Properties of Cotton tensor

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7. Česko-Slovenská studentská vědecká konference ve fyzice

Conventions

Henceforth we shall work

- on a pseudo-Riemannian manifold (symmetric, non-degenerate metric tensor)
- with general dimension n (unless specified otherwise)
- with the Levi-Civita connection (metric, torsion-free)

Conformal transformations of metric tensor

Definition

Let (M, g) be a pseudo-Riemannian manifold. A diffeomorphism $\phi : M \to M$ is called *a conformal transformation* if it satisfies:

$$\phi^* g_{\phi(p)} = e^{2\sigma(p)} g_p \qquad p \in M, \sigma \in C^\infty(M) \equiv \Omega^0(M) \tag{1}$$

- The expression 1 takes form of:

$$g_{\phi(p)}(\phi_* X, \phi_* Y) = e^{2\sigma(p)} g_p(X, Y) \tag{2}$$

  when a pair of tangent vectors $X, Y \in T_p M$ is inserted.

Conformal equivalence

Definition

Let $g, \tilde{g}$ be a pair of metric tensors on a manifold M. The metric $\tilde{g}$ is said to be *conformally equivalent* to g if there exists a conformal transformation between the two metrics.

- An explicit relation for the two metrics is:

$$\tilde{g}_p = e^{2\sigma(p)} g_p \tag{3}$$

- In coordinates:

$$\tilde{g}_{ij} = e^{2\sigma} g_{ij}$$

Theorem

Let $\sigma \in \Omega^0(M)$ and U be the vector field which corresponds to the 1-form $d\sigma$ so that

$$Z(\sigma) = d\sigma(Z) = g(U, Z) \qquad \forall Z \in \Gamma(TM)$$

then under conformal transformation of metric tensor $\tilde{g}_p = e^{2\sigma(p)} g_p$ the Levi-Civita connection[1] *transforms as:*

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y)U \qquad \forall X, Y \in \Gamma(TM)$$

[1] $\tilde{\nabla}$ metric w.r.t. $\tilde{g}$ and ∇ w.r.t. g

Theorem

Let $\sigma \in \Omega^0(M)$ and U be the vector field which corresponds to the one-form $d\sigma$, so that $Z(\sigma) = d\sigma(Z) = g(U, Z)$. Then the Riemann tensor transforms as ($\forall X, Y, Z \in \Gamma(TM)$):

$$\tilde{R}(X, Y)Z =$$

$$= R(X, Y)Z - g(Y, Z)BX + g(BX, Z)Y - g(BY, Z)X + g(X, Z)BY$$

where B is a type (1,1) tensor defined by:

$$B(X) := -X(\sigma)U + \nabla_X U + \frac{1}{2}U(\sigma)X$$

- Contracting transformation formula for the Riemann tensor, we find expressions for $\tilde{\varrho}$ (Ricci (0,2) form), $\tilde{\mathcal{R}}$ (Ricci scalar curvature)
- Substituting back into the formula for the Riemann tensor, we find a quantity with $\tilde{W}^l_{ijk} = W^l_{ijk}$:

Definition

The coordinate expression

$$W^l_{ijk} := R^l_{ijk} + \frac{1}{(n-2)}[\varrho_{ij}\delta^l_k - \varrho_{ik}\delta^l_j + g_{ij}g^{nl}\varrho_{nk} - g_{ik}g^{nl}\varrho_{nj}] +$$

$$+\frac{\mathcal{R}}{(n-1)(n-2)}[g_{ik}\delta^l_j - g_{ij}\delta^l_k]$$

defines a tensor on (M, g), called the *Weyl tensor (conformal curvature tensor)*.

Properties of Weyl tensor

- Invariant of conformal transformations
- Symmetries ($W_{lijk} = -W_{likj}$; $W_{lijk} = -W_{iljk}$; $W_{lijk} = W_{jkli}$)
- Satisfies first Bianchi identity[2]

$$W^l_{ijk} + W^l_{kij} + W^l_{jki} = 0 \qquad \forall i, j, k, l$$

- Completely trace-less
- On (M, g) with dim $M = 3$ identically equal to zero tensor

[2] the same symmetries as Riemann tensor

Theorem

Coordinate expression of the $(1,3)$ *Weyl tensor satisfies:*

$$\nabla_h W^l_{ijk} + \nabla_j W^l_{ikh} + \nabla_k W^l_{ihj} =$$

$$= \frac{1}{n-2}\left(\delta^l_h C_{ijk} + \delta^l_j C_{ikh} + \delta^l_k C_{ihj} + g_{ik} C^l_{jh} + g_{ih} C^l_{kj} + g_{ij} C^l_{hk}\right)$$

where

Definition

The coordinate expression

$$C_{ijk} = \nabla_k \varrho_{ij} - \nabla_j \varrho_{ik} + \frac{1}{2(n-1)}\left(g_{ik}\nabla_j \mathcal{R} - g_{ij}\nabla_k \mathcal{R}\right) \tag{4}$$

defines a $(0,3)$ tensor on (M,g), called the *Cotton tensor*.

Properties of Cotton tensor

- Symmetries ($\forall i, j, k$):

$$C_{ijk} = -C_{ikj}$$

$$C_{ijk} + C_{kij} + C_{jki} = 0$$

- Trace-free
- Identically zero on (M, g) with $\dim M = 2$
- On (M, g) with $\dim M > 3$, Weyl vanishes $\rightarrow$ Cotton vanishes

Cotton tensor under conformal transformations

Theorem

Let (M, g) be a pseudo-Riemannian manifold with $\dim M = n \geq 3$*. Then under a conformal transformation* $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ *of the metric tensor the Cotton tensor of M transforms as follows:*

$$\tilde{C}_{ijk} = C_{ijk} - (n-2)(\partial_a \sigma) W^a_{ijk}$$

- In $\dim M = 3$ solely (!), Weyl tensor is an identically zero tensor, ergo Cotton tensor is an invariant of conformal transformations
$$\tilde{C}_{ijk} = C_{ijk} \qquad \forall i, j, k$$

Obstructions to local conformal flatness

Definition

A pseudo-Riemannian manifold (M, g) is *locally conformally flat* if for any $p \in M$, there exists a neighborhood V of p and a $C^\infty(V)$ function σ such that $(V, \tilde{g} = e^{2\sigma} g)$ is flat.

Theorem

A pseudo-Riemannian manifold (M, g) *with* $\dim M = n$ *is locally conformally flat if and only if*

- *for* $n \geq 4$ *the Weyl tensor of* M *vanishes*
- *for* $n = 3$ *the Cotton tensor of* M *vanishes*

- Let $\dim M = 3$.
- Thanks to anti-symmetry of $C_{ijk} = -C_{ikj}$ it is possible to be percieved as a vector-valued 2-form $(C_i) = C_{ijk} dx^j \wedge dx^k$.
- By using the Hodge star $\star(C_i)$ and then valuing the obtained object on a single element of the basis of T_pM as follows $Y_{ij} = [\star(C_i)]\partial_j$, we arrive at:

Definition

The coordinate expression

$$Y^{ij} = \epsilon^{ikl} \nabla_k \left(\varrho^j_l - \frac{1}{4} \mathcal{R} \delta^j_l \right)$$

defines a $(2,0)$ tensor on (M, g) with $\dim M = 3$, called the *Cotton-York tensor*.

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Thank you for your attention.