Properties of Cotton tensor

Filip Garaj

Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

7. Česko-Slovenská studentská vědecká konference ve fyzice
Conventions

Henceforth we shall work

- on a pseudo-Riemannian manifold (symmetric, non-degenerate metric tensor)
- with general dimension $n$ (unless specified otherwise)
- with the Levi-Civita connection (metric, torsion-free)
Conformal transformations of metric tensor

**Definition**

Let \((M, g)\) be a pseudo-Riemannian manifold. A diffeomorphism \(\phi : M \rightarrow M\) is called a *conformal transformation* if it satisfies:

\[
\phi^* g_{\phi(p)} = e^{2\sigma(p)} g_p \quad p \in M, \sigma \in C^\infty(M) \equiv \Omega^0(M) \quad (1)
\]

- The expression 1 takes form of:

\[
g_{\phi(p)}(\phi_* X, \phi_* Y) = e^{2\sigma(p)} g_p(X, Y) \quad (2)
\]

when a pair of tangent vectors \(X, Y \in T_pM\) is inserted.
Conformal equivalence

Definition
Let \( g, \tilde{g} \) be a pair of metric tensors on a manifold \( M \). The metric \( \tilde{g} \) is said to be \textit{conformally equivalent} to \( g \) if there exists a conformal transformation between the two metrics.

- An explicit relation for the two metrics is:
  \[
  \tilde{g}_p = e^{2\sigma(p)} g_p
  \]  
  \text{ (3)}

- In coordinates:
  \[
  \tilde{g}_{ij} = e^{2\sigma} g_{ij}
  \]
Theorem

Let $\sigma \in \Omega^0(M)$ and $U$ be the vector field which corresponds to the 1-form $d\sigma$ so that

$$Z(\sigma) = d\sigma(Z) = g(U, Z) \quad \forall Z \in \Gamma(TM)$$

then under conformal transformation of metric tensor

$${\tilde{g}}_p = e^{2\sigma(p)}g_p$$

the Levi-Civita connection\(^1\) transforms as:

$${\tilde{\nabla}}_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X, Y)U \quad \forall X, Y \in \Gamma(TM)$$

\(^1\)${\tilde{\nabla}}$ metric w.r.t. $${\tilde{g}}$$ and $\nabla$ w.r.t. $g$
Theorem
Let $\sigma \in \Omega^0(M)$ and $U$ be the vector field which corresponds to the one-form $d\sigma$, so that $Z(\sigma) = d\sigma(Z) = g(U, Z)$. Then the Riemann tensor transforms as ($\forall X, Y, Z \in \Gamma(TM)$):

$$
\tilde{R}(X, Y)Z =
$$

$$
= R(X, Y)Z - g(Y, Z)BX + g(BX, Z)Y - g(BY, Z)X + g(X, Z)BY
$$

where $B$ is a type (1,1) tensor defined by:

$$
B(X) := -X(\sigma)U + \nabla_X U + \frac{1}{2}U(\sigma)X
$$
Contracting transformation formula for the Riemann tensor, we find expressions for \( \tilde{\varrho} \) (Ricci (0,2) form), \( \tilde{\mathcal{R}} \) (Ricci scalar curvature).

Substituting back into the formula for the Riemann tensor, we find a quantity with \( \tilde{W}^l_{ijk} = W^l_{ijk} \):

**Definition**

The coordinate expression

\[
W^l_{ijk} := R^l_{ijk} + \frac{1}{(n-2)} \left[ \varrho_{ij} \delta^l_k - \varrho_{ik} \delta^l_j + g_{ij} g^{nl} \varrho_{nk} - g_{ik} g^{nl} \varrho_{nj} \right] + \frac{\mathcal{R}}{(n-1)(n-2)} \left[ g_{ik} \delta^l_j - g_{ij} \delta^l_k \right]
\]

defines a tensor on \((M, g)\), called the *Weyl tensor* (**conformal curvature tensor**).
Properties of Weyl tensor

- Invariant of conformal transformations
- Symmetries \( W_{lijk} = -W_{likj} ; W_{lijk} = -W_{lijk} ; W_{lijk} = W_{jkli} \)
- Satisfies first Bianchi identity\(^2\)

\[
W_{ijk}^l + W_{kij}^l + W_{jki}^l = 0 \quad \forall i, j, k, l
\]

- Completely trace-less
- On \((M, g)\) with \(\text{dim } M = 3\) identically equal to zero tensor

\(^2\)the same symmetries as Riemann tensor
Theorem

Coordinate expression of the \((1, 3)\) Weyl tensor satisfies:

\[
\nabla_h W^l_{ijk} + \nabla_j W^l_{ikh} + \nabla_k W^l_{ihj} = \\
= \frac{1}{n-2} \left( \delta^l_h C_{ijk} + \delta^l_j C_{ikh} + \delta^l_k C_{ihj} + g_{ik} C^l_{jh} + g_{ih} C^l_{kj} + g_{ij} C^l_{hk} \right)
\]

where

Definition

The coordinate expression

\[
C_{ijk} = \nabla_k \varrho_{ij} - \nabla_j \varrho_{ik} + \frac{1}{2(n-1)} \left( g_{ik} \nabla_j \mathcal{R} - g_{ij} \nabla_k \mathcal{R} \right)
\]

defines a \((0, 3)\) tensor on \((M, g)\), called the Cotton tensor.
Properties of Cotton tensor

- Symmetries ($\forall i, j, k$):

  \[ C_{ijk} = -C_{ikj} \]

  \[ C_{ijk} + C_{kij} + C_{jki} = 0 \]

- Trace-free
- Identically zero on $(M, g)$ with $\text{dim } M = 2$
- On $(M, g)$ with $\text{dim } M > 3$, Weyl vanishes $\rightarrow$ Cotton vanishes
Cotton tensor under conformal transformations

Theorem

Let \((M, g)\) be a pseudo-Riemannian manifold with \(\text{dim } M = n \geq 3\). Then under a conformal transformation \(\tilde{g}_{ij} = e^{2\sigma} g_{ij}\) of the metric tensor the Cotton tensor of \(M\) transforms as follows:

\[
\tilde{C}_{ijk} = C_{ijk} - (n - 2)(\partial_a \sigma) W^a_{ijk}
\]

- In \(\text{dim } M = 3\) solely (!), Weyl tensor is an identically zero tensor, ergo Cotton tensor is an invariant of conformal transformations

\[
\tilde{C}_{ijk} = C_{ijk} \quad \forall i, j, k
\]
Obstructions to local conformal flatness

Definition
A pseudo-Riemannian manifold \((M, g)\) is *locally conformally flat* if for any \(p \in M\), there exists a neighborhood \(V\) of \(p\) and a \(C^\infty(V)\) function \(\sigma\) such that \((V, \tilde{g} = e^{2\sigma} g)\) is flat.

Theorem
A pseudo-Riemannian manifold \((M, g)\) with \(\dim M = n\) is locally conformally flat if and only if
- for \(n \geq 4\) the Weyl tensor of \(M\) vanishes
- for \(n = 3\) the Cotton tensor of \(M\) vanishes
Let $\dim M = 3$.

Thanks to anti-symmetry of $C_{ijk} = -C_{ikj}$ it is possible to be perceived as a vector-valued 2-form $(C_i) = C_{ijk} dx^j \wedge dx^k$.

By using the Hodge star $\star (C_i)$ and then valuing the obtained object on a single element of the basis of $T_pM$ as follows $Y_{ij} = [\star (C_i)] \partial_j$, we arrive at:

**Definition**

The coordinate expression

$$Y^{ij} = \epsilon^{ikl} \nabla_k \left( \varrho^j_l - \frac{1}{4} R \delta^j_l \right)$$

defines a $(2, 0)$ tensor on $(M, g)$ with $\dim M = 3$, called the **Cotton-York tensor**.
References I

M. Nakahara:
*Geometry, Topology and Physics.*

P. Eisenhart:
*Riemannian Geometry.*

J. W. York:
*Gravitational Degrees of Freedom and the Initial-Value Problem.*
References II

J. Jost:
*Riemannian Geometry and Geometric Analysis.*
Springer-Verlag, 978-3-642-21297-0. 2011.

A. Garcia, F.W. Hehl, C. Heinicke, A. Macias:
*The Cotton tensor in Riemannian spacetimes.*
arXiv:gr-qc/0309008, 2004
Thank you for your attention.