Study of hadron structure using angular-dependent Balitsky-Kovchegov equation

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Preface

What I want:

- Give you some insight into the theory behind what I (and some of you) do
- Show you something interresting about the structure of hadrons
- Grant money and coke*

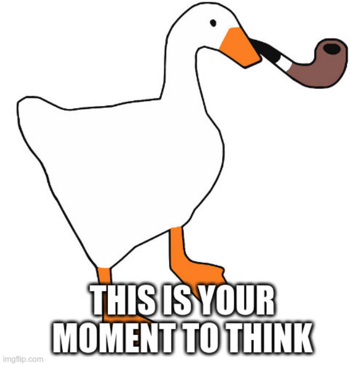
What I do not want:

- Bore you with unnecessary equations
  (I will ommit some technical details in order to give you a general idea)
- To have my sanity questioned

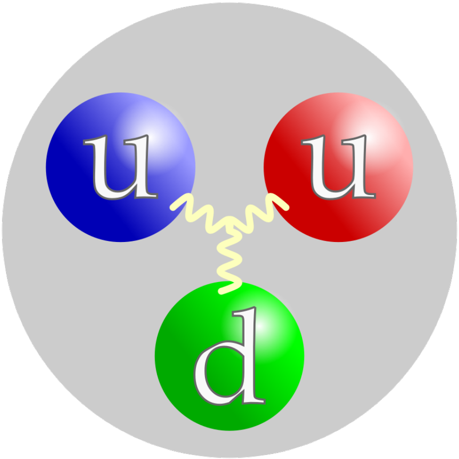
*Kofola is also satisfactory

Quiztime

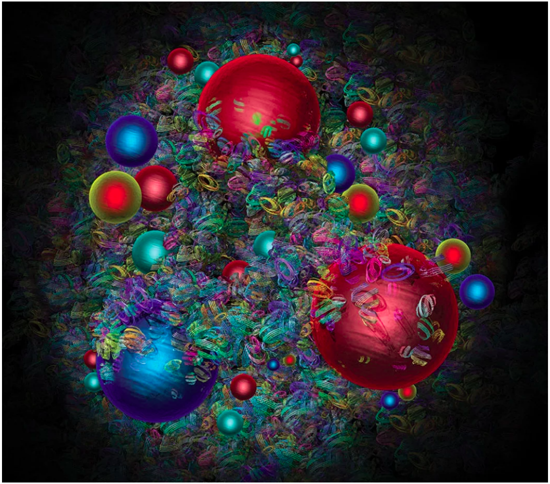
- I will show 2 pictures
- Try to identify the object



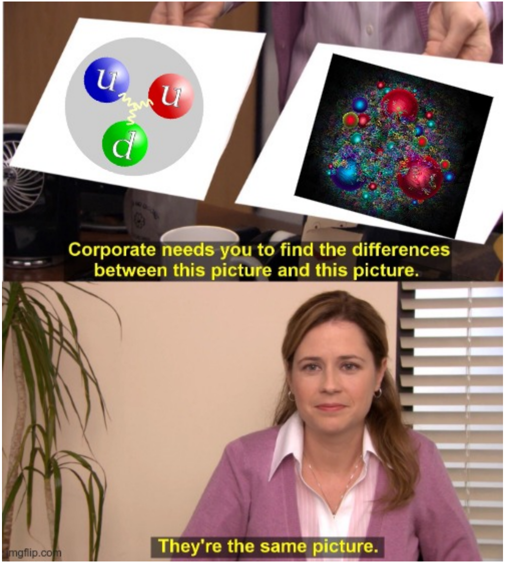
Picture 1



Picture 2



Good job!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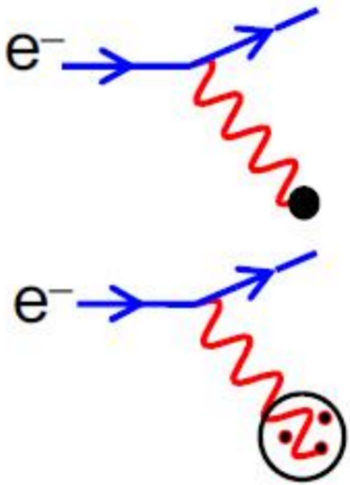
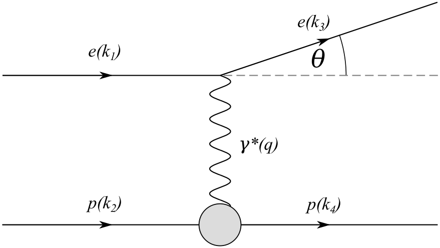
u
d
Corporate needs you to find the differences between this picture and this picture.
They're the same picture.
imgflip.com

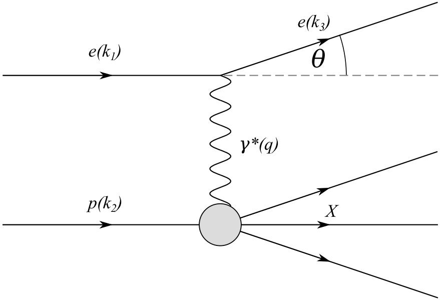
Overview of scattering experiments

- $e^- + p \to e^- + p$ (elastic)
- $e^- + p \to e^- + X$ (inelastic)



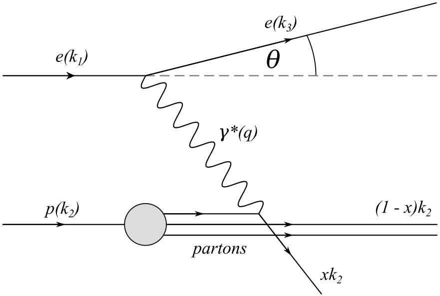
Overview of scattering experiments





Parton model

- Hadrons consist of smaller particles (partons)
- Each parton carries a fraction of the total momentum
- New variable: Bjorken-x
- Also: Virtuality $Q^2 = -q^2$



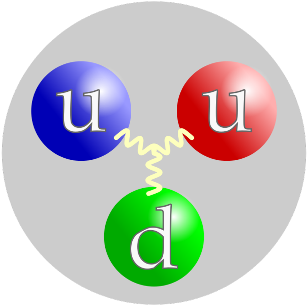
How to quantify the momentum distribution between partons?

- Parton distribution functions (PDFs): $q(x)$ dx
- ... the number of quarks of a flavour ”q” inside the proton, which carry the momentum in the interval $(x, x + dx)$

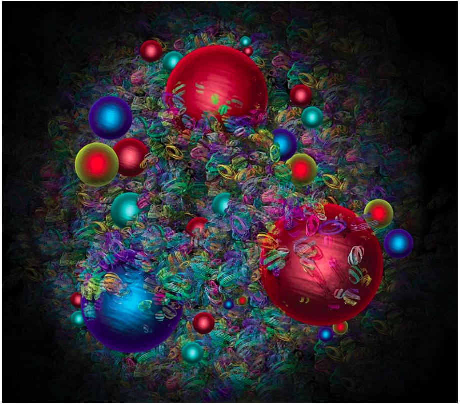
- Structure functions: $F_1(x, Q^2)$ and $F_2(x, Q^2)$
- $F_2(x, Q^2) = 2xF_1(x, Q^2) = x \sum_q e_q^2 q(x)$
- Can be measured in DIS experiments

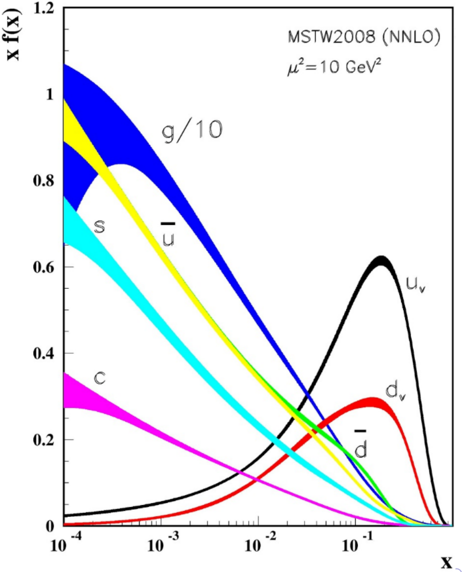
Parton distribution functions

- We can calculate this for the proton: $p = u + u + d$, right?
- Results in only $\sim$50% of the total proton momentum...
- This implies the existence of a sea of partons with small relative momentum fractions



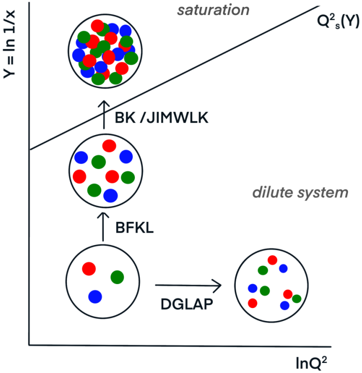
Parton distribution functions





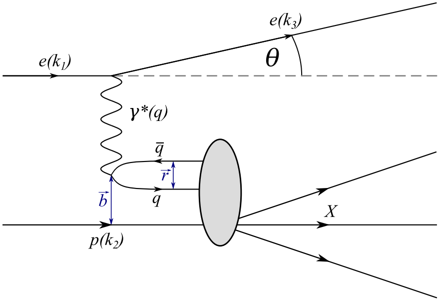
How do we make predictions about the PDFs?

- PDFs are functions of x and Q^2
- To see, how PDFs change, we use evolution equations
- Saturation region



Color dipole model

- Formulated in the Color Glass Condensate (CGC) model
- Helps bring the strong interaction into DIS
- Allows us to make accurate predictions in the saturation region



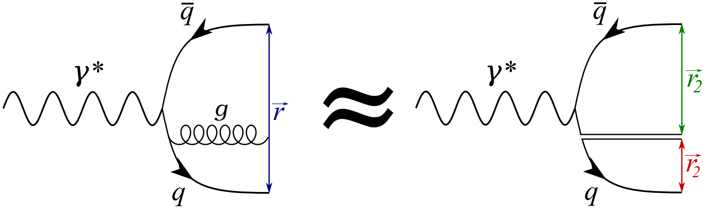
Two effects pressent

Gluon emission

- Linear growth
- Dominates large-x region

Gluon recombination

- Non-linear effect
- Insignificant in the large-x region



Parton saturation

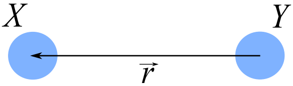
- Gluons can recombine
- The low-x (high density) region is dominated by gluons
- At a certain density, the rate of emission and the rate of recombination are in balance
  $\rightarrow$ *saturation*

Balitsky-Kovchegov equation at leading order

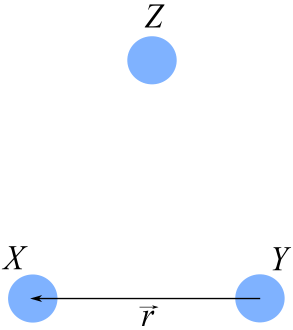
$$\frac{\partial N(\vec{r},\vec{b},Y)}{\partial Y} = \int d^2\vec{r_1} K(\vec{r},\vec{r_1},\vec{r_2})$$
$$\cdot \left[N(\vec{r_1},\vec{b_1},Y) + N(\vec{r_2},\vec{b_2},Y) - N(\vec{r},\vec{b},Y) - N(\vec{r_1},\vec{b_1},Y)N(\vec{r_2},\vec{b_2},Y) \right]$$

- BK equation describes the evolution of scattering amplitudes of the dipole-proton interaction with respect to rapidity $Y = \ln \frac{x_0}{x}$
- Phenomenological interpretation
- Leading order: we assume, that the color dipole can emit 1 gluon at a time

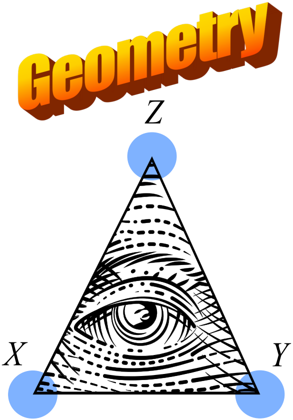
Gluon emission



Gluon emission

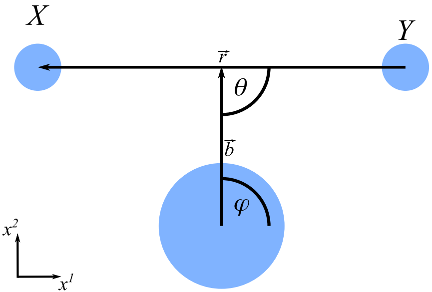


Gluon emission



Kinematics
- where I started to lose my mind (and hair)

- $\vec{x} = \begin{pmatrix} b\cos(\varphi) + \frac{r}{2}\cos(\theta+\varphi) \\ b\sin(\varphi) + \frac{r}{2}\sin(\theta+\varphi) \end{pmatrix}$
- $\vec{y} = \begin{pmatrix} b\cos(\varphi) - \frac{r}{2}\cos(\theta+\varphi) \\ b\sin(\varphi) - \frac{r}{2}\sin(\theta+\varphi) \end{pmatrix}$



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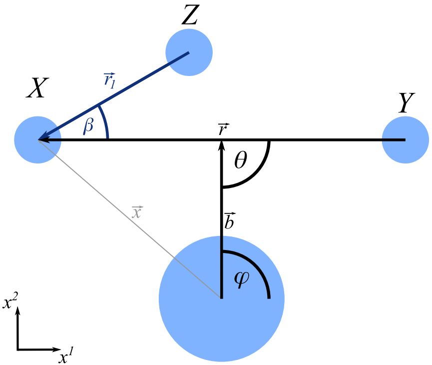
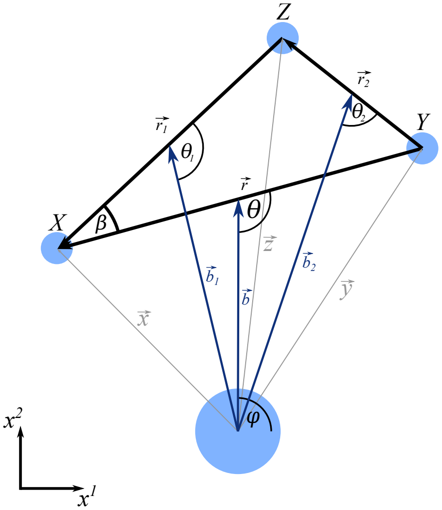
- $\vec{y} = \begin{pmatrix} b\cos(\varphi) - \frac{r}{2}\cos(\theta+\varphi) \\ b\sin(\varphi) - \frac{r}{2}\sin(\theta+\varphi) \end{pmatrix}$
- $\vec{z} = \begin{pmatrix} r_1\sin(\beta+\theta) \\ -r_1\cos(\beta+\theta) \end{pmatrix} + \vec{x}$



Kinematics
- where I started to lose my mind (and hair)

- $\vec{r} = \vec{x} - \vec{y}$
- $\vec{r_1} = \vec{x} - \vec{z}$
- $\vec{r_2} = \vec{z} - \vec{y}$
- $\vec{b} = \frac{1}{2}(\vec{x} + \vec{y})$
- $\vec{b_1} = \frac{1}{2}(\vec{x} + \vec{z})$
- $\vec{b_2} = \frac{1}{2}(\vec{z} + \vec{y})$

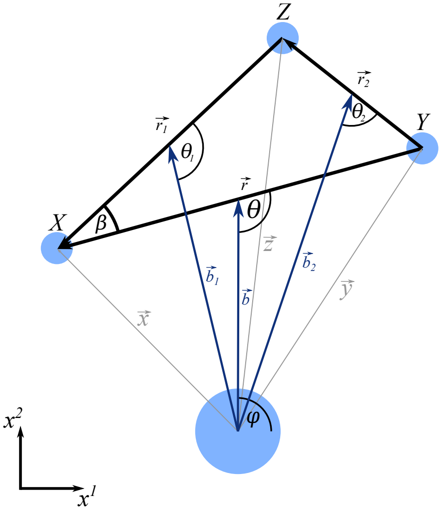
- $r_2 = \sqrt{r^2 + r_1^2 - 2rr_1 \cos\beta}$



Kinematics
- where I started to lose my mind (and hair)

Used this numpy function in python

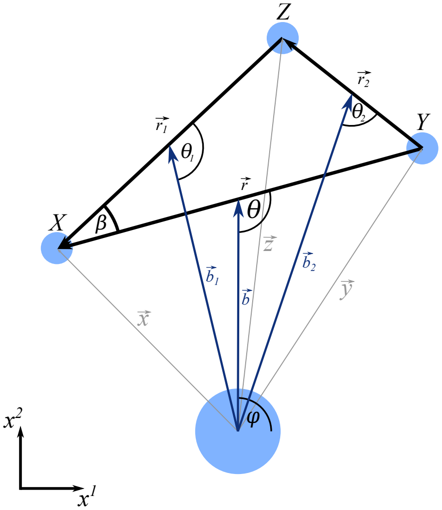
- $\varphi = \text{arctan2}(x_2 + y_2, x_1 + y_1)$
- $\varphi_1 = \text{arctan2}(x_2 + z_2, x_1 + z_1)$
- $\varphi_2 = \text{arctan2}(z_2 + y_2, z_1 + y_1)$
- $\theta = \text{arctan2}(x_2 - y_2, x_1 - y_1) - \varphi$
- $\theta_1 = \text{arctan2}(x_2 - z_2, x_1 - z_1) - \varphi_1$
- $\theta_2 = \text{arctan2}(z_2 - y_2, z_1 - y_1) - \varphi_2$



Kinematics
- where I started to lose my mind (and hair)

- $(\vec{r}, \vec{b}) \to (r, r_1, \beta, b, \theta, \varphi)$
- 1D: $N(r, r_1, \beta)$
- 2D: $N(r, r_1, \beta, b)$
- 3D: $N(r, r_1, \beta, b, \theta)$
- 4D: $N(r, r_1, \beta, b, \theta, \varphi)$

- Benefits of cartesian kinematics (simple, consistent, NLO)



Numerical solution to the BK equation

Things we need:

- Reformulate the equation for numerical methods
- Compute the evolution in rapidity (Bjorken-x) $Y = \ln \frac{x_0}{x}$
- Kernel $K(\vec{r}, \vec{r}_1, \vec{r}_2)$
- Initial conditions $N(\vec{r}, \vec{b}, Y)$
- Obtain scattering amplitudes at $\vec{r}_1, \vec{b}_1$ and $\vec{r}_2, \vec{b}_2$ (interpolation)

A slight modification

$$\frac{\partial N(\vec{r},\vec{b},\eta)}{\partial \eta} = \int d^2\vec{r_1} K(\vec{r},\vec{r_1},\vec{r_2})$$
$$\cdot \left[N(\vec{r_1},\vec{b_1},\eta_1) + N(\vec{r_2},\vec{b_2},\eta_2) - N(\vec{r},\vec{b},\eta) - N(\vec{r_1},\vec{b_1},\eta_1) N(\vec{r_2},\vec{b_2},\eta_2) \right]$$

$$\eta_1 = \eta - \max\left\{0, \ln\frac{r^2}{r_1^2}\right\}, \quad \eta_2 = \eta - \max\left\{0, \ln\frac{r^2}{r_2^2}\right\}$$

- Change from projectile rapidity Y to target rapidity η
- We need to interpolate in η as well

Reformulation

$$\frac{\partial N(\vec{r},\vec{b},\eta)}{\partial \eta} = f\left(N(\vec{r},\vec{b},\eta)\right) = I_1 - N(\vec{r},\vec{b},\eta) I_0 - I_2$$

$$I_0 \equiv \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2)$$

$$I_1 \equiv \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2) \left[N(\vec{r}_1,\vec{b}_1,\eta_1) + N(\vec{r}_2,\vec{b}_2,\eta_2)\right]$$

$$I_2 \equiv \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2) N(\vec{r}_1,\vec{b}_1,\eta_1) N(\vec{r}_2,\vec{b}_2,\eta_2)$$

Numerical integration

$$I_0 = \int d\beta \int r_1 dr_1 K(r, r_1, r_2)$$

$$I_1 = \int d\beta \int r_1 dr_1 K(r, r_1, r_2) \left[N(\vec{r_1}, \vec{b_1}, \eta_1) + N(\vec{r_2}, \vec{b_2}, \eta_2) \right]$$

$$I_2 = \int d\beta \int r_1 dr_1 K(r, r_1, r_2) N(\vec{r_1}, \vec{b_1}, \eta_1) N(\vec{r_2}, \vec{b_2}, \eta_2)$$

Evolution in rapidity

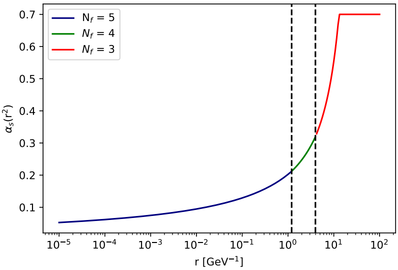
$$N(\vec{r},\vec{b},\eta+h) = N(\vec{r},\vec{b},\eta) + h\left(\frac{k_1}{6}+\frac{k_2}{3}+\frac{k_3}{3}+\frac{k_4}{6}\right)$$

$$\begin{aligned}
k_1 &= f\left(N(\vec{r},\vec{b},\eta)\right)\\
k_2 &= k_1 + \frac{h}{2}k_1 I_0 - \frac{h}{2}k_1 I_1 - \frac{h^2}{4}k_1 I_0\\
k_3 &= k_1 + \frac{h}{2}k_2 I_0 - \frac{h}{2}k_2 I_1 - \frac{h^2}{4}k_2 I_0\\
k_4 &= k_1 + \frac{h}{2}k_3 I_0 - \frac{h}{2}k_3 I_1 - \frac{h^2}{4}k_3 I_0
\end{aligned}$$

$$K(r, r_1, r_2) = \frac{\overline{\alpha}_s}{2\pi} \frac{r^2}{r_1^2 r_2^2} \left(\frac{r^2}{\min(r_1^2, r_2^2)} \right)^{\pm \overline{\alpha}_s A_1}$$

$$\alpha_s(r^2) = \frac{4\pi}{\left(11 - \frac{2}{3} N_f\right) \ln \left(\frac{4C^2}{r^2 \Lambda_{n_f}^2} \right)}$$

$$\overline{\alpha}_s = \frac{N_C}{\pi} \alpha_s(\min(r^2, r_1^2, r_2^2))$$



Initial conditions

$$N(\vec{r}, \vec{b}, Y_0 = 0) = 1 - \exp\left(-\frac{1}{2}\frac{Q_s^2}{4} r^2 T(b_{q_1}, b_{q_2})\right)$$

$$T(b_{q_1}, b_{q_2}) = \exp\left(-\frac{d_1^2}{2B}\right) + \exp\left(-\frac{d_2^2}{2B}\right)$$

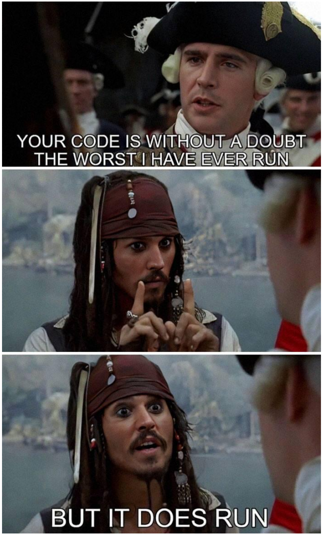
$$d_1^2 = \left(\frac{r}{2}\right)^2 + b^2 - rb\cos\theta$$

$$d_2^2 = \left(\frac{r}{2}\right)^2 + b^2 + rb\cos\theta$$

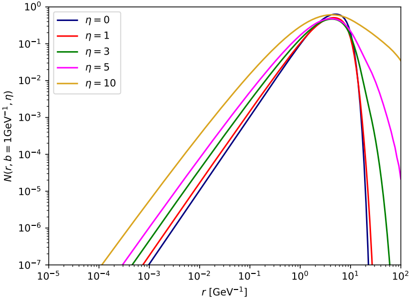
That is enough formulas for now

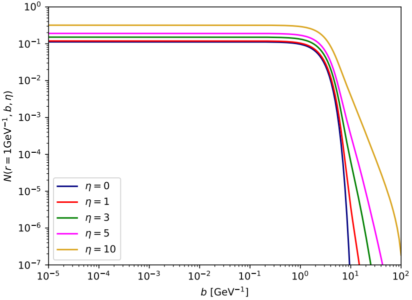
Results (the better looking ones)

- Solution on a grid of variables (r, r_1, β, b)
- $r, r_1, b \in (10^{-5}, 10^2)$ - log scale (183 points)
- $\beta \in (0, 2\pi)$ - lin scale (21 points)
- $\theta \in \{0, \frac{\pi}{2}\}$ - combination
- $\varphi = \frac{\pi}{2}$
- $x_0 = 0.008$
- $h = 0.01$
- RegularGridInterpolator and simpson from scipy module (python)



Scattering amplitudes

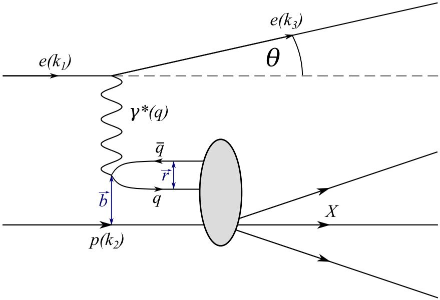




F_2 structure function

Things we need:

- Photon-dipole fluctuation amplitude
- Dipole-proton interaction cross-section
  $\rightarrow$ photon-proton cross-section
  $\rightarrow$ F_2



Wave function

$$|\Psi^*\Psi|_T^f = \frac{3\alpha_{em}}{2\pi^2} e_f^2 \left[\left(z^2 + (1 - z^2)\right) \epsilon^2 K_1^2(\epsilon r) + m_f^2 K_0^2(\epsilon r)\right]$$

$$|\Psi^*\Psi|_L^f = \frac{3\alpha_{em}}{2\pi^2} e_f^2 \left[\left(4Q^2 z^2 (1 - z^2)\right) K_0^2(\epsilon r)\right]$$

$$\epsilon_f^2 = z(1 - z)Q^2 + m_f$$

Proton-dipole cross-section

$$\sigma_{q\bar{q}}(\tilde{x}, \vec{r}) = 2 \int d^2 b N(\vec{r}, \vec{b}, \tilde{x})$$

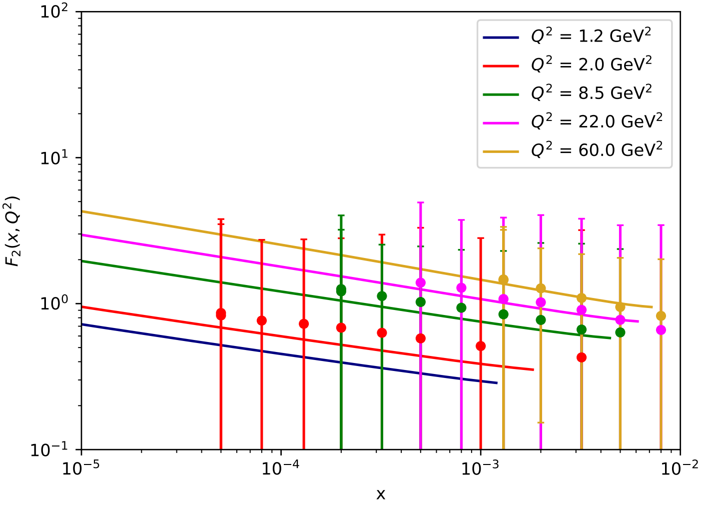
$$\tilde{x} \equiv x \left(1 + \frac{4m_f^2}{Q^2}\right)$$

F_2 structure function

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \left(\sigma_T^{\gamma^q p}(x, Q^2) + \sigma_L^{\gamma^q p}(x, Q^2) \right)$$

$$\sigma_{T,L}^{\gamma^* p}(x, Q^2) = \sum_f \int d\vec{r} \int dz |\Psi^* \Psi|_{T,L}^f \sigma_{q\bar{q}}$$

F_2 structure function



Wigner distribution W

- Also provide information about hadronic structure
- Spatial and momentum distribution of partons
- May not be measure directly
- Fourier transform of W can be obtained in diffractive dijet production in DIS experiments

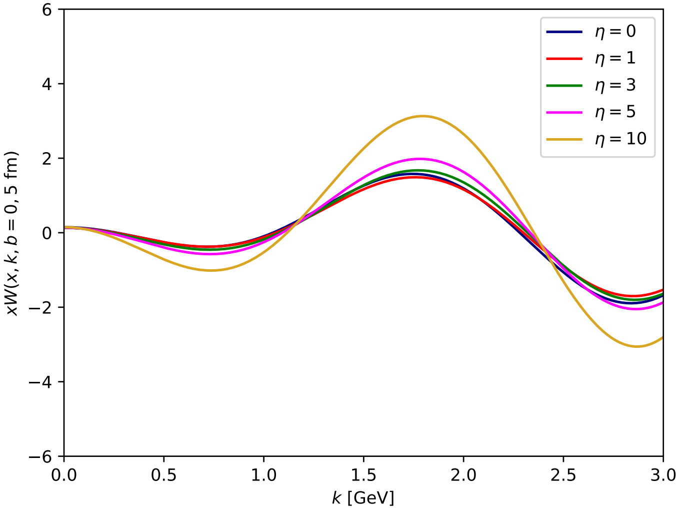
Wigner distribution W

- For the case of 2D BK

$$xW(x,k,b) = -\frac{N_C}{\alpha_s \pi} \int r dr J_0(rk) \left(\frac{1}{4} \nabla_b^2 + k^2 \right) N(x,r,b)$$

$$\nabla_b^2 = \partial_b^2 + \frac{1}{b} \partial_b, \quad \alpha_s = 0.3$$

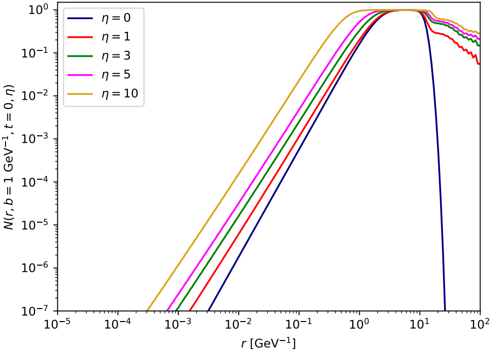
Wigner distribution W

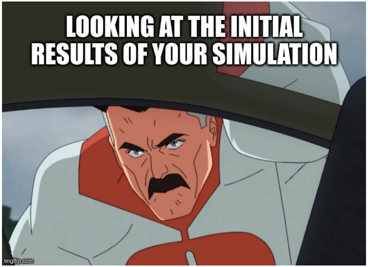


What about 3D BK?

- $N(r, r_1, \beta, b) \to N(r, r_1, \beta, b, \theta)$
- Interpolation over one more varible θ
- ”In theory” should be straightforward
- In practice simulations do what you tell them to do, rarely what you actually want them to do...

The "in theory" part





Conclusions

What I have done:

- Discussed DIS and the models used in calculations
- Presented the numerical solution to the BK equation
- Used the 2D results to calculate F_2 and W

What comes next:

- Resolve issues with 3D BK
- Apply the results on F_2 and W
- Compare 3D and 2D results