on the scaling property of fractal sets and is used only for the analysis of simple self-similar sets. The parameterized type of a fractal dimension is the Renyi dimension [1,3] that is the main object of interest in this paper.

The calculation of the Renyi dimension is based on the Renyi entropy [4] H_α, which is a generalisation of the Shannon (H_1), Hartley (H_0) and collision (H_2) entropies. The α-entropy is defined for $\alpha \geq 0$ as follows

$$H_\alpha(\vec{p}) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{k} p_i^\alpha \tag{1}$$

for $\alpha \neq 1$ and

$$H_1 = \lim_{\alpha \to 1} H_\alpha = -\sum_{i=1}^{k} p_i \ln p_i \tag{2}$$

where k is the number of events and p_i are their probabilities satisfying $\sum_{i=1}^{k} p_i = 1$. The formulas (2) and (1) are frequently used in most sources, but they describe only a finite set of events with a possible extension to a countable case. A more general form which includes also an uncountable case is defined as

$$H_\alpha(\vec{p}) = \frac{1}{1-\alpha} \ln \mathrm{E}\ \left(p^{\alpha-1}\right) \tag{3}$$

and

$$H_1 = \mathrm{E}\ (-\ln p). \tag{4}$$

Based on the definition of α-entropy, the Renyi dimension is defined as

$$D_\alpha = \lim_{\epsilon \to 0^+} \frac{H_\alpha}{-\ln \epsilon} \tag{5}$$

where D_0, D_1 and D_2 are called the capacity, information and correlation dimension, respectively. In this case, ϵ is a scaling parameter that influences the probabilities p_i.

The methods that are used to estimate the Renyi dimension are usually different for different parameters α. The capacity dimension for $\alpha = 0$ is usually estimated via the box-counting method [5] or the Minkowski covering method [1]. The particular type of dimension for $\alpha = 2$ is called the correlation dimension and was introduced first in [6]. There are several methods to estimate the correlation dimension including the traditional approach in [7] or the spectral approach in [8].

The determination of the Renyi dimension is based on Renyi entropy estimates, which is biased in general. The second and more general problem is how to sample the point set. Our approach is focused only on the Lebesgue measure sets where uniform sampling is defined. When these conditions are not guaranteed, such as when the geometric structure of the set is inhomogeneous, we

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Fig. 1. Density of Parzen estimate and corresponding $\mathcal{S}_M$.

can only test the hypothesis of unbiasedness for the given theoretical value D_α i.e., $H_0 : \widehat{D_\alpha} = D_\alpha$. In such cases, it was shown that the correlation sum represents an unbiased estimator of D_2 with respect to approach in [9,10].

Recently, there were efforts to improve the estimation of the capacity dimension of binary images [11,12] and to estimate this dimension of the set of possible singular points in the space-time of suitable weak solutions to the Navier–Stokes equations [13,14]. The correlation dimension is widely used in biomedicine for electroencephalography signal analysis [15,16] or in cardiology [17]. Economical data are also often the subject of the correlation dimension analysis, for example financial markets [18], and especially capital markets [19].

2. Parzen estimate with ball kernel

This section utilizes the Parzen estimate for the derivation of the density function of elements of the Lebesgue measurable set $\mathcal{F} \subset \mathbb{R}^n$. Supposing the existence of n-dimensional distribution function ϕ of points $\vec{x} \in \mathcal{F}$ i.e., $\vec{x} \sim \phi$, it is possible to define a sample of points

$$\Phi = \{\vec{x_1}, \vec{x_2}, \dots, \vec{x_M}\} \subset \mathcal{F} \tag{6}$$

that are uniformly generated from $\mathcal{F}$, i.e., $\vec{x_k} \sim \mathrm{U}(\mathcal{F}) \equiv \phi$. For any point $\vec{x} \in \mathbb{R}^n$, we define its ϵ-neighbourhood, i.e., a ball with radius ϵ as

$$\mathcal{B}(\vec{x}, \epsilon) = \{\vec{y} \in \mathbb{R}^n : \|\vec{y} - \vec{x}\|_2 \le \epsilon\} \tag{7}$$

for any $\epsilon > 0$. The volume V_* of the ball can be expressed as

$$V_* = V_n \cdot \epsilon^n \tag{8}$$

where V_n is the volume of an n-dimensional unit ball. The density estimate will be based on the elementary distribution

$$f_0(\vec{x}, \epsilon) = \frac{\mathrm{I}(\|\vec{x}\|_2 \le \epsilon)}{V_*} \tag{9}$$

using the indicator function $\mathrm{I}(\dots)$. We can use Parzen's [20] formula

$$f(\vec{x}, \Phi, \epsilon) = \frac{1}{M} \sum_{k=1}^{M} f_0(\vec{x} - \vec{x_k}, \epsilon) \tag{10}$$

to obtain a consistent estimate of ϕ. However, we will apply (10) to a discontinuous distribution on $\mathcal{F}$ to obtain new formulas for the Renyi dimension estimation.

The probability density estimate (10) is visualised on Fig. 1 in grayscale. The white area represents the regions where this function equals zero and the darker areas depict the intersection of several balls centred at points from the set sample Φ. The balls can be also used for the traditional definition of the Minkowski sausage [2] as

$$\mathcal{S} = \bigcup_{\vec{x} \in \mathcal{F}} \mathcal{B}(\vec{x}, \epsilon). \tag{11}$$

The sample set Φ is useful for its finite approximation

$$\mathcal{S} \approx \mathcal{S}_M = \bigcup_{k=1}^{M} \mathcal{B}(\vec{x}_k, \epsilon). \tag{12}$$

3. Renyi entropy estimate

Our novel estimate of the Renyi entropy is based on the differential entropy

$$H_\alpha = \frac{1}{1-\alpha} \ln \int_{\vec{x} \in \mathbb{R}^n} f^\alpha(\vec{x}) \mathrm{d}\vec{x} \tag{13}$$

for $\alpha \in \mathbb{R}_0^+ \setminus \{1\}$ and the Parzen estimate $f(\vec{x})$ that is scaled by $\epsilon > 0$. To avoid negative entropy values, we define the modified Renyi entropy as

$$H_\alpha^*(\Phi, \epsilon) = \frac{\ln J(\Phi, \alpha, \epsilon) - \ln J_0(\alpha, \epsilon)}{1-\alpha}. \tag{14}$$

for $\alpha \ge 0$ and $\alpha \ne 1$ where

$$J(\Phi, \alpha, \epsilon) = \int_{\vec{x} \in \mathbb{R}^n} f^\alpha(\vec{x}, \Phi, \epsilon) \mathrm{d}\vec{x} \tag{15}$$

and

$$J_0(\alpha, \epsilon) = \int_{\vec{x} \in \mathbb{R}^n} f_0^\alpha(\vec{x}, \epsilon) \mathrm{d}\vec{x} = V_*^{1-\alpha}. \tag{16}$$

Using the expected value of $v(\vec{x})$ for $\vec{x} \sim g$ as

$$\mathop{\mathrm{E}}_{\vec{x} \sim g} v(\vec{x}) = \int_{\vec{x} \in \mathbb{R}^n} v(\vec{x}) g(\vec{x}) \mathrm{d}\vec{x}, \tag{17}$$

the first term can be simplified as

$$\begin{aligned} J = J(\Phi, \alpha, \epsilon) &= \int_{\vec{x} \in \mathbb{R}^n} f^{\alpha-1}(\vec{x}, \Phi, \epsilon) \cdot f(\vec{x}, \Phi, \epsilon) \mathrm{d}\vec{x} \\ &= \mathop{\mathrm{E}}_{\vec{x} \sim f_\Phi} f^{\alpha-1}(\vec{x}, \Phi, \epsilon) \end{aligned} \tag{18}$$

We define the degeneracy of $\vec{x} \in \mathbb{R}^n$ as

$$G(\vec{x}, \Phi, \epsilon) = \sum_{k=1}^{M} \mathrm{I}(\|\vec{x} - \vec{x_k}\|_2 \le \epsilon) \tag{19}$$

holding that $G(\vec{x}, \Phi, \epsilon) \in \{0, \dots, M\}$. Recall that the probability density function $f(\vec{x})$ is

$$f(\vec{x}, \Phi, \epsilon) = \frac{1}{M \cdot V_*} \sum_{k=1}^{M} \mathrm{I}(\|\vec{x} - \vec{x_k}\|_2 \le \epsilon) = \frac{G(\vec{x}, \Phi, \epsilon)}{M \cdot V_*}. \tag{20}$$

Therefore

$$J = \mathop{\mathrm{E}}_{\vec{x} \sim f_\Phi} \left(\frac{G(\vec{x}, \Phi, \epsilon)}{M \cdot V_*} \right)^{\alpha-1} = M^{1-\alpha} V_*^{1-\alpha} \mathop{\mathrm{E}}_{\vec{x} \sim f_\Phi} G^{\alpha-1}(\vec{x}, \Phi, \epsilon) \tag{21}$$

and subsequently also the modified Renyi entropy is

$$H_\alpha^*(\Phi, \epsilon) = \frac{\ln J - \ln J_0}{1-\alpha} \tag{22}$$

$$= \frac{(1-\alpha)\ln M + (1-\alpha)\ln V_* + \ln \mathrm{E}\, G^{\alpha-1}(\vec{x}, \Phi, \epsilon) - (1-\alpha)\ln V_*}{1-\alpha} \tag{23}$$

The resulting modified entropy equals

$$H_\alpha^*(\Phi, \epsilon) = \ln M + \frac{\ln \mathrm{E}\, G^{\alpha-1}(\vec{x}, \Phi, \epsilon)}{1-\alpha} \tag{24}$$

for $\alpha > 0$ and $\alpha \ne 1$.

4. Basic properties and particular cases

In this and the following sections, the degeneracy of $\vec{x} \in \mathbb{R}^n$ will be denoted as $G(\vec{x})$ instead of $G(\vec{x}, \Phi, \epsilon)$. When $\vec{x} \in \mathbb{R}^n$, the degeneracy $G(\vec{x}) \in \{0, \ldots, M\}$, but for $\vec{x} \in \mathcal{S}_M$, the degeneracy fulfils $G(\vec{x}) \in \{1, \ldots, M\}$. The modified Renyi entropy follows $0 \le H^*_\alpha \le \ln M$. This entropy is a translational and rotational invariant, as it is easy to prove from (19) and (24). For the particular cases of α, one can derive the

- Modified Hartley entropy for $\alpha = 0$ as
$$H^*_0 = \ln M + \ln \mathrm{E} G^{-1}(\vec{x}), \tag{25}$$
- Modified Shannon entropy as a limit for $\alpha \to 1$ i.e.
$$H^*_1 = \lim_{\alpha\to 1} H^*_\alpha = \ln M - \mathrm{E}\ \ln G(\vec{x}), \tag{26}$$
- Modified collision entropy for $\alpha = 2$ as
$$H^*_2 = \ln M - \ln \mathrm{E} G(\vec{x}), \tag{27}$$
- Modified minimum entropy as a limit for $\alpha \to \infty$ as
$$H^*_\infty = \lim_{\alpha\to+\infty} H^*_\alpha = \ln M - \ln \max G(\vec{x}). \tag{28}$$

where the expected values are over $\vec{x} \sim f$. If the derivative $\frac{\partial H^*}{\partial \alpha}$ exists, it is always less or equal to zero, as it is easy to prove.

Moreover, the modified Renyi entropy H^*_α can be used for an alternative definition of the dimension as

$$D^*_\alpha = \lim_{\epsilon\to 0^+} \frac{H^*_\alpha(\epsilon)}{-\ln \epsilon} \tag{29}$$

for a given $\mathcal{F}$ as an analogy to formula (5).

5. Monte carlo approach

Basic properties of the finite sample Φ have been collected in the previous sections. Their direct application to the given data set is useful for the estimation of $H^*_\alpha(\epsilon)$ and consequently for the D^*_α estimation. Using the operator U of uniform sampling, the approximation of the Renyi entropy can be achieved via a Monte Carlo technique in the following way:

1. At first, the sample index is generated uniformly $k \sim \mathrm{U}(\{1, \ldots, M\})$.
2. The point $\vec{x}$ is generated uniformly from the ϵ-ball centred at $\vec{x_k}$ as $\vec{x} \sim \mathrm{U}(\mathcal{B}(\vec{x_k}, \epsilon))$.
3. The subsequent degeneration is calculated using (19).

The entropy H^*_α is calculated as an average of the degenerations using (24), (26) or (28) depending on α. The first two steps generate $\vec{x} \sim f$, of course. Assuming the entropy estimate H^*_α fulfils $H^*_\alpha \propto \epsilon^{-D^*_\alpha}$ for small $\epsilon > 0$, we can use it for the estimation of D^*_α using the model

$$H^*_\alpha(\epsilon) = A - D^*_\alpha \ln \epsilon \tag{30}$$

for small ϵ and satisfying linear dependency H^*_α on $\ln \epsilon$. The aim of this study is to demonstrate that D^*_α is an unbiased estimate of D_α for large M.

6. Relationship to capacity and correlation dimension

The capacity (D_0) and correlation (D_2) dimensions are defined for any Lebesgue measurable set $\mathcal{F}$. The only possibility how to compare D^*_α with D_α is to come back from the sample Φ to the original set $\mathcal{F}$. The sample Φ is a finite set with $D_\mathrm{H} = D_0 = D_2 = 0$, of course. We will study the particular cases of D^*_α for $\alpha = 0$ and $\alpha = 2$ in the case of the measurable set $\mathcal{F}$.

6.1. Relationship to D_0

The Renyi dimension is the characteristic that has an important relationship to the Minkowski-Bouligard capacity dimension. The capacity dimension can be defined [1] based on the Minkowski sausage as

$$D_0 = n - \lim_{\epsilon\to 0^+} \frac{\ln \mathrm{vol}(\mathcal{S})}{\ln \epsilon} \tag{31}$$

where $\mathcal{S}$ is defined in (11) and $\mathrm{vol}(\mathcal{S}) = \int_{\vec{x}\in\mathcal{S}}$ is its volume. Supposing the existence of D_0, we can directly calculate

$$J_0 = \int_{\mathbb{R}^n} f^0_0(\vec{x}) \mathrm{d}\vec{x} = V_* = V_n \cdot \epsilon^n, \tag{32}$$

and also the density

$$f(\vec{x}) = \mathop{\mathrm{E}}_{\vec{y}\sim\mathrm{U}(\mathcal{F})} f_0(\vec{x} - \vec{y}) \tag{33}$$

and

$$f^0(\vec{x}) = \mathrm{I}\left(\mathop{\mathrm{E}}_{\vec{y}\sim\mathrm{U}(\mathcal{F})} f_0(\vec{x} - \vec{y}) > 0 \right) = \mathrm{I}\left(\bigvee_{\vec{y}\in\mathcal{F}} \|\vec{x} - \vec{y}\|_2 \le \epsilon \right) = \mathrm{I}(\vec{x} \in \mathcal{S}). \tag{34}$$

Therefore, the function J can be expressed as

$$J = \int_{\mathbb{R}^n} f^0(\vec{x}) \mathrm{d}\vec{x} = \mathrm{vol}(\mathcal{S}). \tag{35}$$

The resulting modified Hartley entropy equals

$$\begin{aligned} H^*_0(\epsilon) &= \ln \frac{\int_{\mathbb{R}^n} f^0(\vec{x}) \mathrm{d}\vec{x}}{\int_{\mathbb{R}^n} f^0_0(\vec{x}) \mathrm{d}\vec{x}} = \ln \mathrm{vol}(\mathcal{S}) - \ln V_* \\ &= \ln \mathrm{vol}(\mathcal{S}) - \ln V_n - n \ln \epsilon. \end{aligned} \tag{36}$$

Now, it is clear that

$$\begin{aligned} D^*_0 &= \lim_{\epsilon\to 0^+} \frac{H^*_0(\epsilon)}{-\ln \epsilon} = \lim_{\epsilon\to 0^+} \frac{\ln \mathrm{vol}(\mathcal{S}) - \ln V_n - n \ln \epsilon}{-\ln \epsilon} \\ &= n - \lim_{\epsilon\to 0^+} \frac{\ln \mathrm{vol}(\mathcal{S})}{\ln \epsilon} = D_0. \end{aligned} \tag{37}$$

Therefore, D^*_0 obtained from the modified Hartley entropy $H^*_0(\epsilon)$ is equivalent to the capacity dimension D_0 of the measurable set $\mathcal{F}$.

6.2. Relationship to D_2

The correlation dimension of $\mathcal{F}$ is defined as

$$D_2 = \lim_{\epsilon\to 0^+} \frac{\ln C(\epsilon)}{\ln \epsilon} \tag{38}$$

where

$$C(\epsilon) = \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim\mathrm{U}(\mathcal{F})} \mathrm{I}(\|\vec{y} - \vec{z}\|_2 \le \epsilon) \tag{39}$$

is the correlation integral. Supposing the existence of D_2, recall that

$$f_0(\vec{x}) = \frac{\mathrm{I}(\|\vec{x}\|_2 \le \epsilon)}{V_n \epsilon^n} \tag{40}$$

and by means of integrating the elementary distribution over the space we get

$$J_0 = \int_{\mathbb{R}^n} f^2_0(\vec{x}) \mathrm{d}\vec{x} = \frac{1}{V_n^2 \epsilon^{2n}} V_n \epsilon^n = V_n^{-1} \epsilon^{-n}. \tag{41}$$

In the finite case, we have

$$f(\vec{x}) = \frac{1}{m} \sum_{k=1}^{m} f_0(\vec{x} - \vec{x_k}), \tag{42}$$

which can be generalized to

$$f(\vec{x}) = \mathop{\mathrm{E}}_{\vec{y}\sim\mathrm{U}(\mathcal{F})} f_0(\vec{x} - \vec{y}). \tag{43}$$

Therefore,

$$f^2(\vec{x}) = \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} f_0(\vec{x}-\vec{y})\cdot f_0(\vec{x}-\vec{z}) \tag{44}$$

and

$$J = \int_{\mathbb{R}^n} f^2(\vec{x})\mathrm{d}\vec{x} = \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} Z(\vec{y},\vec{z}) \tag{45}$$

where

$$Z(\vec{y},\vec{z}) = \int_{\mathbb{R}^n} f_0(\vec{x}-\vec{y})\cdot f_0(\vec{x}-\vec{z})\mathrm{d}\vec{x}. \tag{46}$$

Denoting the distance $d = \|\vec{y}-\vec{z}\|_2$, we can evaluate

$$\begin{aligned} Z(\vec{y},\vec{z}) &= V_n^{-2}\epsilon^{-2n}\int_{\mathbb{R}^n} \mathrm{I}(\|\vec{x}-\vec{y}\|_2 \le \epsilon)\mathrm{I}(\|\vec{x}-\vec{z}\|_2 \le \epsilon)\mathrm{d}\vec{x} \\ &= V_n^{-2}\epsilon^{-2n} W_n(d,\epsilon) \end{aligned} \tag{47}$$

where $W_n(d, \epsilon)$ is the volume of two hyper-ball intersections in the case of the mutual center distance d and radii ϵ. Using n-dimensional analytical geometry, we obtain

$$\begin{aligned} W_n(d,\epsilon) &= 2\int_{d/2}^{\epsilon} V_{n-1}\left(\epsilon^2 - r^2\right)^{\frac{n-1}{2}}\mathrm{d}r \\ &= 2V_{n-1}\epsilon^n \int_{d/2\epsilon}^{1}\left(1-r^2\right)^{\frac{n-1}{2}}\mathrm{d}r \end{aligned} \tag{48}$$

and after substitution $r = \cos\phi$, we get

$$W_n(d,\epsilon) = 2V_{n-1}\epsilon^n \int_0^{\arccos(d/2\epsilon)} \sin^n\phi \;\; \mathrm{d}\phi. \tag{49}$$

Moreover,

$$W_n(0,\epsilon) = 2V_{n-1}\epsilon^n \int_0^{\pi/2} \sin^n\phi \;\; \mathrm{d}\phi = V_n\cdot\epsilon^n \tag{50}$$

which is also the volume of the n-dimensional ball of radius ϵ. Therefore, we can express the Z function as

$$Z(\vec{y},\vec{z}) = V_n^{-1}\epsilon^{-n}\frac{\int_0^{\arccos(d/2\epsilon)}\sin^n\phi \;\; \mathrm{d}\phi}{\int_0^{\pi/2}\sin^n\phi \;\; \mathrm{d}\phi} \tag{51}$$

and the entropy is

$$H_2^*(\epsilon) = -\ln \mathop{\mathrm{E}}_{\vec{x},\vec{y}\sim \mathrm{U}(\mathcal{F})} S_n(\|\vec{x}-\vec{y}\|_2,\epsilon) \tag{52}$$

where

$$S_n(d,\epsilon) = \frac{\int_0^{\arccos(d/2\epsilon)}\sin^n\phi \;\; \mathrm{d}\phi}{\int_0^{\pi/2}\sin^n(\phi)\mathrm{d}\phi} \tag{53}$$

for $0 \le d < 2\epsilon$ and $S_n(d,\epsilon) = 0$ for $d \ge 2\epsilon$. Let

$$Q_n = \frac{\int_0^{\pi/3}\sin^n\phi \;\; \mathrm{d}\phi}{\int_0^{\pi/2}\sin^n\phi \;\; \mathrm{d}\phi} \in (0,1) \tag{54}$$

be the value of $S_n(\epsilon, \epsilon)$. When $0 \le d \le \epsilon$, we can estimate the ratio as

$$Q_n \le S_n(d,\epsilon) \le 1. \tag{55}$$

For $\epsilon < d \le 2\epsilon$, we have $0 \le S_n(d, \epsilon) < Q_n$. Therefore, we can underestimate

$$S_n(d,\epsilon) \ge \mathrm{I}(d \le \epsilon)\cdot Q_n \tag{56}$$

and an adequate upper estimate is

$$S_n(d,\epsilon) \le \mathrm{I}(d\le\epsilon) + (\mathrm{I}(d \le 2\epsilon) - \mathrm{I}(d\le\epsilon))\cdot Q_n \tag{57}$$

$$\begin{aligned} &(1-Q_n)\cdot\mathrm{I}(d\le\epsilon) + Q_n\cdot\mathrm{I}(d\le 2\epsilon) \le (1-Q_n)\cdot\mathrm{I}(d\le 2\epsilon) \\ &\quad + Q_n\cdot\mathrm{I}(d\le 2\epsilon) = \mathrm{I}(d\le 2\epsilon). \end{aligned} \tag{58}$$

We can continue in the estimation to obtain

$$\begin{aligned} &\mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} \mathrm{I}(\|\vec{y}-\vec{z}\|_2 \le \epsilon)\cdot Q_n \le \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} S_n(\|\vec{y}-\vec{z}\|_2,\epsilon) \\ &\quad \le \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} \mathrm{I}(\|\vec{y}-\vec{z}\|_2 \le 2\epsilon) \end{aligned} \tag{59}$$

and therefore

$$L(\epsilon) \le \mathop{\mathrm{E}}_{\vec{y},\vec{z}\sim \mathrm{U}(\mathcal{F})} S_n(\|\vec{y}-\vec{z}\|_2,\epsilon) \le U(\epsilon) \tag{60}$$

where the lower bound equals

$$L(\epsilon) = Q_n\cdot C(\epsilon) \tag{61}$$

and the appropriate upper bound equals

$$U(\epsilon) = C(2\epsilon). \tag{62}$$

For all $0 < \epsilon < 1$, the following inequalities hold

$$\frac{\ln U(\epsilon)}{\ln\epsilon} \le \frac{H_2^*(\epsilon)}{-\ln\epsilon} \le \frac{\ln L(\epsilon)}{\ln\epsilon}. \tag{63}$$

We can calculate

$$\lim_{\epsilon\to 0^+}\frac{\ln L(\epsilon)}{\ln\epsilon} = \lim_{\epsilon\to 0^+}\left(\frac{\ln Q_n}{\ln\epsilon} + \frac{\ln C(\epsilon)}{\ln\epsilon}\right) = D_2 \tag{64}$$

and also for the upper bound

$$\begin{aligned} &\lim_{\epsilon\to 0^+}\frac{\ln U(\epsilon)}{\ln\epsilon} = \lim_{\epsilon\to 0^+}\frac{\ln C(2\epsilon)}{\ln\epsilon} = \lim_{\epsilon\to 0^+}\frac{\ln C(\epsilon)}{\ln\frac{\epsilon}{2}} \\ &= \lim_{\epsilon\to 0^+}\frac{\ln C(\epsilon)}{\ln\epsilon}\cdot\lim_{\epsilon\to 0^+}\frac{\ln\epsilon}{\ln\epsilon - \ln 2} = D_2. \end{aligned} \tag{65}$$

Therefore,

$$D_2^* = \lim_{\epsilon\to 0^+}\frac{H_2^*(\epsilon)}{-\ln\epsilon} = D_2. \tag{66}$$

As a conclusion, when D_0, D_2 exist for a given set $\mathcal{F}$, the equalities $D_0^* = D_0$, $D_2^* = D_2$ have been proven.

7. Experimental part

Computer experiments can be realized only on the finite sample Φ with three aims:

- verify hypothesis $H_0 : \widehat{D}_0^* = D_0$ experimentally,
- verify hypothesis $H_0 : \widehat{D}_2^* = D_2$ experimentally,
- evaluate $\widehat{D}_\alpha^*$ in other cases where D_α is known theoretically or is referenced.

The estimation of the Renyi dimension D_α^* will be performed for $\alpha \in [0;\, 2]$. Supposing the model (30) with additional Gaussian noise $e \sim \mathcal{N}(0;\sigma^2)$ in the form

$$H_\alpha^* = A - D_\alpha^*\ln\epsilon + e \tag{67}$$

we can use the least squares method for the D_α^* estimation using different values ϵ_i for $i = 1, \ldots, N$. We suggest to use a geometrically increasing series of ϵ_i generated by the formula

$$\epsilon_i = 10^{f_{\min}+(i-1)\Delta f} \tag{68}$$

with $N = \lfloor (f_{\max} - f_{\min})/\Delta f\rfloor + 1$.

The novel algorithm was tested on sets with known capacity dimensions. Four traditional deterministic fractal sets were studied using recursive random point generation of depth 100:

- Cantor set [1] with the contraction parameter $0 < a < 1/2$ and $n = 1$ with the Hausdorff dimension

$$D_{\mathrm{H}} = -\frac{\ln 2}{\ln a} \tag{69}$$

Table 1
Capacity dimension estimation.

System	a	D_0	$\widehat{D_0^*}$	sd	p-value	$f_{\min}$	$f_{\max}$
Cantor set	1/4	0.5000	0.5059	0.0070	0.3993	−3.0	−1.0
Cantor set	1/3	0.6309	0.6327	0.0034	0.5965	−3.0	−1.0
Cantor dust	1/4	1.0000	0.9834	0.0157	0.2937	−2.0	0.0
Cantor dust	1/3	1.2619	1.2547	0.0133	0.5883	−2.0	0.0
Even numbers set	–	0.6990	0.7030	0.0148	0.7870	−4.0	−1.0
Sierpinki carpet	1/3	1.8928	1.8894	0.0059	0.2843	−2.0	−1.0
Sierpinki carpet	1/4	1.5000	1.4901	0.0148	0.2514	−2.0	−1.0

Table 2
Correlation dimension estimation.

System	a	D_2	$\widehat{D_2^*}$	sd	p-value	$f_{\min}$	$f_{\max}$
Cantor set	1/4	0.500	0.4974	0.0034	0.2236	−3.0	−1.0
Cantor set	1/3	0.6309	0.6286	0.0047	0.3124	−3.0	−1.0
Cantor dust	1/4	1.0000	0.9863	0.0221	0.2676	−2.0	−1.0
Cantor dust	1/3	1.2619	1.2630	0.0269	0.4840	−2.0	0.0
Even numbers set	–	0.6990	0.6991	0.0038	0.4896	−4.0	−1.0
Sierpinki carpet	1/3	1.8928	1.8964	0.0083	0.3325	−2.0	−1.0
Sierpinki carpet	1/4	1.5000	1.5053	0.0064	0.2032	−2.0	−1.0

- Cantor dust [21] with the contraction parameter $0 < a < 1/2$ and $n = 2$ with the Hausdorff dimension

$$D_H = -2\frac{\ln 2}{\ln a} \tag{70}$$

- Even the digits set [1] that contains numbers from (0, 1) with even digits and the Hausdorff dimension

$$D_H = -\frac{\log 5}{\log 10} \tag{71}$$

- Sierpinski carpet [1] with the contraction parameter $0 < a < 1/2$ and $n = 2$ with the Hausdorff dimension

$$D_H = -\frac{\ln 8}{\ln a} \tag{72}$$

Since all the mentioned sets are self-similar and fulfil the open set condition [22], their Hausdorff dimension equals the Renyi dimension for any eligible parameter α, e.g. $D_H = D_0 = D_2$. The results of the capacity dimension estimation are provided in Table 1 and the estimates in the case of the correlation dimension are in Table 2 for $\Delta f = 0.05$ and $M = 10^5$. The theoretical capacity (correlation) dimension is denoted D_0 (D_2), whereas its estimate is $\widehat{D_0^*}$ ($\widehat{D_2^*}$) together with its standard deviation sd. The range for the choice of $\ln \epsilon$ is recommended to be in the interval $[f_{\min}; f_{\max}]$.

A one-sample, two-sided t-test has been used to prove the unbiasedness of the dimension estimates level 0.05. As seen in Tabs. 1 and 2 the hypotheses $H_0 : \widehat{D}_\alpha^* = D_\alpha$ have been accepted in all cases.

The graph of De Wijs's fractal [23] with the parameter a is a kind of multifractal that has the Renyi dimension dependent on the dimension parameter α. The corresponding Renyi dimension equals

$$D_\alpha = \frac{1}{1-\alpha}\log_2\left(a^\alpha + (1-a)^\alpha\right) \tag{73}$$

for $0 < a < 1/2$ and $\alpha \in [0; 1) \cup (1, \infty)$ with the particular case

$$D_1 = \lim_{\alpha\to 1} D_\alpha = -a\log_2 a - (1-a)\log_2(1-a). \tag{74}$$

The D_α^* has been estimated for $\alpha \in \{0, 1/2, 1, 3/2, 2\}$ and the testing results are included in Table 3.

Table 3
De Wijs's fractal dimensions.

α	a	D_α	$\widehat{D_\alpha^*}$	sd	p-value	$f_{\min}$	$f_{\max}$
0	$a = 1/3$	1.0000	0.9908	0.0058	0.1127	−6.0	−4.0
0	$a = 1/4$	1.0000	0.9890	0.0087	0.2062	−6.0	−4.0
0	$a = 1/6$	1.0000	0.9780	0.0143	0.1240	−6.0	−4.0
1/2	$a = 1/3$	0.9581	0.9574	0.0062	0.4550	−5.5	−3.5
1/2	$a = 1/4$	0.9000	0.8921	0.0103	0.2215	−5.5	−3.5
1/2	$a = 1/6$	0.8035	0.7895	0.0159	0.1893	−5.0	−3.0
1	$a = 1/3$	0.9183	0.9158	0.0060	0.6769	−4.0	−2.0
1	$a = 1/4$	0.8250	0.8259	0.0098	0.9269	−4.0	−2.0
1	$a = 1/6$	0.6500	0.6387	0.0217	0.6026	−3.0	−1.0
3/2	$a = 1/3$	0.8814	0.8749	0.0099	0.2557	−4.0	−2.0
3/2	$a = 1/4$	0.7376	0.7255	0.0153	0.2145	−4.0	−2.0
3/2	$a = 1/6$	0.5419	0.5234	0.0209	0.1880	−3.0	−1.0
2	$a = 1/3$	0.8480	0.8359	0.0189	0.5220	−3.0	−1.0
2	$a = 1/4$	0.6781	0.6687	0.0205	0.6466	−3.0	−1.0
2	$a = 1/6$	0.4695	0.4552	0.0235	0.5429	−2.0	0.0

The one-sample, two-sided t-test has been also used to prove the unbiasedness of the De Wijs's fractal, the hypotheses $H_0 : \widehat{D}_\alpha^* = D_\alpha$ have been again accepted in all cases.

One of the traditional methods on how to estimate the capacity dimension D_0 is called box-counting [24]. It is based on counting points from the sample Φ using an n-dimensional rectangular grid of size $a > 0$. Using the grid, there are always k non-empty boxes consisting of $M_1, M_2, \dots, M_k \in \mathbb{N}$ points satisfying $\sum_{j=1}^k M_j = M$. The basic form of box-counting calculates the Hartley entropy estimate according to (1) as $\widehat{H_0}(\alpha) = \ln k$ which is the logarithm of covering an element number. The box-counting estimate of D_0 is obtained from the model (67). It is also possible to estimate the general Renyi entropy D_α using the approximation $p_j = M_j/M$ and formulas (2) and (1).

Discrete dynamic systems with chaotic behaviour generate fractal trajectories and attractors with a nonlinear character. The investigation of this kind of sets can be performed in two ways – the first option is to investigate the dimension in the original state space, the second option is to use Whitney's theorem [25] and estimate it in a reconstructed space. Generally, the n-dimensional discrete dynamical process has an internal state $\vec{x}_j \in \mathbb{R}^n$ and output $y_j \in \mathbb{R}$ for $j \in \mathbb{N}_0$. Using reconstruction length $W \in \mathbb{N}$, we define a sliding sample $\vec{\xi}_j = (y_j, \dots, y_{j+W-1}) \in \mathbb{R}^W$ for $j \in \mathbb{N}_0$, first. Whitney's embedding theorem can be rewritten from continuous to discrete time as follows: When $W \geq 2N + 1$, then the reconstructed series $\{\vec{\xi}_j\}_{j=0}^\infty$ has the same properties as $\{\vec{x}_j\}_{j=0}^\infty$. Therefore, any Renyi dimension D_α of the reconstructed attractor is the same as in the case of the state space.

Table 4 shows the comparison of the dimension estimation using the box-counting (denoted as box-count) method and the new method of the modified Renyi entropy (denoted as m. Renyi).

Table 4
Discrete dynamical system analysis.

system	α	D_α	data	method	$\widehat{D_\alpha}$	sd	p-value
Henon map	1	1.2583	OD	m. Renyi	1.2608	0.0156	**0.4363**
				box-count	1.2428	0.0113	**0.0851**
			RD	m. Renyi	1.2590	0.0056	**0.4503**
				box-count	1.2489	0.0031	0.0012
Henon map	2	1.2201	OD	m. Renyi	1.2243	0.0174	**0.4046**
				box-count	1.2161	0.0109	**0.3568**
			RD	m. Renyi	1.2230	0.0026	**0.1323**
				box-count	1.2172	0.0014	0.0192
Lozi map	1	1.4042	OD	m. Renyi	1.4131	0.0197	**0.3257**
				box-count	1.3915	0.0174	**0.2327**
			RD	m. Renyi	1.4098	0.0044	**0.1016**
				box-count	1.3945	0.0032	0.0012
Lozi map	2	1.3845	OD	m. Renyi	1.3937	0.0144	**0.2614**
				box-count	1.3786	0.0161	**0.3570**
			RD	m. Renyi	1.3885	0.0031	**0.0985**
				box-count	1.3749	0.0041	0.0096

The comparison has been performed for original state data (OD) and reconstructed data (RD). For the experiment, the Henon map [26,27] with the parameters $a = 0.4,\ b = 0.3$ and the starting points $x_{0,1} = 0, x_{0,2} = 0.9$ and the Lozi map [28,29] with the parameters $a = 1.7,\ b = 0.5$ and the starting points $x_{0,1} = -0.1, x_{0,2} = 0.1$ were used for the simulation for $\alpha \in \{1,\ 2\}$, reconstruction length $W = 5$, range for modified Renyi method as $f_{\min} = -2.0$ and $f_{\max} = -2.0$ and for all experiments, the set Φ contained $M = 10^6$ elements. The experiment was also conducted for bigger lengths of the reconstruction window, but it didn't have a significant impact on the results and their precision.

When the systems are investigated in the state space (OD) of low dimension ($n = 2$), the box-counting offered more accurate estimates with smaller standard deviation than the novel method. However, the p-values indicate unbiasedness in both cases. Another behaviour of estimation methods has been observed in the case of state reconstruction (RD) when the space dimension is large ($n = 5$). Therefore, the box-counting estimates of event probabilities are biased due to data sparsity. As seen in Table 4, all the box-counting estimates after reconstruction are biased. The sparsity effect is not present in the case of new method, where the p-values are higher with similar standard deviation. Therefore, the modified Renyi dimension is more suitable for reconstructed systems in higher-dimensional space, where the unbiasedness is present and the estimation accuracy is higher.

8. Conclusion

The paper presents new term modified entropy that has been defined using the Parzen formula with a ball kernel. The new entropy measure can be calculated for all finite samples Φ using a degeneracy function. The Monte Carlo approach enables the estimation of the proposed modified entropy which is later useful for the dimension estimation. The relationship between D_0, D_2 and their estimates from the modified entropy have been both theoretically and numerically proven for an arbitrary measurable set $\mathcal{F}$. Moreover, numerical simulations on selected fractal sets verified the unbiasedness of the D^*_α estimates.

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References

[1] Falconer K. Fractal geometry: mathematical foundations and applications. Wiley; 2014. ISBN 111994239X.
[2] Kirkby MJ. The fractal geometry of nature. Earth Surf Processes Landforms 1983;8(4) 406–406. doi:10.1002/esp.3290080415.
[3] Edgar G. Measure, topology, and fractal geometry (undergraduate texts in mathematics). Springer; 2007. ISBN 0387747486.
[4] Renyi A. Probability theory (north-Holland series in applied mathematics AND mechanics). Elsevier Science Publishing Co Inc.,U.S.; 1970. ISBN 0720423600.
[5] Sarkar N, Chaudhuri BB. An efficient differential box-counting approach to compute fractal dimension of image. IEEE Trans Syst Man Cybern 1994;24(1):115–20. doi:10.1109/21.259692.
[6] Grassberger P, Procaccia I. Measuring the strangeness of strange attractors. Physica D 1983;9(1–2):189–208. doi:10.1016/0167-2789(83)90298-1.
[7] Grassberger P, Procaccia I. Characterization of strange attractors. Phys Rev Lett 1983;50(5):346–9. doi:10.1103/physrevlett.50.346.
[8] Dlask M, Kukal J. Application of rotational spectrum for correlation dimension estimation. Chaos Solitons Fractals 2017;99:256–62. doi:10.1016/j.chaos.2017.04.026.
[9] Grassberger P. Finite sample corrections to entropy and dimension estimates. Phys Lett A 1988;128(6–7):369–73. doi:10.1016/0375-9601(88)90193-4.
[10] Smith LA. Intrinsic limits on dimension calculations. Phys Lett A 1988;133(6):283–8. doi:10.1016/0375-9601(88)90445-8.
[11] Yan J, Sun YY, Cai SS, Hu XP. An improved box-counting method to estimate fractal dimension of images. J Appl Anal Comput 2016;6(4):1114–25. doi:10.11948/2016073.
[12] Liu Y, Chen L, Wang H, Jiang L, Zhang Y, Zhao J, Wang D, Zhao Y, Song Y. An improved differential box-counting method to estimate fractal dimensions of gray-level images. J Vis Commun Image Represent 2014;25(5):1102–11. doi:10.1016/j.jvcir.2014.03.008.
[13] Sadowski W. A remark on the box-counting dimension of the singular set for the Navier-Stokes equations. Commun Math Sci 2013;11(2):597–602.
[14] Wang Y, Wu G. On the box-counting dimension of the potential singular set for suitable weak solutions to the 3d Navier–Stokes equations. Nonlinearity 2017;30(5):1762–72. doi:10.1088/1361-6544/aa6444.
[15] Shayegh F, Sadri S, Amirfattahi R, Ansari-Asl K. A model-based method for computation of correlation dimension, lyapunov exponents and synchronization from depth-EEG signals. Comput Methods Programs Biomed 2014;113(1):323–37. doi:10.1016/j.cmpb.2013.08.014.
[16] Rawal K, Saini BS, Saini I. Adaptive correlation dimension method for analysing heart rate variability during the menstrual cycle. Australas. Physical Eng Sci Med 2015;38(3):509–23. doi:10.1007/s13246-015-0369-y.
[17] Kalauzi A, Vuckovic A, Bojić T. Topographic distribution of EEG alpha attractor correlation dimension values in wake and drowsy states in humans. Int J Psychophysiol 2015;95(3):278–91. doi:10.1016/j.ijpsycho.2014.11.008.
[18] Nie C-X. Correlation dimension of financial market. Physica A 2017;473:632–9. doi:10.1016/j.physa.2017.01.045.
[19] Yue-chun Z, Guang-shui L. The research on determinants of capital structure based on fractal dimensionality reduction. In: Hua L, editor. 2012 International conference on management science and engineering. International Conference on Management Science and Engineering-Annual Conference Proceedings; IEEE; Univ Texas, Sch Management; Harbin Inst Technol, Sch Management; IEEE Technol Management Council; Natl Nat Sci Fdn China (NSFC); Harbin Inst Technol; 2012. p. 1553–9. ISBN 978-1-4673-3015-2; 978-1-4673-3014-5; 19th Annual International Conference on Management Science and Engineering (ICMSE), Dallas, TX, SEP 20–22, 2012.
[20] Parzen E. On estimation of a probability density function and mode. Annals Math Stat 1962;33(3):1065–76. doi:10.1214/aoms/1177704472.
[21] OLSON EJ, ROBINSON JC, SHARPLES N. Generalised cantor sets and the dimension of products. Math Proc Cambridge Philos Soc 2015;160(01):51–75. doi:10.1017/s0305004115000584.
[22] Bandt C, Hung NV, Rao H. On the open set condition for self-similar fractals. Proc Am Math Soc 2005;134(05):1369–74. doi:10.1090/s0002-9939-05-08300-0.
[23] Takayasu H. Fractals in the physical sciences (nonlinear science). Manchester Univ Pr; 1990. ISBN 0719024854.
[24] Rajković N, Krstonošić B, Milošević N. Box-counting method of 2d neuronal image: method modification and quantitative analysis demonstrated on images from the monkey and human brain. Comput Math Methods Med 2017;2017:1–9. doi:10.1155/2017/8967902.
[25] Whitney H. Differentiable manifolds. Annals Math 1936;37(3):645. doi:10.2307/1968482.
[26] Cao Y-L. Strange attractor of henon map and its basin. Science in China. Series A 1995;38.
[27] Hénon M. A two-dimensional mapping with a strange attractor. Comm Math Phys 1976;50(1):69–77.
[28] Misiurewicz M, Štimac S. Symbolic dynamics for lozi maps. Nonlinearity 2016;29(10):3031–46. doi:10.1088/0951-7715/29/10/3031.
[29] Lozi R. Un attracteur etrange du type attracteur de henon. Le J de Physique Colloques 1978;39(C5) C5-9–C5-10. doi:10.1051/jphyscol:1978505.