Study of the Dipole Scattering Amplitude using the Balitsky-Kovchegov evolution equation

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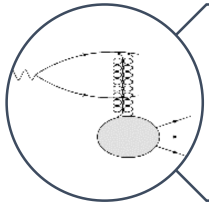
Outline



The rcBK equation

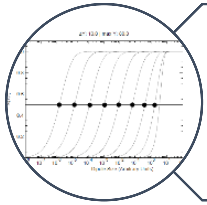
Precision

Running time

Optimal setup

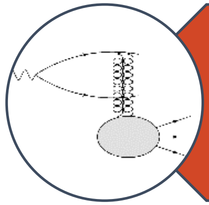
F_2

Impact parameter independent solutions



Geometric scaling

Introduction and the rcBK equation



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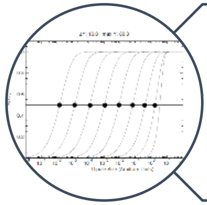


Geometric scaling

Introduction

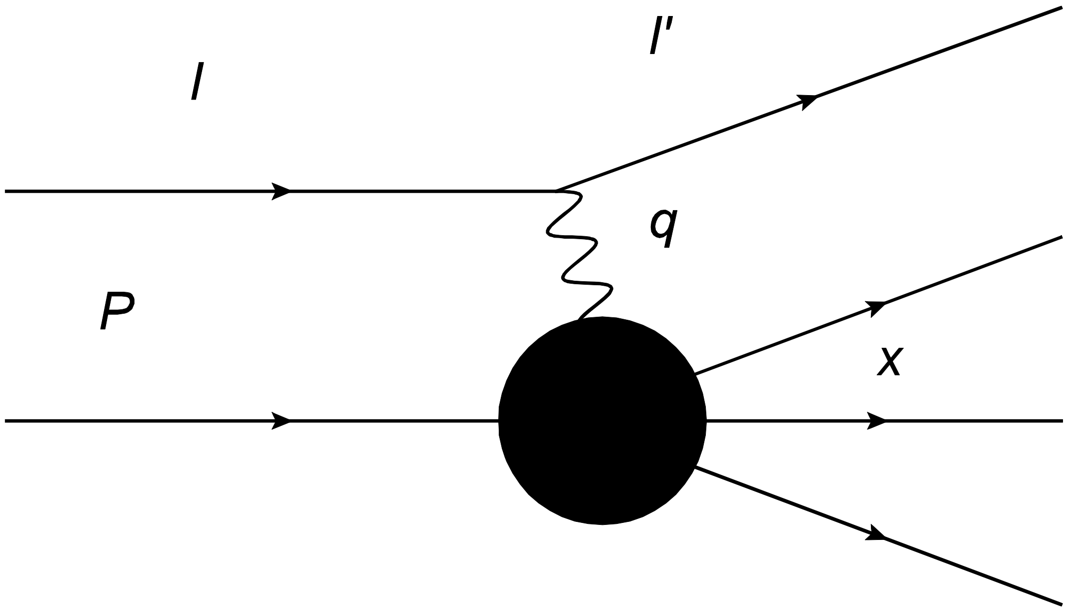
- One of the most important ways of studying **the inner structure of protons** is the **electron-proton scattering** at large accelerators (HERA-Germany).

- This **inner structure** can be composed of **quarks, anti-quarks and gluons** and can be parametrized as the **protons structure function** $\boldsymbol{F_2}$, which can be directly measured.

- These observables can be also calculated from the so called evolution equations.

- **Balitsky-Kovchegov (BK)** evolution equation describes such systems and predicts the values of $\boldsymbol{F_2}$ **and** $\boldsymbol{\sigma_{reduced}}$.

[1]



[1] M. H., "Balitsky-Kovchegov equation," *University of Jyvaskyla*, pp. 1–69, 2011.

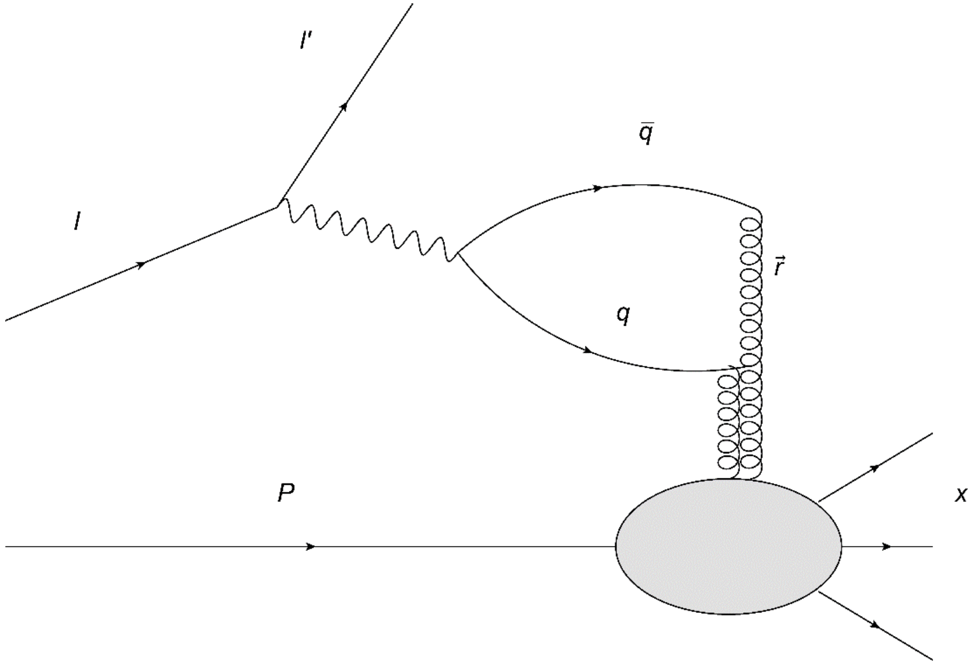
Color dipole approach to DIS

A virtual photon is emitted by incoming lepton

Virtual photon fluctuates into a quark and anti-quark pair

This dipole interacts with the target hadron

Outgoing state is produced



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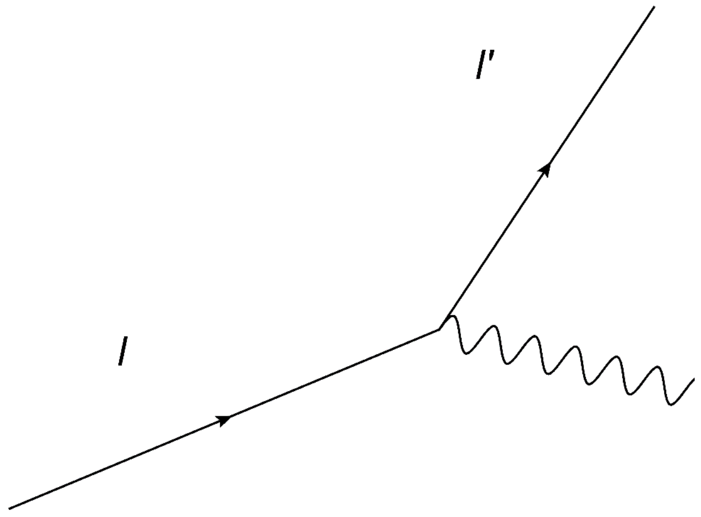

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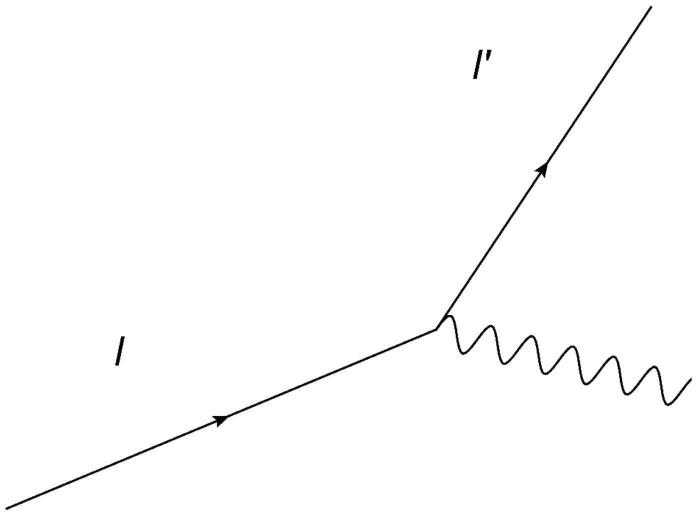

This process is not a part of the observables that we compute.

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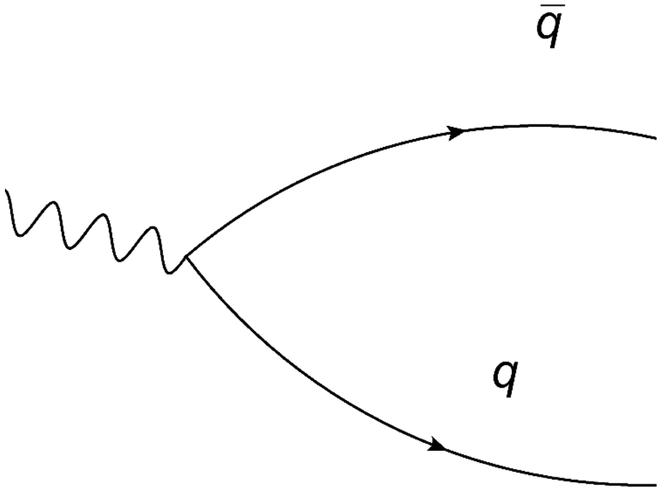

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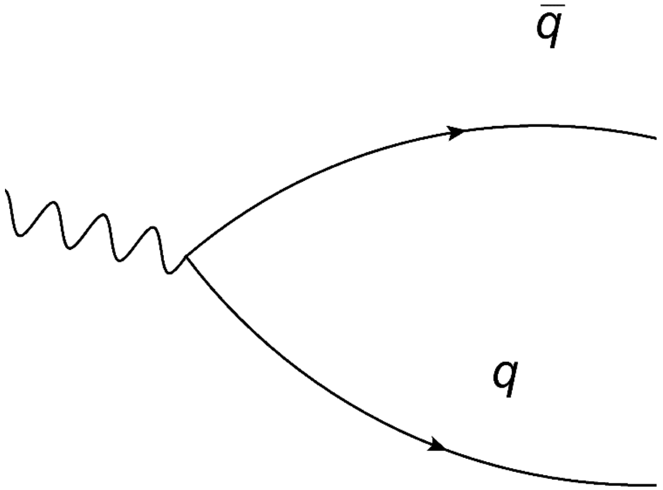
- Virtual photon fluctuates into a quark and anti-quark pair
- This dipole interacts with the target hadron
- Outgoing state is produced



$$|\Psi_T(z,\vec{r},Q^2)|^2 = \frac{3\alpha_{em}}{2\pi^2}\sum_i e_{q_i}^2((z^2 + (1-z)^2)\,\varepsilon^2 K_1^2(\varepsilon r) + m_{q_i}^2 K_0^2(\varepsilon r))$$

$$|\Psi_L(z,\vec{r},Q^2)|^2 = \frac{3\alpha_{em}}{2\pi^2}\sum_i e_{q_i}^2(4Q^2 z^2(1-z)^2)\; K_0^2(\varepsilon r))$$

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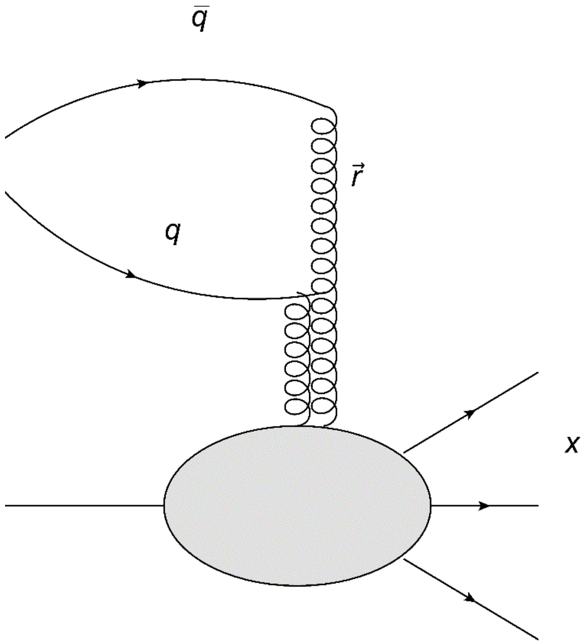

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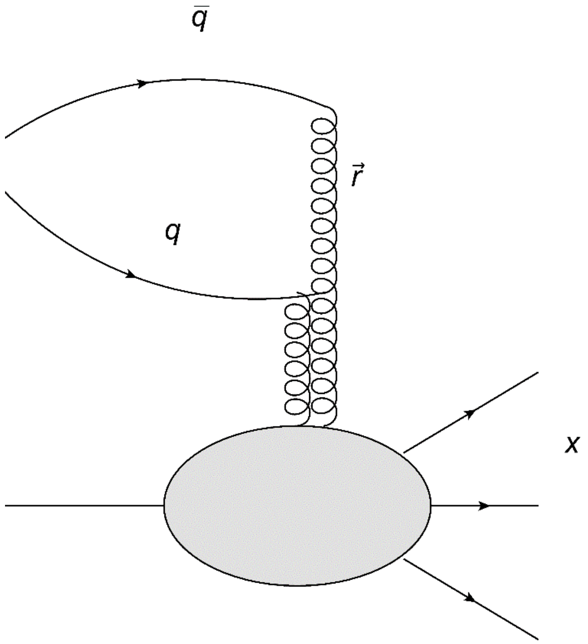

$$\sigma_{dip}(\vec{r}, x) = 2 \int d\vec{b}\, N(\vec{r}, x, \vec{b})$$

Where $N(\vec{r}, x, \vec{b})$ is the scattering amplitude.

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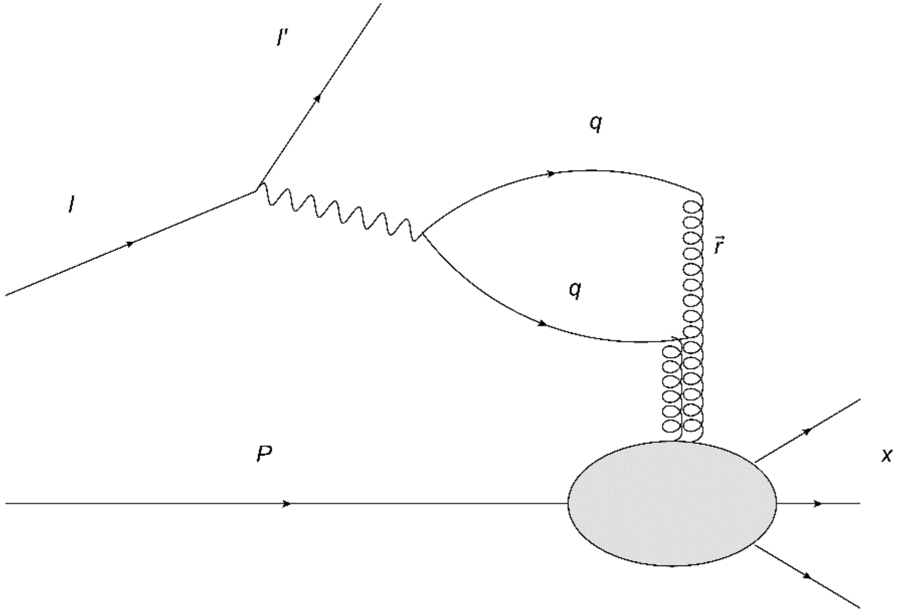

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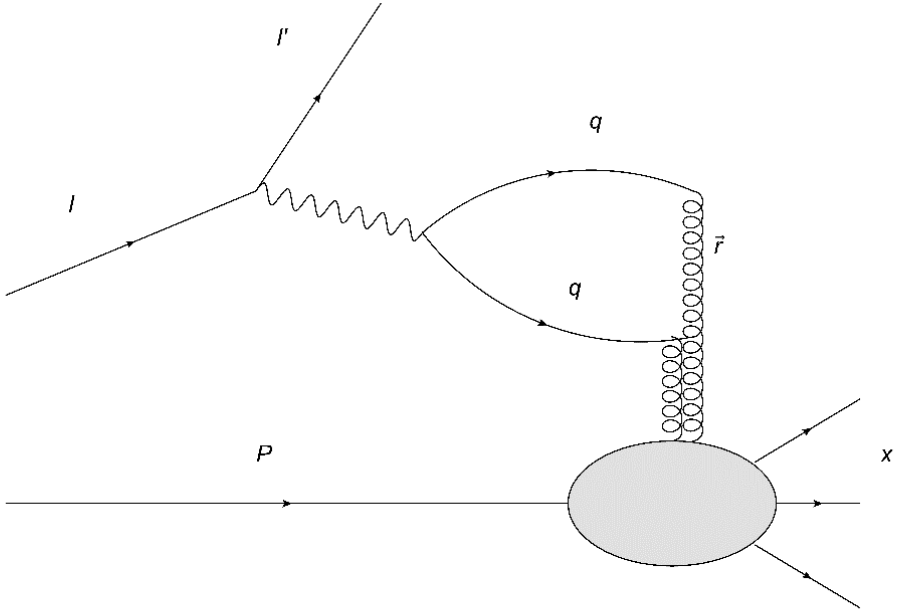


$$F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}} \iint d\vec{r}dz\left(|\Psi_T(z,\vec{r},Q^2)|^2 + |\Psi_L(z,\vec{r},Q^2)|^2\right)\sigma_{dip}(\vec{r},x)$$

Color dipole approach to DIS

For the computation of the structure function F_2 we need:

Wave functions $\vert\Psi_T(z,\vec{r},Q^2)\vert^2$ and $\vert\Psi_L(z,\vec{r},Q^2)\vert^2$	• These we can obtain from QED
Cross section of the interaction between the dipole and the hadron $\sigma_{dip}(\vec{r},x)$	• For this we need the scattering amplitude $N(\vec{r},x,\vec{b})$

The **BK evolution equation** can be used as a tool to compute $N(\vec{r},x,\vec{b})$

The rcBK evolution equation

The BK equation with NLO kernel and running coupling reads

$$\frac{\partial N(\vec{r},Y)}{\partial lnY} = \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2)(N(\vec{r}_1,Y) + N(\vec{r}_2,Y) - N(\vec{r},Y) - N(\vec{r}_1,Y)N(\vec{r}_2,Y))$$

Where for the kernel stands

$$K(\vec{r},\vec{r}_1,\vec{r}_2) = \frac{\alpha_s(r^2)N_c}{2\pi}\left[\frac{r^2}{r_1^2 r_2^2} + \frac{1}{r_1^2}\left(\frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} - 1\right) + \frac{1}{r_2^2}\left(\frac{\alpha_s(r_2^2)}{\alpha_s(r_1^2)} - 1\right)\right]$$

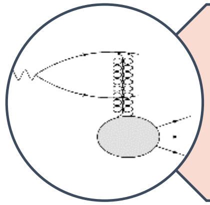
And running coupling is

$$\alpha_s(r^2) = \frac{12\pi}{\left(11 - \frac{2}{3}n_f\right)\ln\left(\frac{4C^2}{\mathrm{r}^2\Lambda_{\mathrm{QCD}}^2}\right)}$$

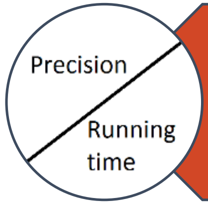
$\vec{r}_2 = \vec{r}$-$\vec{r}_1$ $\quad C = 2.52 \quad \Lambda_{QCD} = 0.241$ GeV $\quad n_f = N_c = 3 \quad \gamma = 1.135$ [3]

[3] J. L. Albacete, N. Armesto, J. G. Milhano, P. Quiroga-Arias, and C. A. Salgado, “AAMQS: A non-linear QCD analysis of new HERA data at small-x including heavy quarks," Eur. Phys. J., vol. C71, p. 1705, 2011.

Optimal setup of the computation



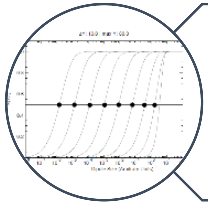
The rcBK equation



Optimal setup

F_2

Impact parameter independent solutions



Geometric scaling

Solving the rcBK equation

For the numerical computation we used:

- 25 steps over one order of magnitude where the interval of $|\vec{r}|$ runs from 10^{-7} to 10^{2}
- 20 steps over the interval of $[0,2\pi]$ in θ which is the angle between $\vec{r}$ and $\vec{r_1}$

- Discretization of the dipole size vector and rapidity interval
- Simpson rule for integration
- Runge-Kutta method of fourth order
- Linear interpolation (in log-scale) for obtaining the values of $N(r_2, Y)$

$$\int_a^b f(x)dx = \frac{h}{3}(f(a_0) + 4f(a_1) + 2f(a_2) + \ \dots\ + 4f(a_{m-1}) + f(a_m))$$

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1)$$
$$k_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2)$$
$$k_4 = f(x_n + h, y_n + hk_3)$$

$$L_1(x) = f(a_0) + (f(a_1) - f(a_0))\frac{(x - a_0)}{(a_1 - a_0)}$$

Optimal setup

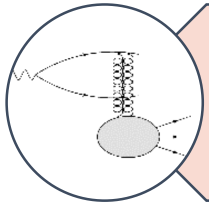
We **varied the parameters** that go in the computation in order to **determine their influence** on the overall precision and CPU time.

$$D(r,Y) = \frac{|N_{original}(r,Y) - N_{new}(r,Y)|}{N_{original}(r,Y)}$$

We fixed the proportional difference below 1% for the relevant interval

Step in rapidity (RK method)	• $\Delta Y = 0.01$	Interval over r (Simpson rule)	• 25 steps over one order of magnitude
Order of the Runge-Kutta method	• Fourth order	Interval over θ (Simpson rule)	• 20 steps over $[0,2\pi]$
Order of interpolation	• Linear interpolation	Interval over the momentum fraction z	• 10000 steps over the interval [0,1]

Impact parameter independent rcBK solutions



The rcBK equation

Precision / Running time

Optimal setup

F_2

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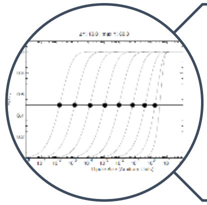
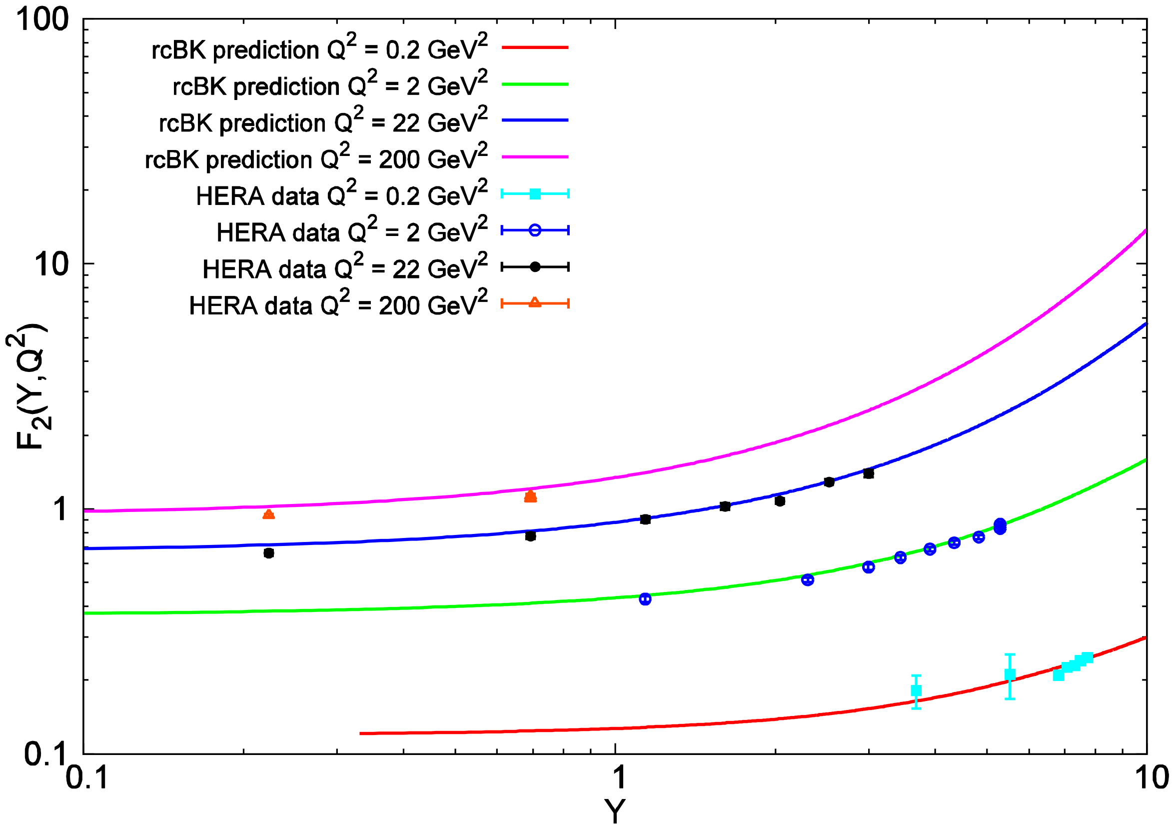


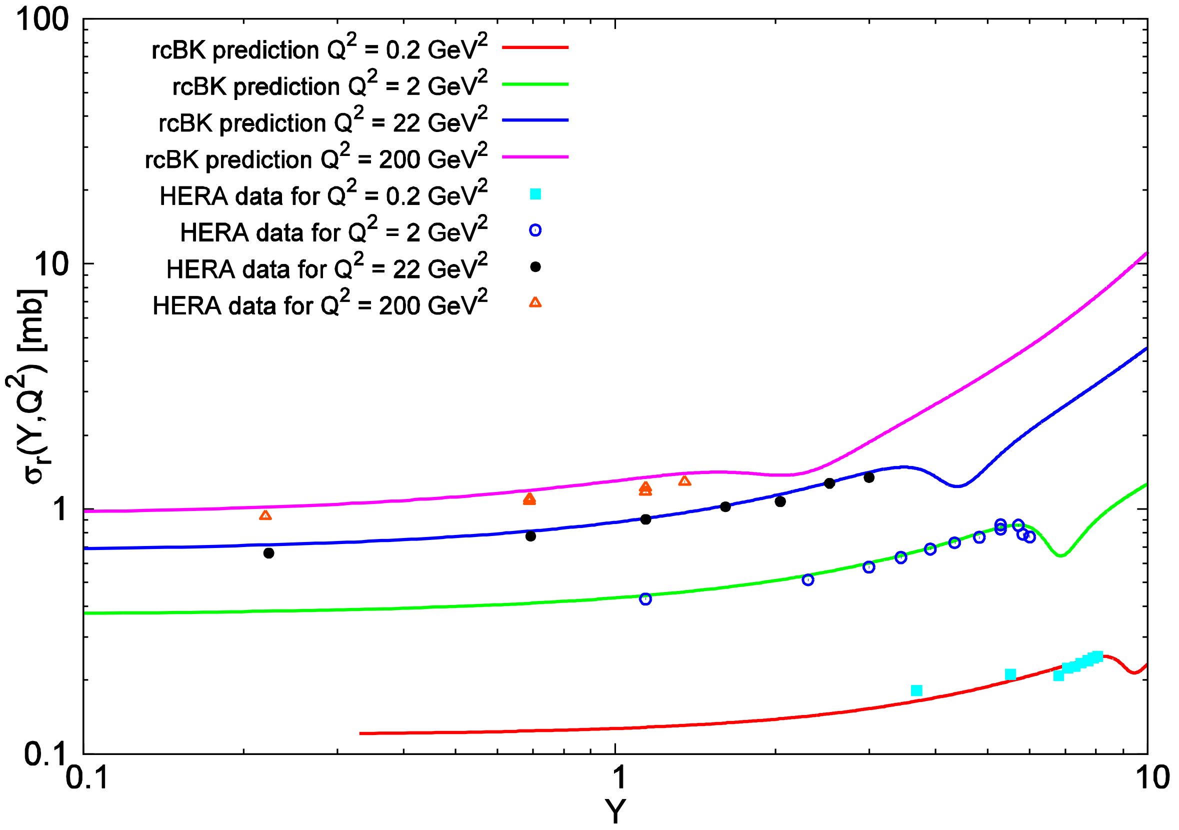
Geometric scaling

Impact parameter independent rcBK solutions

Structure function $F_2(Y, Q^2)$



Reduced cross section $\sigma(Y, Q^2)$



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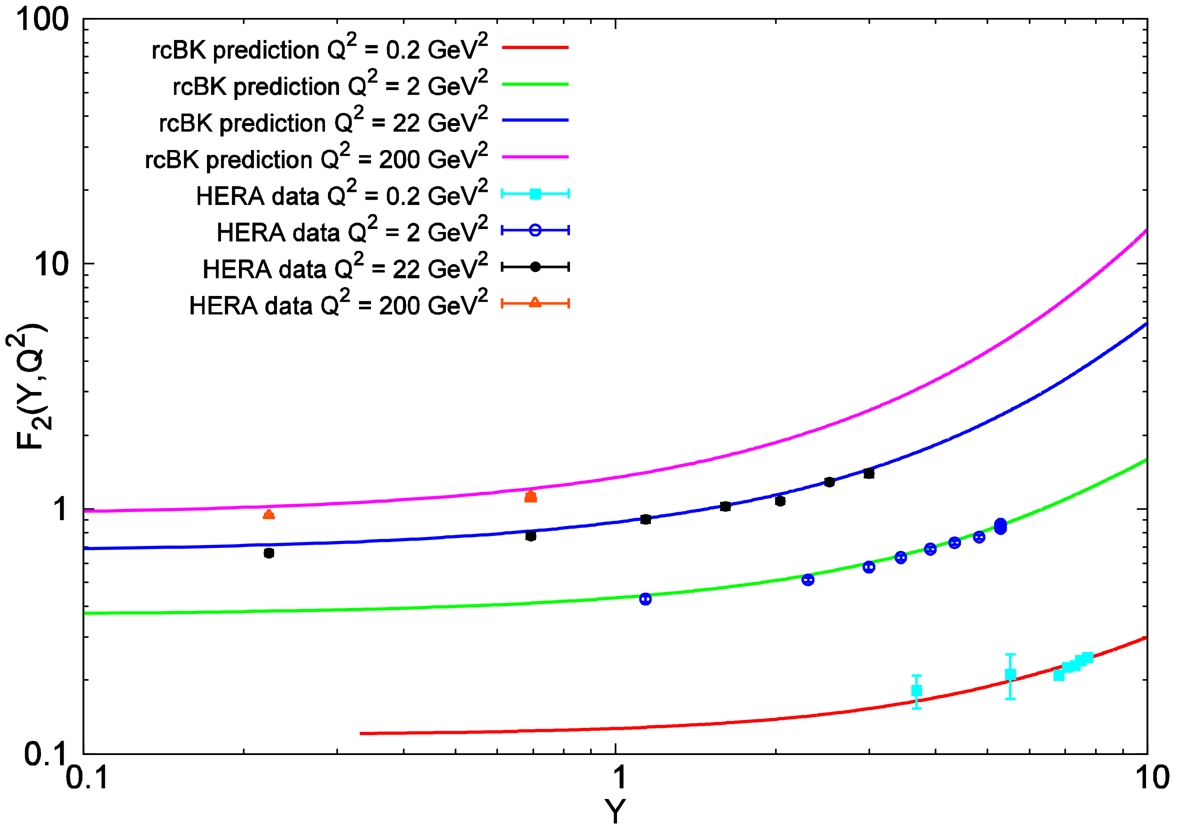
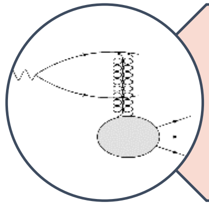


- **100s** with an average personal computer (it took **hours for other groups**)
- The **mean square error** of the prediction was **below 1.5%** of the experimentally measured value
- This is important for fitting the **initial conditions**, **impact parameter dependent** rcBK equation.

Geometric scaling



The rcBK equation

Precision

Running time

Optimal setup

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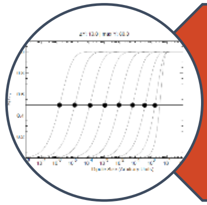


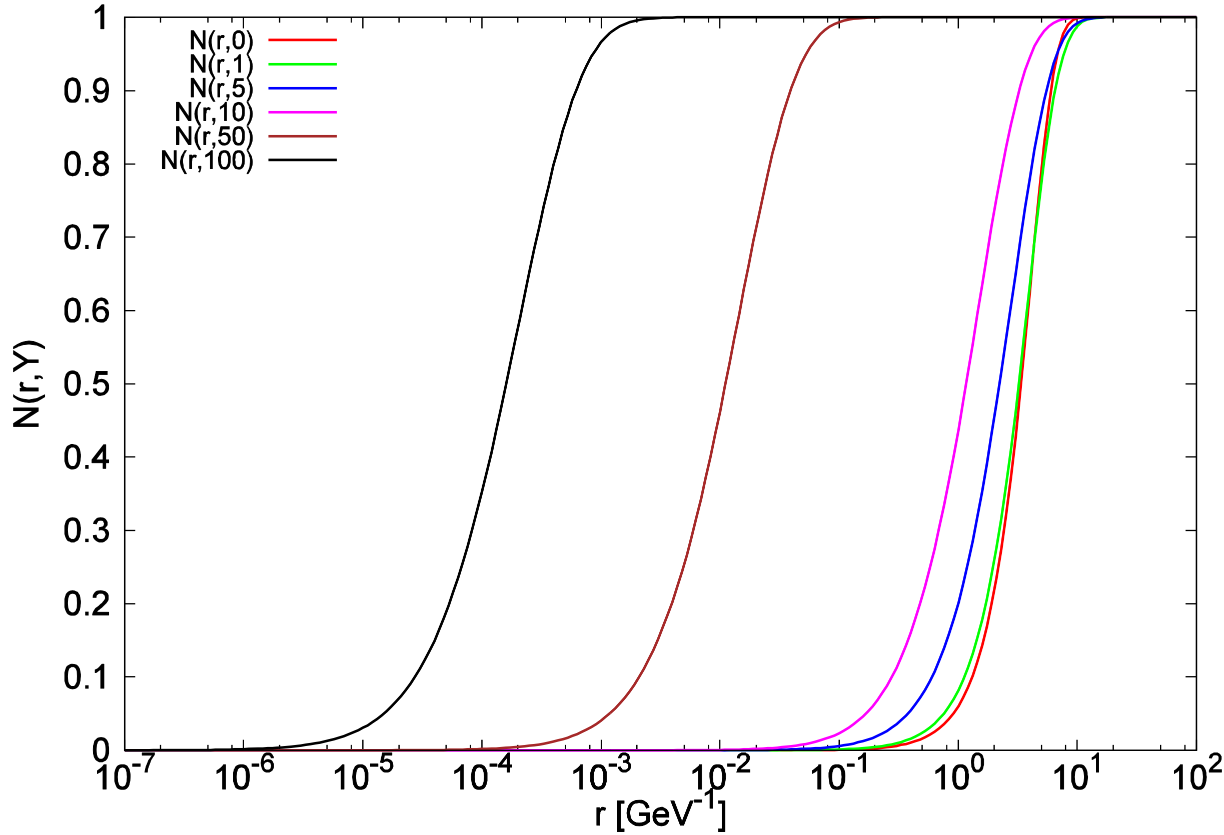
Geometric scaling

Geometric scaling

Of course, are these results really relevant or just fitted to data?

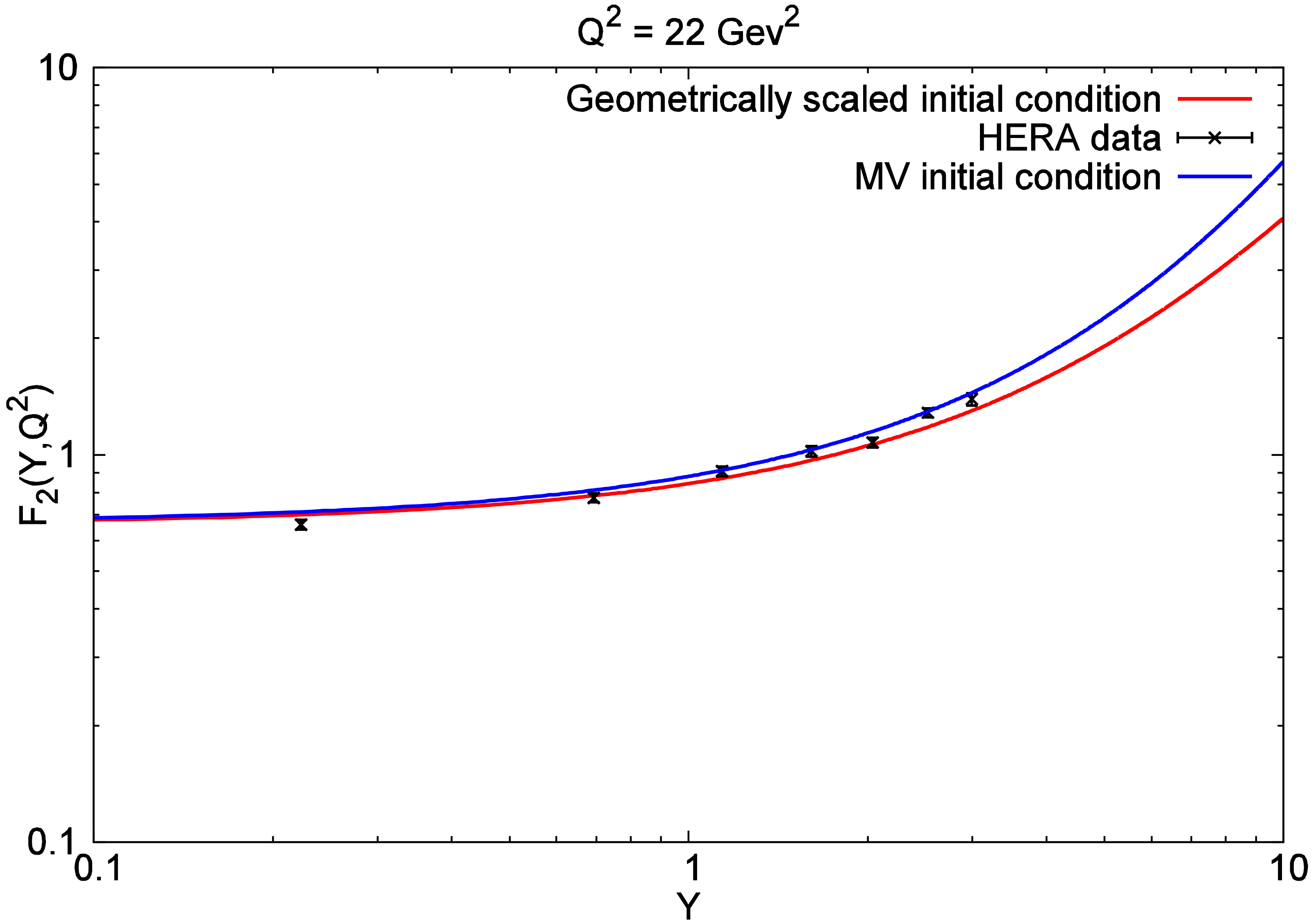
The rcBK equation **exhibits** the phenomenon **geometric scaling**

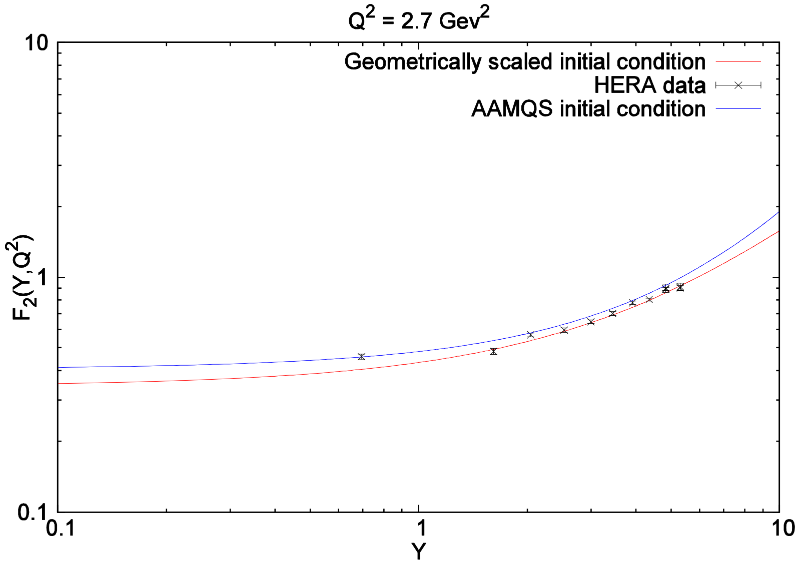
$$\xcancel{N^{MV}(r,0) = 1 - \exp\left[\frac{-(r^2 Q_{s0}^2)^{\gamma}}{4} \ln\left(\frac{1}{r^2 \Lambda_{QCD}^2} + e\right)\right]} \longrightarrow \; ?$$

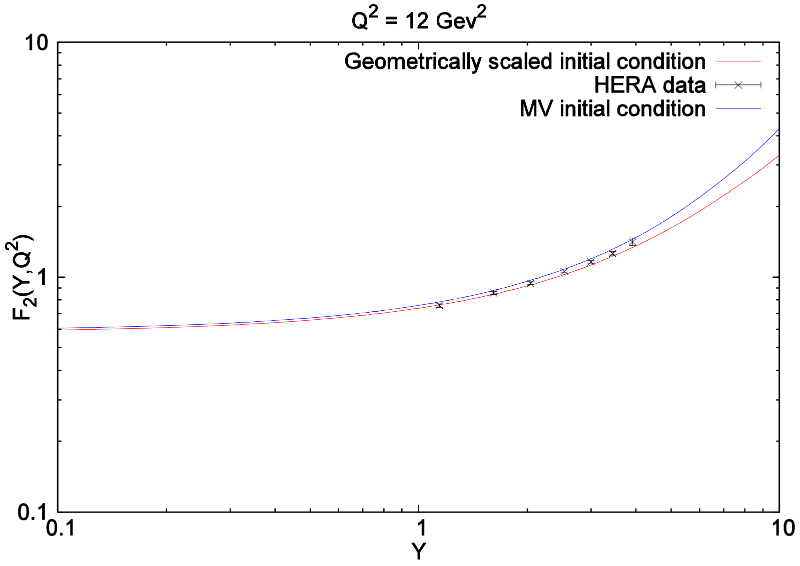


The **MV initial condition** depends on **four parameters**. The **geometrically scaled** initial condition would **require** much **less**.

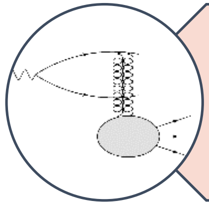
Geometrically scaled rcBK prediction







Conclusions



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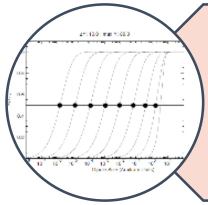


Geometric scaling

CONCLUSIONS

Conclusions

The **rcBK** evolution equation **was solved** on a wide interval of $\boldsymbol{Q^2}$, values of $\boldsymbol{F_2(Y, Q^2)}$ and $\boldsymbol{\sigma(Y, Q^2)}$ were **computed**.

Step in rapidity (RK method)	Order of the Runge-Kutta method	Order of interpolation	Interval over r (Simpson rule)	Interval over θ (Simpson rule)	Interval over the momentum fraction z
• $\Delta Y = 0.01$	• Fourth order	• Linear interpolation	• 25 steps over one order of magnitude	• 20 steps over $[0,2\pi]$	• 10000 steps over the interval [0,1]

- A **geometrically scaled initial condition** was obtained from the rcBK evolution equation intrinsic properties.
- The **ideal** choice for **the rescaling parameter** was determined to be $Q_S^2 = 0.07$ GeV ^{2}.
- This approach **reduced** dramatically **the number of free parameters** in necessary for this model and can be used to obtain a more physical description of the system.
- This initial condition was then used to **predict values of structure function** in regions that were **not yet measured** and future measurements (possibly at the LHC) will determine the **validity of this approach to the dipole model** and rcBK equation in particular.

Thank you for your attention

I presented the Optimal setup section at POETIC6 conference in Ecole Polytechnique in Paris.
The Optimal setup was then published by the European Physics Journal.
The Geometrical scaling section will be submitted for publishing in the next few weeks.

How do we compute the rcBK evolution equation?

Solving the impact parameter independent rcBK

- Choose an **equidistant grid** and precompute the initial condition.
- **Precompute** the values of **Kernel and $\boldsymbol{r_2}$** into a three dimensional array.
- **For each** value of $\boldsymbol{r}$, you need to compute the values of $Kernel,\ Split$ and $Recomb$ (**integrate over $\vec{\boldsymbol{r}}_1$**).
- For this you will need the **values of $\boldsymbol{N(r_2, Y)}$** that since $r_2 = \sqrt{r^2 + r_1^2 - rr_1\cos(\theta)}$ **do not** always **fall on** the precomputed **grid**.
- Therefore you need to **linearly interpolate** with the neighboring values. If $N(r_2, Y)$ falls outside the considered interval, its value is fixed as 0 or 1 (smaller or greater then the values of the interval).
- First**, for each** value of $\boldsymbol{r_1}$ **integrate over** the interval of $\boldsymbol{\theta}$.
- Then **integrate** this half-integrated function **over $\boldsymbol{r_1}$**.
- This then allows you to calculate the **Runge-Kutta coeffitients**.
- Then with RK method you can determine the amount of **change in the point $\boldsymbol{r}$** to the function $N(r, Y)$ and **store it in an array**.
- Once all values of r are accounted for, you can add the array that stores the change to the function $N(r, Y)$ and therefore **acquire $\boldsymbol{N(r\ , Y + h)}$**.
- This you can **repeat** until you reach the desired value of rapidity .

Solving the impact parameter dependent rcBK

$\vec{r} = \vec{x}_0 - \vec{x}_1$ $\vec{b} = \frac{\vec{x}_0 + \vec{x}_1}{2}$

$\vec{r}_1 = \vec{x}_0 - \vec{x}_2$ $\vec{b}_1 = \frac{\vec{x}_0 + \vec{x}_2}{2}$

$\vec{r}_2 = \vec{x}_2 - \vec{x}_1$ $\vec{b}_2 = \frac{\vec{x}_2 + \vec{x}_1}{2}$

θ_{br} is fixed in the computation

$\theta_{br_1} = \theta_{br} + \theta_{rr_1}$

$\theta_{br_2} = \theta_{br} + \theta_{rr_2}$

θ_{rr_1} is a parameter in the numerical computation

$$\theta_{rr_2} = arccos\frac{|\vec{r}^2| - |\vec{r}_1^2| + |\vec{r}_2^2|}{2\vec{r}\vec{r}_2}$$

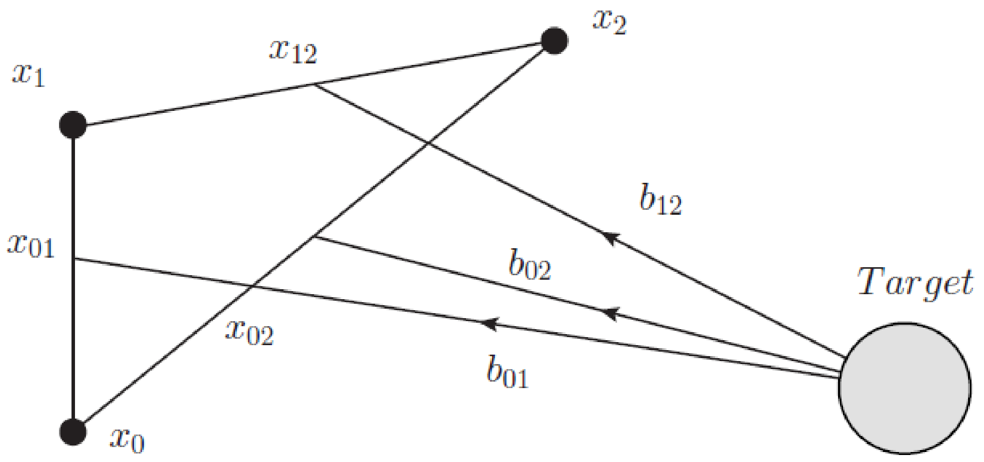
In this approach, we need to obtain the **values of impact parameters** in order **to** evaluate the integral of rcBK equation and **compute the RK method**

$$\frac{\partial N(\vec{r},\vec{b},Y)}{\partial lnY} = \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2)(N(\vec{r}_1,\vec{b}_1,Y) + N(\vec{r}_2,\vec{b}_2,Y) - N(\vec{r},\vec{b},\ Y) - N(\vec{r}_1,\vec{b}_1,Y)N(\vec{r}_2,\vec{b}_2,Y))$$

RK method goes for all values of $|\vec{b}|$

$$|\vec{b}_1| = \left|\vec{b} + \frac{\vec{r}_2}{2}\right| = \sqrt{|\vec{b}|^2 + \frac{|\vec{r}_2|^2}{4} - |\vec{b}||\vec{r}_2|cos\theta_{br_2}}$$

$$|\vec{b}_2| = \left|\vec{b} - \frac{\vec{r}_1}{2}\right| = \sqrt{|\vec{b}|^2 + \frac{|\vec{r}_1|^2}{4} - |\vec{b}||\vec{r}_1|cos\theta_{br_1}}$$



How do we compute the rcBK evolution equation fast?

Speeding up the computation

- Integrating only over $[0, \pi]$ in θ and multiplying by two since it only comes in cosine
- Precomputing the values of r_2 and kernel into a large three-dimensional array
- Using the fact that the BK equation does not depend explicitly on Y and dividing the integral into

$$Kernel = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r}) \qquad Split = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r})(N(\vec{r}_1, Y) + N(\vec{r}_2, Y)) \qquad Recomb = \int d\vec{r}_1 K(\vec{r}_1, \vec{r}_2, \vec{r}) N(\vec{r}_1, Y) N(\vec{r}_2, Y)$$

And from these, we can compute the Runge Kutta coefficients as follows

$$k_1 = Split - N(\vec{r}, Y) Kernel - Recomb$$

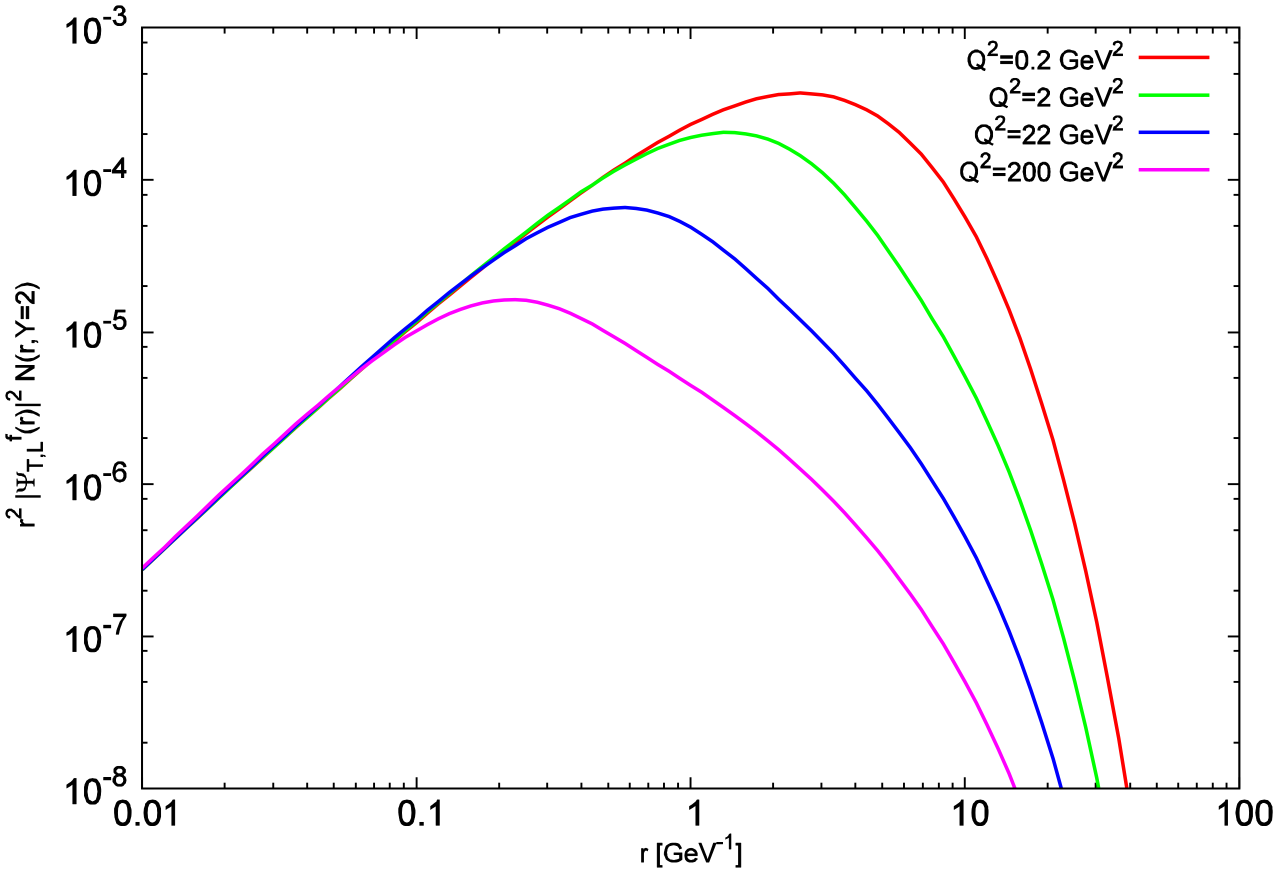
$$k_3 = k_2 + \frac{1}{2} h k_2 Kernel - \frac{1}{2} h k_2 Split - \frac{1}{4} h^2 k_2^2\, Kernel$$

$$k_2 = k_1 + \frac{1}{2} h k_1 Kernel - \frac{1}{2} h k_1 Split - \frac{1}{4} h^2 k_1^2\, Kernel$$

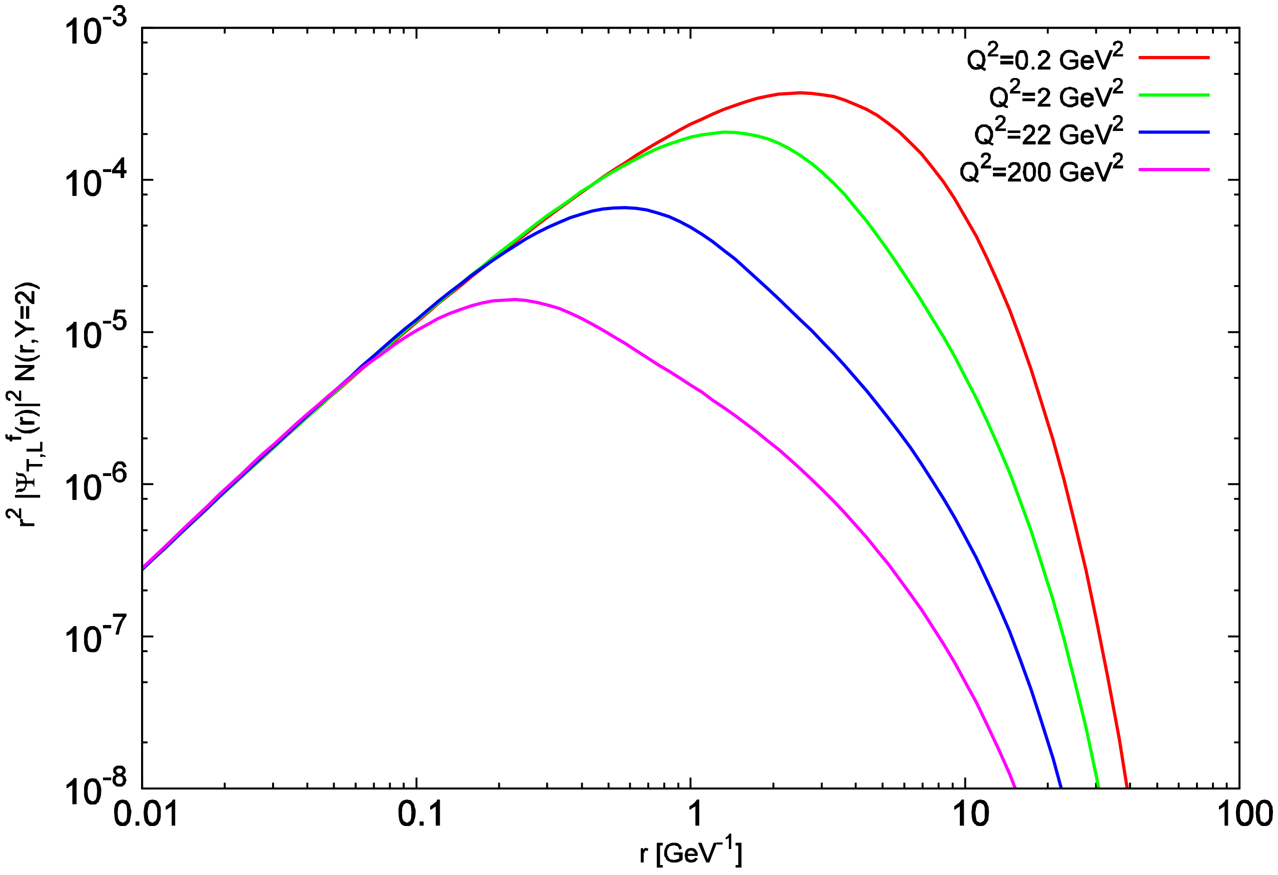
$$k_4 = k_3 + \frac{1}{2} h k_3 Kernel - \frac{1}{2} h k_3 Split - \frac{1}{4} h^2 k_3^2\, Kernel$$

How do we determine the optimal setup?

Interval of main interest



The unintegrated structure function at rapidity $Y = 2$

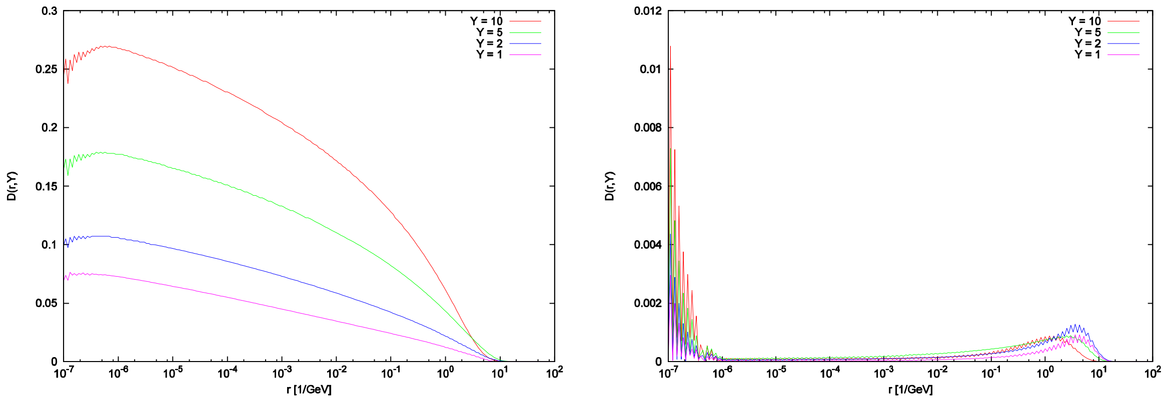


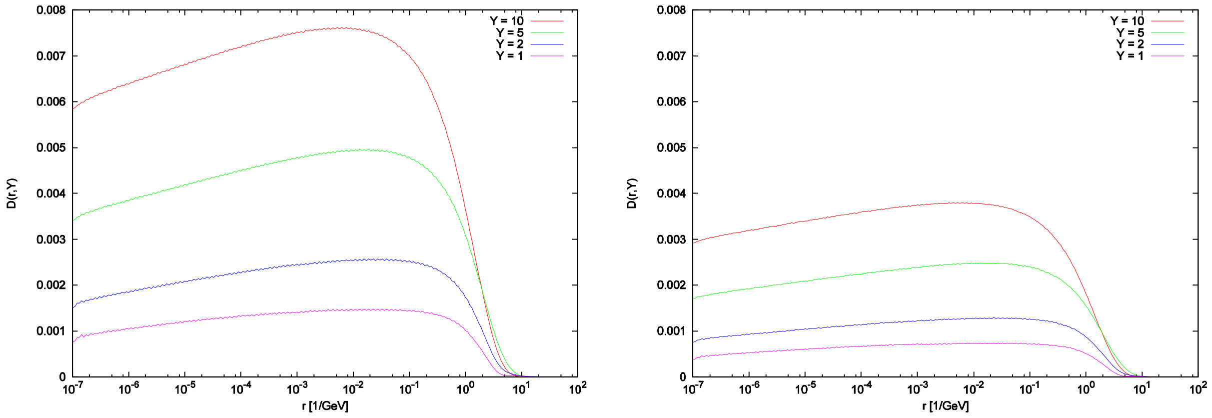
The unintegrated structure function at rapidity $Y = 10$

The interval of main interest is then $r \sim [0.1; 20]$

Determining the optimal setup

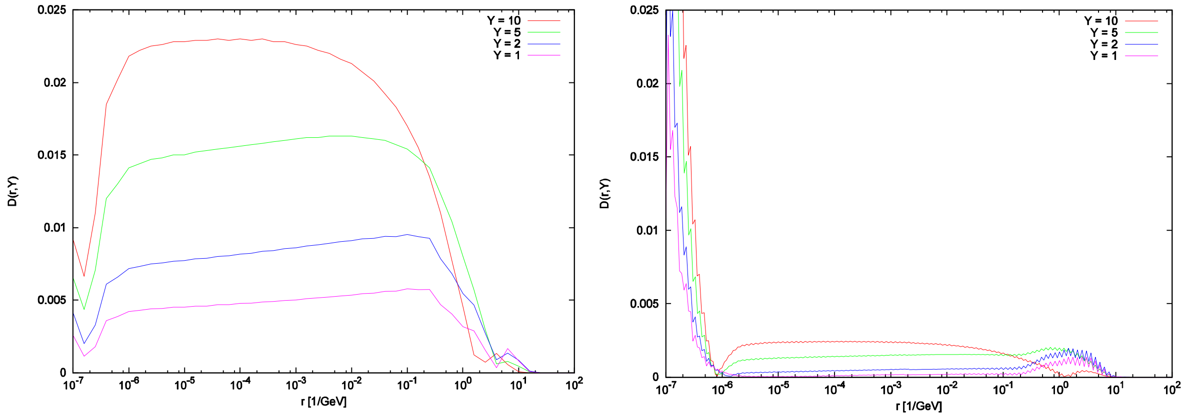
$$D(r,Y) = \frac{|N_{original}(r,Y) - N_{new}(r,Y)|}{N_{original}(r,Y)}$$





Order of the Runge-Kutta method variation

Variation of steps over $[0, \pi]$ in θ $(5 - 10, 10 - 20)$.



Variation of steps over one order of magnitude in r $(10 - 25, 25 - 50)$.

Impact parameter dependent Solutions

Impact parameter dependent solutions

The impact parameter dependent rcBK evolution equation reads:

$$\frac{\partial N(\vec{r},\vec{b},Y)}{\partial lnY} = \int d\vec{r}_1 K(\vec{r},\vec{r}_1,\vec{r}_2)(N(\vec{r}_1,\vec{b}_1,Y) + N(\vec{r}_2,\vec{b}_2,Y) - N(\vec{r},\vec{b},\ Y) - N(\vec{r}_1,\vec{b}_1,Y)N(\vec{r}_2,\vec{b}_2,Y))$$

with **NLO kernel** $K(\vec{r},\vec{r}_1,\vec{r}_2) = \frac{\alpha_s(r^2)N_c}{2\pi}\left[\frac{r^2}{r_1^2 r_2^2} + \frac{1}{r_1^2}\left(\frac{\alpha_s(r_1^2)}{\alpha_s(r_2^2)} - 1\right) + \frac{1}{r_2^2}\left(\frac{\alpha_s(r_2^2)}{\alpha_s(r_1^2)} - 1\right)\right]\theta\left(r_1^2 - \frac{1}{m^2}\right)\ \theta\left(r_2^2 - \frac{1}{m^2}\right)$

where the cutoff parameter $m = 0.35\text{GeV}$

$N^0(r,b,0) = 1 - \exp(-cr^2\exp(-db^2))$ where $c = 0.0643\text{GeV}^2$ and $d = 0.125\text{GeV}^2$

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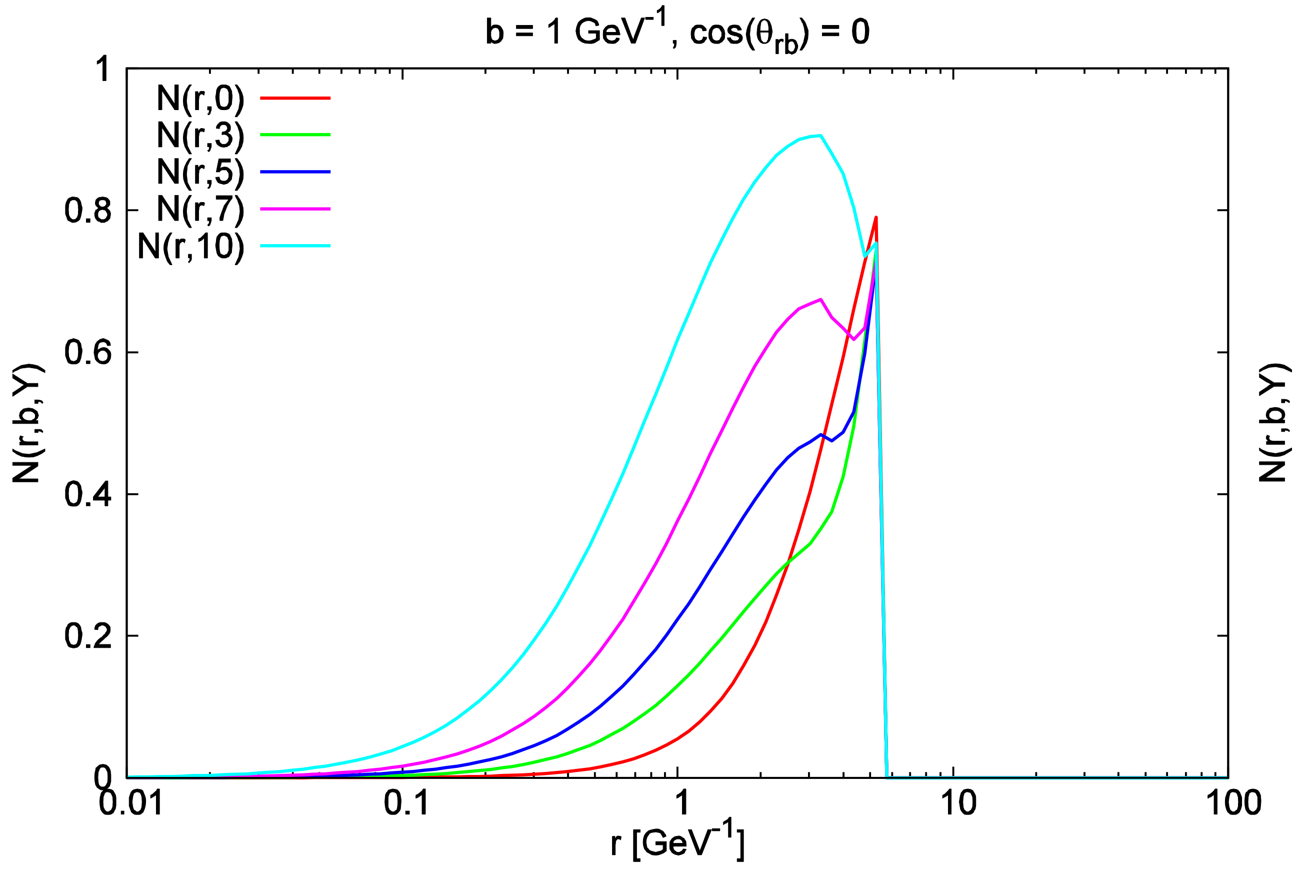
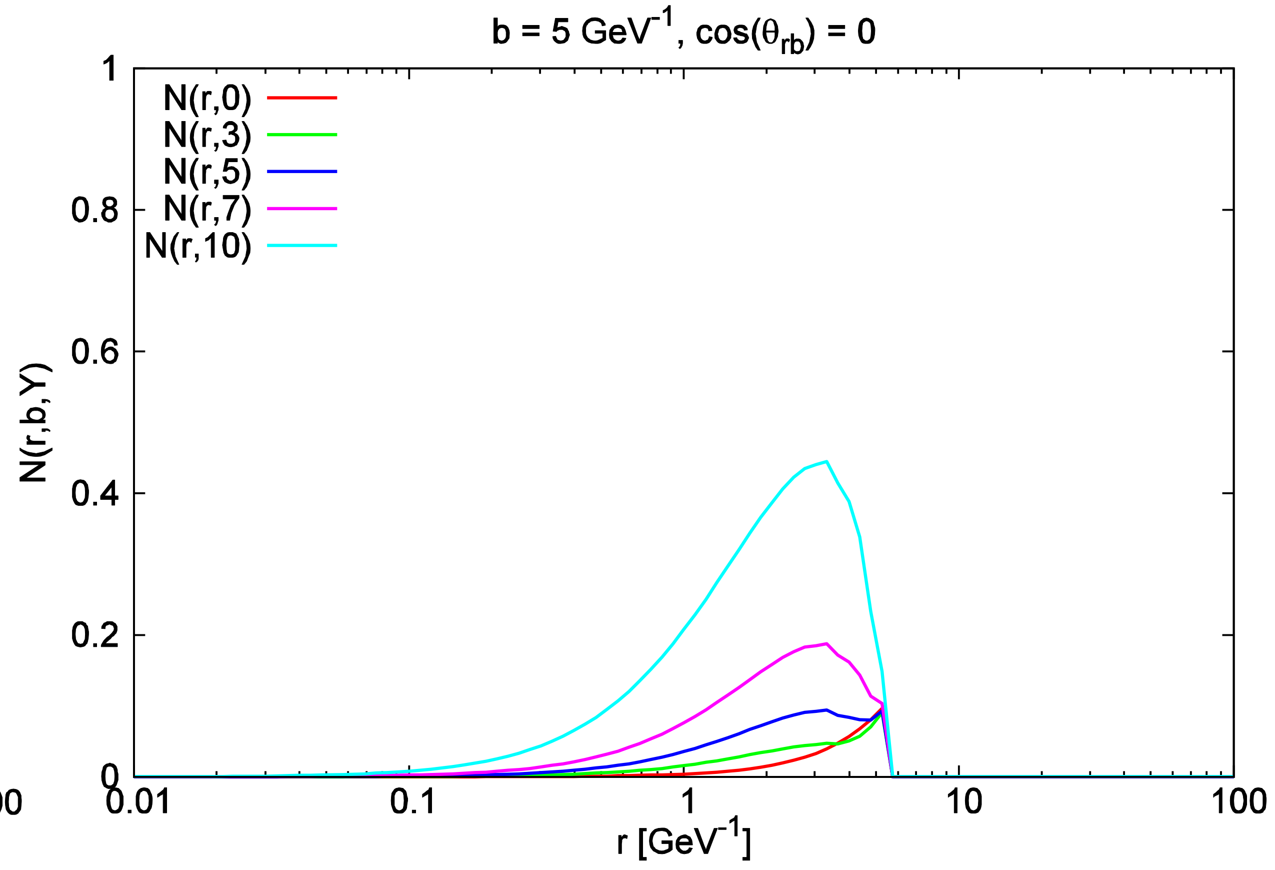
- Impact parameter dependence introduces **two additional dimensions** to the computation ($|\vec{\boldsymbol{b}}|$ and $\boldsymbol{\theta_{rb}}$)
- This elongates the **running time** of the evolution **up to 5 hours** on a regular PC (for **other groups**, computation took **weeks**).

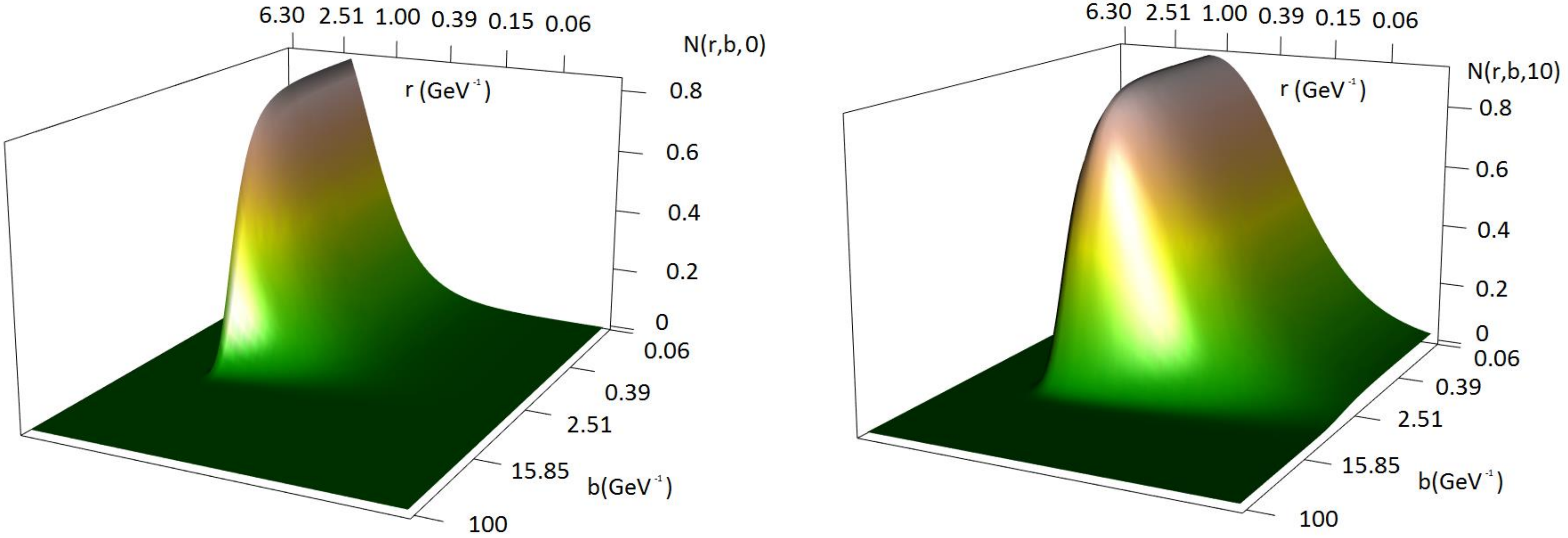
Impact parameter dependent solutions





The initial condition was cut at $\frac{2}{m}$ since $\vec{r} = \vec{r}_1 + \vec{r}_2$

Impact parameter dependent solutions



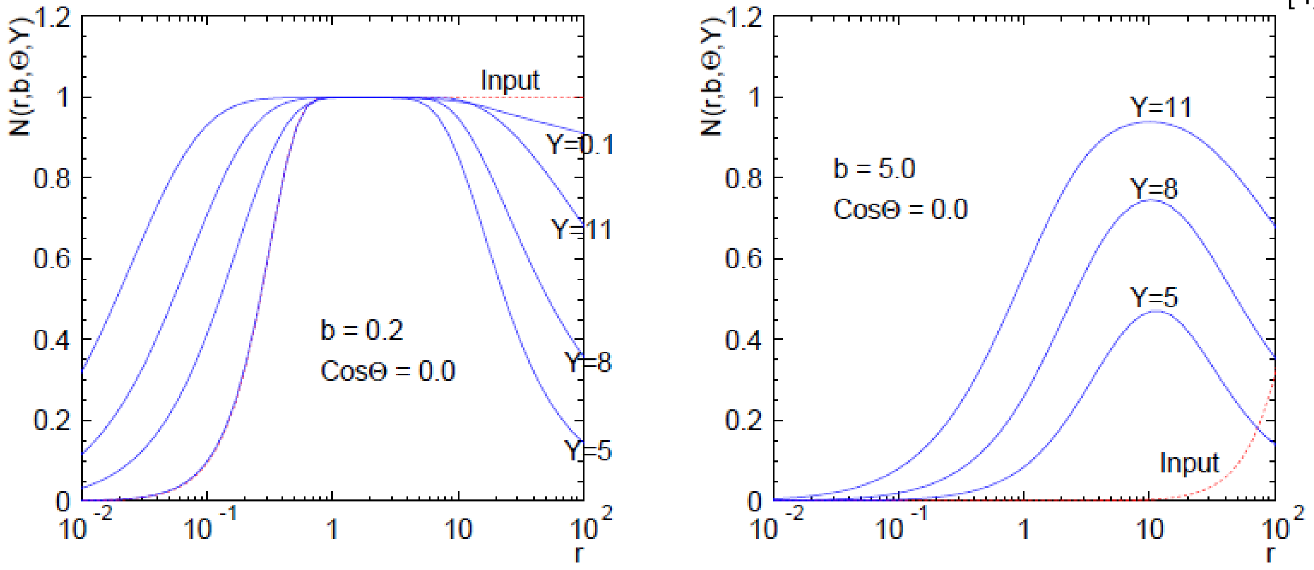
Scattering amplitude with respect to r and b.

Impact parameter dependent solutions

We were the **first ones to show** that the **Runge-Kutta** method can be **used for solving** this equation.

This approach enabled me to **reduce the computation time** by nearly a **factor of 100**.

[4]



- Shows a **completely different shape** even for the values of small b.
- **Decreases for** high values of r. In this approach, **dipoles** that are **too large** do not interact with the target hadron as easily.

[4] K. J. Golec-Biernat and A. M. Stasto, "On solutions of the Balitsky-Kovchegov equation with impact parameter," *Nucl. Phys.*, vol. B668, pp. 345–363, 2003.

Impact parameter dependent solutions