Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Properties and Applications of Cotton Tensor

Vlastnosti a význam Cottonova tensoru

Bachelor's Thesis

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Author's declaration:
I declare that this Bachelor's Thesis is entirely my own work and I have listed all the used sources in the bibliography.

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Abstrakt: Weylův tensor nabývá důležitosti jakožto invariant konformních transformací metrického tensoru a je součástí rozkladu Riemannova tensoru. Ekvivalence lokální konformní plochosti variety a nulovosti Weylova tensoru je důležitým výsledkem Riemannovy geometrie. Na třídimenzionální varietě toto neplatí, protože Weylův tensor je za takových předpokladů identicky rovný nule. Existuje jiný tensor - Cottonův, který slouží jako obstrukce lokální konformní plochosti třídimenzionální (pseudo-)Riemannovské variety. Vlastnosti Cottonova tensoru jsou velmi blízké vlastnostem Weylova tensoru, obzvláště důležitá je konformní invariance Cottonova tensoru (ve třech dimenzích). Představujeme detailní studii vlastností tensorů, které zastávají důležitou roli v moderní matematice a teoretické fyzice.

Klíčová slova: Cottonův tensor, Cottonův-Yorkův tensor, konformní transformace metrického tensoru, křivost, lokální konformní plochost, Weylův tensor

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Abstract: The Weyl tensor arises as an invariant of conformal transformations of the metric tensor and it is a completely trace-less constituent of decomposition of the Riemann tensor. The vanishing of the Weyl tensor being equivalent to local conformal flatness of the manifold is an important result of Riemannian geometry. However in three dimensions this does not hold, for the Weyl tensor vanishes identically. There is another known tensor, namely the Cotton tensor, that serves as an obstruction to local conformal flatness of a three-dimensional (pseudo-Riemannian) manifold. The Cotton tensor possesses properties that are very similar to those of the Weyl tensor, in particular it is conformally invariant (in three dimensions). The Weyl tensor and the Cotton tensor play an important role in modern mathematics and theoretical physics and we present a thorough survey of their properties.

Key words: conformal transformations of metric tensor, Cotton tensor, Cotton-York tensor, curvature, local conformal flatness, Weyl tensor

Contents

Introduction

The aim of this thesis is to present a self-contained study of geometric and algebraic properties of the Weyl and the Cotton tensors on a pseudo-Riemannian manifold in a unified notation. The Weyl tensor that first appeared in [9] is of great importance to certain subjects of study in theoretical physics, namely the general theory of relativity and other theories that include gravity ([11]). The Cotton tensor (first appearance [10]) is a less known object, however it is closely tied to the conformal geometry and the Weyl tensor by its properties and plays an important role in the Hamiltonian formulation of general relativity ([12]).

On the background of a pseudo-Riemannian manifold equipped with the Levi-Civita connection we define a conformal transformation of the metric tensor as well as review upon some of its basic properties. After doing so, we are to ask, how do fundamental objects of differential geometry (e.g. the Riemann tensor, its traces and the Levi-Civita connection) change. This is necessary, for the Weyl tensor most naturally arises as an invariant of conformal transformations of the metric tensor. The Weyl tensor is a completely trace-free constituent of decomposition of the Riemann curvature tensor and it satisfies the same symmetries (e.g. first Bianchi identity) as the Riemann tensor. We give proofs to all these classical results as well as prove that on a three-dimensional manifold the Riemann tensor is possible to be expressed solely in terms of the metric tensor, the Ricci form and the Ricci scalar curvature. This is equivalent to the fact that the Weyl tensor vanishes identically for three-dimensional manifolds.

The Cotton tensor ([10]) in a general dimension arises as a constituent of the non-zero right-hand side when we study the first covariant derivatives of the Weyl tensor. We find all the basic symmetries of the Cotton tensor and show that for the dimensions of the manifold higher than three (three excluding) the vanishing of the Weyl tensor is a sufficient condition for the Cotton tensor to vanish. After doing so we prove that the Cotton tensor is invariant under conformal transformations of the metric tensor.

The vital property of the Weyl tensor is the fact that it is an obstruction (its vanishing is a necessary and sufficient condition) to local conformal flatness of a pseudo-Riemannian manifold in dimensions four and higher. The Cotton tensor is an obstruction in a similar sense on a three-dimensional manifold. We summarize this in a theorem with a rather involved proof using the Ricci identity as an integrability condition ([4]).

Using the Hodge star we convert the Cotton tensor into an equivalent tensor of lower order on a three-dimensional manifold. This equivalent tensor, called the Cotton-York tensor, is due to York ([8]). We study its properties and show that it is divergence-free.

Chapter 1

Preliminary terms in differential geometry

In this opening chapter we shall summarize the necessary framework of differential geometry along with notation conventions that we are to use throughout the whole thesis. The definitions and results that are contained in this chapter are largely standard yet possess appropriate amount of generality which shall be made use of in the chapters that are to follow. Unless specified otherwise content of this chapter is courtesy of [1] and [2].

1.1 Differentiable manifolds, vector bundles

Definition 1.1.1. Let M be a topological space.

- Let $U^\circ = U \subset M$ and $V^\circ = V \subset \mathbb{R}^n$. A homeomorphism $\varphi : U \to V$ is called a *(local coordinate) chart* on M. Here the open set U is called a *coordinate neighborhood.*

- An open covering $\{U_\alpha\}_{\alpha\in I}$ of the space M equipped with charts $\varphi_\alpha : U_\alpha \to V_\alpha$ is called an *atlas* on M.

- We say that atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ on M is *differentiable* if all the maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

  are of the class $C^\infty(\mathbb{R}^n)$ for all $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Compositions of the type $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called *transition maps.*

- A chart is said to be *compatible* with an atlas if its transition maps for all charts contained in the atlas with intersecting coordinate neighborhoods are of the class $C^\infty(\mathbb{R}^n)$.

- An atlas is called a *differentiable structure* if it contains all the compatible maps.

Definition 1.1.2. A topological space M that is Hausdorff, paracompact and equipped with a differentiable structure is called a *differentiable manifold.*

Definition 1.1.3. A *tangent vector* X on a differentiable manifold M at a point $p \in M$ is a map $X : C^\infty(M) \to \mathbb{R}$ satisfying:

1. $X(\alpha f + g) = \alpha X f + X g \qquad \forall f, g \in C^\infty(M), \forall \alpha \in \mathbb{R}$
2. $\forall f, g \in C^\infty(M)$ there exists an open neighborhood U of the point $p \in M$ such that $f|_U = g|_U$ implies $Xf = Xg$
3. $X(fg) = (Xf)g(p) + f(p)(Xg) \qquad \forall f, g \in C^\infty(M)$

Remark 1.1.4. A space of all tangent vectors at a point p is a vector space of the same dimension n that are the open sets of $\mathbb{R}^n$ mapped bijectively by charts. Such space is called the *tangent space* at p, denoted by T_pM.

Definition 1.1.5. A *(differentiable) vector bundle* of rank n consists of a *total space* E, a base M and a *projection* $\pi : E \to M$, where E and M are differentiable manifolds, π is differentiable, each *fiber* $E_x := \pi^{-1}(x)$ for $x \in M$ carries the structure of an n-dimensional vector space, and the following *local triviality* requirement is satisfied:
For each $x \in M$, there exists neighborhood U and a diffeomorphism

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$

with the property that for every $y \in U$

$$\varphi_y := \varphi|_{E_y} : E_y \to \{y\} \times \mathbb{R}^n$$

is a vector space isomorphism (a bijective linear map). Such pair (φ, U) is called a *bundle chart.*

Remark 1.1.6. A vector bundle that is globally isomorphic to $M \times \mathbb{R}^n$ (n =rank) is called *trivial.*

Definition 1.1.7. A *tangent bundle* is a disjoint union of all $T_pM,\ p \in M$:

$$TM = \coprod_{p \in M} T_pM = \{X_p \in T_pM | p \in M\}$$

equipped with projection $\pi : TM \to M$ such that $\pi(X_p) = p$. An *atlas* $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in I}$ is on TM constructed from an atlas $\{U_\alpha, \varphi_\alpha = \{x^i_\alpha\}\}_{\alpha \in I}$ on M for $V_\alpha := \pi^{-1}(U_\alpha)$ and $\psi_\alpha := V_\alpha \to \varphi_\alpha(U\alpha) \times \mathbb{R}^n$ as follows

$$\psi_\alpha(X_p) = (x^1_\alpha(p), ..., x^n_\alpha(p), X^1_\alpha(p), ..., X^n_\alpha(p))$$

Remark 1.1.8. A tangent bundle is a vector bundle.

Definition 1.1.9. Let (E, π, M) be a vector bundle. A *section* of E is a differentiable map $s : M \to E$ with $\pi \circ s = \mathrm{id}_M$. The space of sections of E is denoted by $\Gamma(E)$.

Definition 1.1.10. A section of the tangent bundle TM of M is called a *vector field* on M.

Definition 1.1.11. Let M be a differentiable manifold, $x \in M$. The vector space dual to the tangent space T_pM is called *the cotangent space* of M at the point p and denoted by T_p^*M. The vector bundle over M whose fibers are the cotangent spaces of M is called the *cotangent bundle* of M and denoted T^*M. Sections of T^*M are called *one-forms*.

Remark 1.1.12. We shall hold onto the conventional notation and denote the basis vectors of a tangent bundle by $\{\partial_i\} \equiv \left\{\frac{\partial}{\partial x^i}\right\}$. Thus the coordinate expression of a vector field becomes $X = X^i\partial_i$. In a very simmilar manner a basis of a cotangent bundle is $\{dx^i\}$ and a coordinate expression for an arbitrary one-form becomes $\omega = \omega_i dx^i$. The two bases $\{\partial_i\}$ and $\{dx^i\}$ are dual to each other in a sense that $\langle dx^i, \partial_j\rangle = dx^i(\partial_j) = \delta^i_j$ for all i, j. This also defines a bilinear inner product $\langle\cdot,\cdot\rangle : T_p^*M \times T_pM \to \mathbb{R}$ as follows:

$$\langle\omega, V\rangle = \omega_i V^j \langle dx^i, \partial_j\rangle = \omega_i V^j \delta^i_j = \omega_i V^i \qquad \forall\omega \in T_p^*M, V \in T_pM$$

1.2 Tensors, differential forms

Definition 1.2.1. A p times *contravariant* and q times *covariant tensor* or a tensor of a type (p, q) on a differentiable manifold M is a section of

$$\underbrace{TM \otimes ... \otimes TM}_{p \text{ times}} \quad \otimes \quad \underbrace{T^*M \otimes ... \otimes T^*M}_{q \text{ times}}$$

Definition 1.2.2. A *tensor* T of a type (p, q) is a multilinear map (in stronger sense of multiplying one of its arguments by a $C^\infty(M)$ function) that maps p elements of TM and q elements of T^*M to $\mathbb{R}$.

Remark 1.2.3. The two definitions of a tensor are equivalent.

Definition 1.2.4. Let M be a differentiable manifold. A *pseudo-Riemannian* metric g is a $(0, 2)$ tensor on M such that it is:

- Symmetric $\quad g(U, V) = g(V, U) \qquad \forall U, V \in \Gamma(TM)$
- Non-degenerate $\quad g(U, V) = 0 \quad \forall V \in TM \quad \to \quad V = 0$

In addition to that, if the metric tensor is:

- Positive definite $\quad g(U, V) \geq 0 \quad \forall U, V \in TM$

and the equality holds only for $V = 0$, it is called a *Riemannian metric.*

Remark 1.2.5. By g_p we shall denote a metric at a particular point p of M. Since g_p is a map $T_pM \otimes T_pM \to \mathbb{R}$ it is possible to define a linear map $g_p(U, \cdot) : T_pM \to \mathbb{R}$ by $V \to g_p(U, V)$. Then $g_p(U, \cdot)$ is identified with a one-form $\omega_U \in T_p^*M$. Similarly, $\omega \in T_p^*M$ induces $V_\omega \in T_pM$ by $g_p(V_\omega, U) := \langle\omega, U\rangle$ for all $U \in T_pM$. (Here the inner product $\langle\cdot,\cdot\rangle$ is the one defined in remark 1.1.12.) Thus, the metric g_p gives rise to an isomorphism between T_pM and T_p^*M.

Definition 1.2.6. Let (E, π, M) be a vector bundle. A *bundle metric* is given by a family of scalar products on the fibers E_x, depending smoothly on $x \in M$.

Theorem 1.2.7. Each vector bundle (E, π, M) of rank n with a bundle metric has structure group $O(n)$. In particular, there exist bundle charts $(f, U), f : \pi^{-1}(U) \to U \times \mathbb{R}^n$, for which $\forall x \in U, f^{-1}(x, (e_1, ..., e_n))$ is an orthonormal basis of E_x $(e_1, ..., e_n$ is an orthonormal basis of $\mathbb{R}^n)$.

Theorem 1.2.8. Each vector bundle can be equipped with a bundle metric.

Definition 1.2.9. A local orthonormal basis of T_xM of the type obtained in theorem 1.2.7 is called an *orthonormal frame field.*

Definition 1.2.10. The *wedge product* $\wedge$ of k one-forms is the totally antisymmetric tensor product

$$dx^{\mu_1} \wedge ... \wedge dx^{\mu_k} = \sum_{\pi \in S_k} \operatorname{sgn} \pi \; dx^{\mu_{\pi(1)}} \wedge ... \wedge dx^{\mu_{\pi(k)}}$$

Remark 1.2.11. We put

$$\Lambda^k_p(M) := \underbrace{T^*_p M \wedge ... \wedge T^*_p M}_{k \text{ times}}$$

A vector bundle over M whose fibers are the spaces $\Lambda^k_p(M)$ for $p \in M$ is denoted $\Lambda^k(M)$.

Definition 1.2.12. The space of sections of $\Lambda^k(M)$ is denoted by $\Omega^k(M)$ so that $\Omega^k(M) = \Gamma(\Lambda^k(M))$. Elements of $\Omega^k(M)$ are called *(exterior) k-forms.*

Remark 1.2.13. Let $\dim M = n$, we put

$$\Lambda^\bullet_p(M) := \Lambda^0_p(M) \oplus \Lambda^1_p(M) \oplus ... \oplus \Lambda^n_p(M)$$

where $\Lambda^0_p(M) = C^\infty(U)$, $p \in U \subset M$ and $\oplus$ denotes the direct sum. A vector bundle over M whose fibers are the spaces $\Lambda^\bullet_p(M)$ for $p \in M$ is denoted $\Lambda^\bullet(M)$. The space of sections of $\Lambda^\bullet(M)$ is denoted by $\Omega^\bullet(M)$ so that $\Omega^\bullet(M) = \Gamma(\Lambda^\bullet(M))$.

Definition 1.2.14. We define an *exterior product* for the elements of $\Lambda^\bullet_p(M)$ followingly. Let $\omega \in \Lambda^q_p(M), \xi \in \Lambda^r_p(M)$ and $V_1, ..., V_{q+r} \in T_pM$, we put:

$$(\omega \wedge \xi)(V_1, ..., V_{q+r}) := \frac{1}{q! \cdot r!} \sum_{\pi \in S_{q+r}} \operatorname{sgn} \pi \;\; \omega(V_{\pi(1)}, ..., V_{\pi(q)}) \xi(V_{\pi(q+1)}, ..., V_{\pi(q+r)})$$

Theorem 1.2.15 (Properties of exterior product)**.** The exterior product satisfies:

1. linearity (in both arguments)

$$(\alpha\omega_1 + \omega_2) \wedge \theta = \alpha\omega_1 \wedge \theta + \omega_2 \wedge \theta \qquad \forall \omega_1, \omega_2, \theta \in \Lambda^\bullet_p(M), \alpha \in \mathbb{R}$$

2. asociativity

$$(\omega \wedge \theta) \wedge \tau = \omega \wedge (\theta \wedge \tau) \qquad \forall \tau, \omega, \theta \in \Lambda^\bullet_p(M)$$

3. antisymmetry (in the following sense)

$$\omega \wedge \sigma = (-1)^{k \cdot m} \sigma \wedge \omega \qquad \omega \in \Lambda_p^m(M), \sigma \in \Lambda_p^k(M)$$

Definition 1.2.16. The *exterior derivative* is a linear map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying:

1. $d(\sigma \wedge \omega) = d\sigma \wedge \omega + (-1)^k \sigma \wedge d\omega \qquad \sigma \in \Omega^k(M), \omega \in \Omega^\bullet(M)$
2. $d^2\omega = (d \circ d)\omega = 0 \qquad \omega \in \Omega^\bullet(M)$
3. $df(X) = Xf \qquad f \in \Omega^0(M) = C^\infty(M), X \in \Gamma(TM)$

Definition 1.2.17. A differential form $\omega \in \Omega^\bullet(M)$ is said to be:

- *closed* if $\quad d\omega = 0$
- *exact* if $\quad \exists \sigma \in \Omega^\bullet(M)$ such that $\omega = d\sigma$

Remark 1.2.18. A differential form that is exact is closed.

Theorem 1.2.19 (Poincaré lemma)**.** If a coordinate neighborhood U of a manifold M is contractible to a point $p \in M$, any closed r-form on U is also exact.

Remark 1.2.20. Any closed form is by Poincaré lemma exact at least locally (on a certain coordinate neighborhood).

1.3 Lie algebras, induced maps, Lie groups

Definition 1.3.1. For vector fields X, Y on M, the *Lie bracket* $[X, Y]$ is defined as the vector field:

$$[X, Y] := X \circ Y - Y \circ X$$

Proposition 1.3.2 (Cartan identity)**.** Let $\omega \in \Omega^1(M)$ and $X, Y \in \Gamma(TM)$. Then the following identity holds:

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

Definition 1.3.3. A *Lie algebra* (over $\mathbb{R}$) is a real vector space V equipped with a bilinear map $l(.,.) : V \times V \to V$, satisfying:

1. $l(X, X) = 0 \quad \forall X \in V$
2. $l(X, l(Y, Z)) + l(Y, l(Z, X)) + l(Z, l(X, Y)) = 0 \quad \forall X, Y, Z \in V$

Corollary 1.3.4. The space of vector fields on M, equipped with the Lie bracket is a Lie algebra.

Definition 1.3.5. Let M, N be two diferentiable manifolds, and let $\Phi : M \to N$ be a differentiable map.

- A *tangent map* Φ_* at a point $p \in M$ induced by the map Φ is a map $\Phi_* : T_pM \to T_{\Phi(p)}N$ defined by:

$$(\Phi_*(X))f = X(f \circ \Phi) \qquad X \in T_pM, f \in C^\infty(U_{\Phi(p)})$$

- A *cotangent map* Φ^* at a point $p \in N$ induced by the map Φ is a map $\Phi^* : T^*_{\Phi(p)}N \to T^*_pM$ defined by:

$$(\Phi^*(\omega))X = \omega(\Phi_* X) \qquad X \in T_pM, \omega \in T^*_{\Phi(p)}N$$

- A map $\Phi^* : \Omega^\bullet \to \Omega^\bullet$ defined for an arbitrary k-form $\omega \in \Omega^k(N)$ by:

$$(\Phi^*(\omega))(X_1, ..., X_k) := \omega(\Phi(p))(\Phi_*(X_1|_p), ..., \Phi_*(X_k|_p)) \quad X_1, ..., X_k \in TM$$

  is called a *pullback*.

- If the map Φ is a diffemorphism of the manifolds M, N, we define a map $\Phi_* : TM \to TM$ by

$$(\Phi_* X)(\Phi(p)) := \Phi_*(X|_p) \qquad X \in TM$$

  that is called a *pushforward*.

Lemma 1.3.6. Let $\psi : M \to N$ be a diffeomorphism, X, Y vector fields on M. Then,

$$[\psi_* X, \psi_* Y] = \psi_*[X, Y] \tag{1.1}$$

Thus, ψ_* induces a Lie algebra isomorphism.

Definition 1.3.7. A *Lie group* G is a differentiable manifold which is endowed with a group structure such that the group operations:

1. $\cdot : G \times G \to G, \quad (g_1, g_2) \to g_1 \cdot g_2$
2. $^{-1} : G \to G, \quad g \to g^{-1}$

are differentiable.

Definition 1.3.8. Let G be a Lie group. For $g \in G$ we define the left translation

$$L_g : G \to G : h \to gh \quad \forall h \in G$$

and the right translation

$$R_g : G \to G : h \to hg \quad \forall h \in G$$

Remark 1.3.9. The translations L_g and R_g are diffeomorphisms of a Lie group G.

Definition 1.3.10. A vector field X on G is called *left invariant* if for all $g, h \in G$

$$L_{g_*} X|_h = X|_{gh}$$

Theorem 1.3.11. Let G be a Lie group. For every $V \in T_eG$ (where e shall denote the identity element of the group G throughout the rest of the chapter):

$$X|_g = L_{g_*}V \qquad \forall g \in G$$

defines a left invariant vector field on G, and we thus obtain an isomorphism between T_eG and the space of left invariant vector fields on G.

Remark 1.3.12. By lemma 1.3.6 for $g \in G$ and vector fields X, Y on G, we have:

$$[L_{g_*}X, L_{g_*}Y] = L_{g_*}[X, Y]$$

Corollary 1.3.13. The vector space T_eG carries the structure of a Lie algebra.

Definition 1.3.14. The *Lie algebra* $\mathfrak{g}$ *of* G is the vector space T_eG equipped with the Lie algebra structure of corollary 1.3.13.

Definition 1.3.15. Let G be a Lie group. A *principal G-bundle* consists of a base M, which is a differentiable manifold and a differentiable manifold P, the total space of the bundle, and a differentiable projection $\pi : P \to M$ with an action of G on P satisfying:

1. G acts freely on P from the right - $(p, g) \in P \times G$ is mapped to $pg \in P$ and $pg \neq p$ for $g \neq e$. The so-called G-action then defines an equivalence relation on $P : q \sim p$ if and only if $\exists g \in G$ such that $q = pg$.

2. M is the quotient of P by this equivalence relation and $\pi : P \to M$ maps $p \in P$ to its equivalence class. By (1.), each fiber $\pi^{-1}(x)$ can then be identified with G.

3. P is locally trivial in the following sense: For each $x \in M$, there exists a neighborhood U of x and a diffeomorphism

   $$\varphi : \pi^{-1}(U) \to U \times G$$

   of the form $\varphi(p) = (\pi(p), \psi(p))$ which is G-equivariant, i.e.

   $$\varphi(pg) = (\pi(p), \psi(p)g) \qquad \forall g \in G$$

1.4 Miscellaneous tools of tensor analysis

Definition 1.4.1. Let M be a differentiable manifold. We define a totally antisymmetric *Levi-Civita tensor density* ε by

$$\varepsilon_{\mu_1\mu_2...\mu_m} := \begin{cases} +1 & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an even permutation of } (12...m) \\ -1 & \text{if } (\mu_1\mu_2...\mu_m) \text{ is an odd permutation of } (12...m) \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

Remark 1.4.2. Apparently

$$\varepsilon^{\mu_1\mu_2...\mu_m} = g^{\mu_1\nu_1}g^{\mu_2\nu_2}...g^{\mu_m\nu_m}\varepsilon_{\nu_1\nu_2...\nu_m} = \det(g^{-1})\varepsilon_{\mu_1\mu_2...\mu_m} = \frac{1}{\det g}\varepsilon_{\mu_1\mu_2...\mu_m}$$

Lemma 1.4.3. The Levi-Civita tensor density ε_{ijk} satisfies:

$$\varepsilon_{ijk}\varepsilon^{imn} = \delta_j^m \delta_k^n - \delta_k^m \delta_j^n$$

Remark 1.4.4. It is possible to define a totally antisymmetric *Levi-Civita tensor* ϵ with the same algebraic properties (e.g. lemma 1.4.3) as the Levi-Civita tensor density except for the fact, that it does transform as a tensor, by putting:

$$\epsilon^{ijk} := \frac{1}{\sqrt{|\det g|}}\varepsilon^{ijk} \quad \text{and} \quad \epsilon_{ijk} := \sqrt{|\det g|}\varepsilon_{ijk} \tag{1.3}$$

Therefore we have:

$$\epsilon_{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_k^m \delta_j^n \tag{1.4}$$

Definition 1.4.5. Let (M, g) be a pseudo-Riemannian orientable manifold with $\dim M = m$. Let us define a linear operation $\star : \Omega^r(M) \to \Omega^{m-r}(M)$ by its action on a basis r-form:

$$\star(dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r}) := \frac{\sqrt{|\det g|}}{(m-r)!}\varepsilon^{\mu_1\mu_2...\mu_r}_{\nu_{r+1}...\nu_m} dx^{\nu_{r+1}} \wedge ... \wedge dx^{\nu_m}$$

This operation is called the *Hodge star (dual)*.

Remark 1.4.6. For an arbitrary r-form

$$\omega = \frac{1}{r!}\omega_{\mu_1\mu_2...\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_r} \qquad \in \Omega^r(M)$$

we have

$$\star\omega = \frac{\sqrt{|\det g|}}{r!(m-r)!}\omega_{\mu_1\mu_2...\mu_r}\varepsilon^{\mu_1\mu_2...\mu_r}_{\nu_{r+1}...\nu_m} dx^{\nu_{r+1}} \wedge ... \wedge dx^{\nu_m} \tag{1.5}$$

Remark 1.4.7. The Hodge star is an isomorphism of $\Omega^r(M)$ and $\Omega^{m-r}(M)$.

Definition 1.4.8. [3] The *Kulkarni-Nomizu product* of two symmetric $(0, 2)$ tensors H and Q is the $(0, 4)$ tensor $H \owedge Q$ given ($\forall V, X, Y, Z \in \Gamma(TM)$) by[1]:

$$H \owedge Q(V, X, Y, Z) = H(V, Z)Q(X, Y) + H(X, Y)Q(V, Z) - H(V, Y)Q(X, Z) - H(X, Z)Q(V, Y)$$

Remark 1.4.9. In coordinates, this becomes:

$$(H \owedge Q)_{lijk} = H_{jl}Q_{ik} + H_{ik}Q_{jl} - H_{ij}Q_{kl} - H_{kl}Q_{ij} \qquad \forall i, j, k, l \tag{1.6}$$

Lemma 1.4.10. [3] The Kulkarni-Nomizu product is symmetric

$$H \owedge Q = Q \owedge H$$

Proposition 1.4.11. [3] Let M be a differentiable manifold with $\dim M > 2$. The map $Q \owedge \cdot$ from $\Gamma(TM \otimes TM)$ into $\Gamma(TM \otimes TM \otimes TM \otimes TM) \equiv \Gamma(\bigotimes^4 TM)$ defined by

$$H \to Q \owedge H \qquad \forall H \in \Gamma(TM \otimes TM)$$

is injective.

[1] Here we adopt a different sign convention than in [3] in order for coordinate expression of the Kulkarni-Nomizu product to be in line with our overall coordinate notation.

Chapter 2

Connection and canonical tensors

In this chapter we shall develop the theory of connections on (pseudo-)Riemannian manifolds. For the chapters on conformal transformations it would suffice to define a connection on a tangent bundle solely however the theory we build here is more general in a sense that connections are constructed over vector bundles. Unless specified differently, the theory presented here is again courtesy of [1] and [2].

2.1 Connection on vector bundle

Definition 2.1.1. Let M be a differentiable manifold, E a vector bundle over M. A *(linear) connection* is a map

$$D : \Gamma(TM) \otimes \Gamma(E) \to \Gamma(E)$$

with the properties:

1. D is f-linear in $V \in \Gamma(TM)$:

$$D_{V+W}\sigma = D_V\sigma + D_W\sigma \qquad \forall W \in \Gamma(TM), \sigma \in \Gamma(E)$$

$$D_{fV}\sigma = fD_V\sigma \qquad f \in C^\infty(M), \sigma \in \Gamma(E)$$

2. D is $\mathbb{R}$-linear in $\sigma \in \Gamma(E)$:

$$D_V(\alpha\sigma + \tau) = \alpha D_V\sigma + D_V\tau \qquad V \in \Gamma(TM), \alpha \in \mathbb{R}, \forall\tau \in \Gamma(E)$$

   and it satisfies the following product rule:

$$D_V(f\sigma) = V(f) \cdot \sigma + fD_V\sigma \qquad V \in \Gamma(TM), f \in C^\infty(M)$$

Remark 2.1.2. By a property (1) in the previous definition, we may consider D as a map from $\Gamma(E)$ to $\Gamma(E) \otimes \Gamma(T^*M)$ by putting $D\sigma(V) := D_V\sigma$ for all $\sigma \in \Gamma(E)$.

Definition 2.1.3. Let $p_0 \in M$ and let U be an open neighborhood of p_0 such that a chart for M and a bundle chart for E are defined on U. We then obtain coordinate vector fields $\partial_1, ..., \partial_d$ (where $d = \dim M$), and through the identification

$$E|_U \cong U \times \mathbb{R}^n \qquad (n = \text{fiber dimension of } E)$$

a basis of $\mathbb{R}^n$ yields a basis $\mu_1, ..., \mu_n$ of sections of $E|_U$. For a connection D, we define functions called Christoffel symbols Γ^k_{ij} $(j, k = 1, ..., n, i = 1, ..., d)$ by:

$$D_{\partial_i}\mu_j \equiv D_i\mu_j =: \Gamma^k_{ij}\mu_k$$

Remark 2.1.4. Let now $\mu \in \Gamma(E)$; locally, we write $\mu(y) = a^k(y)\mu_k(y)$. Also let $c(t)$ be a smooth curve in U. Putting $\mu(t) := \mu(c(t))$ we define a section of E along c. Furthermore, let $V(t) = \dot{c}(t) = \frac{d}{dt}c(t) = [\dot{c}(t)]^i\partial_i$. Then by the definition of a connection:

$$D_{V(t)}\mu(t) = D_{\dot{c}^i(t)\partial_i}[a^k(c(t))\mu_k(c(t))] = \dot{c}^i(t)[\partial_i(a^k(c(t)))\mu_k(t) + a^k(c(t))D_i\mu_k(t)] =$$

$$= \underbrace{\dot{c}^i(t)\frac{\partial a^k(c(t))}{\partial x^i}}_{\dot{a}^k(t)}\mu_k(t) + a^k(c(t))\dot{c}^i(t)\Gamma^j_{ik}(c(t))\mu_j(t)$$

The first memeber is completely independent of D. Christoffel symbols $\Gamma^j_{ik}(c(t))$ here have indices j, k running from 1 to n (where n is the fibre dimension of E) and an index i running from 1 to $d = \dim M$. The index i describes the application of the tangent vector $\dot{c}^i(t)\partial_i$. We can thus consider $\{\Gamma^j_{ik}\}_{i,j,k}$ as an $(n \times n)$-matrix valued 1-form on U (from the previous definition) :

$$\{\Gamma^j_{ik}\}_{i,j,k} \in \Gamma(\mathfrak{gl}(n, \mathbb{R}) \otimes T^*M|_U)$$

Here, Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ is a space of $(n \times n)$-matrices with real coefficients. In a more abstract manner, we now write on U

$$D = d + A \tag{2.1}$$

where d is an exterior derivative and $A \in \Gamma(\mathfrak{gl}(n, \mathbb{R}) \otimes T^*M|_U)$. A can be also considered as an $(n \times n)$-matrix with values in section of the cotangent bundle of M; A applied to the tangent vector ∂_i becomes $\{\Gamma^j_{ik}\}_{i,j,k}$. The application of A to $a^j\mu_j$ is given by ordinary matrix multiplication. We have:

$$D(a^j\mu_j) = d(a^j)\mu_j + a^jA\mu_j$$

Remark 2.1.5. We now write $A\mu_j = A^k_j\mu_k$, where each A^k_j now is a 1-form $A^k_j = \Gamma^k_{ij}dx^i$. Let $\mu^*_1, ..., \mu^*_n$ be the basis dual to $\mu_1, ..., \mu_n$ on the bundle E^* dual to E, explicitly $(\mu_i, \mu^*_j) = \delta_{ij}$ where $(\cdot, \cdot) : E \otimes E^* \to \mathbb{R}$ is the bilinear pairing[1] between E and E^*.

[1] Here the pairing between a vector bundle and a bundle dual to it is defined in an analogic way to that of the one between tangent bundle and its dual (see remarks 1.1.12 and 1.2.5). We shall consistently use two types of brackets - $\langle ., . \rangle$ for the one on $T^*M \times TM$ and $(., .)$ for the one on $E^* \times E$ - to differentiate between them.

Definition 2.1.6. Let D be a connection on E. The *connection D^* dual* to D on the dual bundle E^* is defined by the requirement:

$$d(\mu, \nu^*) = (D\mu, \nu) + (\mu, D^*\nu^*) \qquad \forall \mu \in \Gamma(E), \nu^* \in \Gamma(E^*)$$

Remark 2.1.7. Let us (via $D = d + A$) compute :

$$0 = d(\mu_i, \mu_j^*) = (\underbrace{d\mu_i}_{=0} + A_i^k \mu_k, \mu_j^*) + (\mu_i, \underbrace{d^*\mu_j^*}_{=0} + (A^*)_j^k \mu_k^*) = A_i^k \delta_{ij} + (A^*)_j^k \delta_{ik} = A_{ij} + (A^*)_{ji}$$

Hence

$$A^* = -A^T \tag{2.2}$$

where upper "index" T denotes matrix transposition. From this we get:

$$D_i^* \mu_j^* = -\Gamma_{ik}^j \mu_k^* \tag{2.3}$$

Definition 2.1.8. Let E_1, E_2 be vector bundles over M with connections D_1, D_2 respectively. The *induced connection* D on $E := E_1 \otimes E_2$ is defined by the requirement:

$$D(\mu_1 \otimes \mu_2) = (D_1\mu_1) \otimes \mu_2 + \mu_1 \otimes (D_2\mu_2) \qquad \mu_i \in \Gamma(E_i),\ i = 1, 2$$

Remark 2.1.9. In particular, we obtain an induced connection on $\mathrm{End}(E) = E \otimes E^*$ denoted by D. Let $\sigma = \sigma_j^i \mu_i \otimes \mu_j^*$. We compute:

$$D(\sigma_j^i \mu_i \otimes \mu_j^*) = [(d + A)\sigma_j^i \mu_i] \otimes \mu_j^* + \sigma_j^i \mu_i \otimes [(d^* + A^*)\mu_j^*] =$$

$$= (d\sigma_j^i)\mu_i \otimes \mu_j^* + \sigma_j^i \underbrace{d(\mu_k)}_{=0} \otimes \mu_j^* + \sigma_j^i A_i^k \mu_k \otimes \mu_j^* + \sigma_j^i \mu_i \otimes \underbrace{d^*\mu_j^*}_{=0} + \underbrace{\sigma_j^i \mu_i \otimes (A^*)_j^k \mu_k^*}_{=-\sigma_j^i A_k^j \mu_i \otimes \mu_k^*}$$

We obtain the following identity:

$$D(\sigma) = d\sigma + [A, \sigma] \qquad \forall \sigma \in \mathrm{End}(E) = E \otimes E^* \tag{2.4}$$

Remark 2.1.10. Henceforth we shall write as an abbreviation:

$$\Omega^p(E) := \Gamma(E) \otimes \Omega^p(M)$$

where as usual M is a differentiable manifold and E a vector bundle.

2.2 Curvature

Definition 2.2.1. The *curvature* of a connection D on a vector bundle E is the map

$$F := D \circ D : \Omega^0(E) \to \Omega^2(E)$$

Definition 2.2.2. The connection is called *flat* if its curvature satisfies $F = 0$.

Remark 2.2.3. We compute by 2.1 for $\mu \in \Gamma(E)$

$$F(\mu) = (d + A) \circ (d + A)\mu = (d + A)(d\mu + A\mu) =$$

$$= \underbrace{d^2\mu}_{=0} + (dA)\mu + (-1)^1 A \wedge d\mu + A \wedge d\mu + (A \wedge A)\mu = (dA + A \wedge A)\mu$$

Therefore we have the identity:

$$F = dA + A \wedge A \tag{2.5}$$

Remark 2.2.4. We now want to express the identity 2.5 in coordinates. Because $A = A_j dx^j$, we have:

$$\begin{aligned}
F &= d(A_j dx^j) + (A_i dx^i) \wedge (A_j dx^j) = \\
&= \frac{\partial A_j}{\partial x^i} dx^i \wedge dx^j + A_i A_j dx^i \wedge dx^j = \\
&= \left(\frac{\partial A_j}{\partial x^i} + A_i A_j \right) dx^i \wedge dx^j = \\
&= \frac{1}{2} \left(\frac{\partial A_j}{\partial x^i} + A_i A_j \right) dx^i \wedge dx^j + \frac{1}{2} \left(\frac{\partial A_i}{\partial x^j} + A_j A_i \right) dx^j \wedge dx^i = \\
&= \frac{1}{2} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + \underbrace{A_i A_j - A_j A_i}_{=[A_i, A_j]} \right) dx^i \wedge dx^j
\end{aligned}$$

The identity in coordinates now yields:

$$F = \frac{1}{2} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} + [A_i, A_j] \right) dx^i \wedge dx^j \tag{2.6}$$

Remark 2.2.5. We now want to compute DF. F is a map from $\Omega^0(E)$ to $\Omega^2(E)$, i.e.

$$F \in \Omega^2(E) \otimes (\Omega^0(E))^* = \Omega^2(\mathrm{End}(E)) := \Omega^2(E \otimes E^*)$$

We thus consider F as a 2-form with values in $\mathrm{End}(E)$. Now we have (by using identities 2.4 and 2.5)

$$DF = dF + [A, F] = d(dA + A \wedge A) + [A, dA + A \wedge A] =$$

$$= \underbrace{d^2 A}_{=0} + dA \wedge A + (-1)^1 A \wedge dA + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A =$$

$$= dA \wedge A - A \wedge dA + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A = 0$$

Theorem 2.2.6 (Second Bianchi Identity)**.** The curvature of a connection D satisfies

$$DF = 0$$

Remark 2.2.7. We will denote the connection on a tangent bundle TM by $\nabla : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM)$. For such connection, the Christoffel symbols are given by $\nabla_{\partial_i}\partial_j \equiv \nabla_i\partial_j = \Gamma^k_{ij}\partial_k$.

Remark 2.2.8. In order to find the coordinate expression for the curvature map, we shall consider F as an element of $\Omega^2(\mathrm{End}(E))$ and by R denote :

$$\begin{aligned} F : \Omega^0(E) &\to \Omega^2(E) \\ \mu &\to R(\cdot,\cdot)\mu \end{aligned}$$

Henceforth, we shall write (for $k, l = 1, ..., n$ and $i, j = 1, ..., d$):

$$R^k_{lij}\mu_k := R(\partial_i, \partial_j)\mu_l \tag{2.7}$$

where (by equality 2.6 and remark 2.1.4)

$$R(\cdot,\cdot)\mu_l = F\mu_l = \frac{1}{2}\left(\partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\right) dx^i \wedge dx^j \otimes \mu_k$$

i.e.

$$R^k_{lij} = \partial_i\Gamma^k_{jl} - \partial_j\Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il} \tag{2.8}$$

which is the coordinate expression of R.

Lemma 2.2.9. The quantity R is a tensor.

2.3 Riemann tensor, connection on tangent bundles

Definition 2.3.1. Let M be a differentiable manifold equipped with a connection D. We call R the *Riemann (curvature) tensor* of the connection D on the manifold M.

Theorem 2.3.2. The Riemann curvature tensor R of a connection D satisfies:

$$R(X, Y)\mu = D_X D_Y \mu - D_Y D_X \mu - D_{[X,Y]}\mu \tag{2.9}$$

for all vector fields X, Y on M and all $\mu \in \Gamma(E)$.

Corollary 2.3.3. The Riemann tensor R satisfies

$$R(X, Y) = -R(Y, X) \qquad \forall X, Y \in \Gamma(TM)$$

Corollary 2.3.4. Coordinate expression of Riemann tensor R satisfies:

$$R^k_{lij} = -R^k_{lji} \qquad \forall i, j, k, l \tag{2.10}$$

Remark 2.3.5. Henceforth we shall denote by ∇ a connection on the tangent bundle TM. For such connection, the Christoffel symbols are given by:

$$\nabla_{\partial_i}\partial_j \equiv \nabla_i\partial_j = \Gamma^k_{ij}\partial_k$$

Remark 2.3.6. With respect to ∇ on $\Gamma(TM)$ the Riemann tensor takes form of:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad \forall X,Y,Z \in \Gamma(TM) \tag{2.11}$$

Remark 2.3.7. In a very similar manner to that of remark 2.1.7 and especially equation 2.3, where a connection D induces a connection D^* on the dual bundle E^*, the connection of the previous remark 2.3.5 induces the dual connection (on T^*M), which we shall denote ∇ as well. Its Christoffel symbols satisfy:

$$\nabla_i dx^j = -\Gamma^j_{ik} dx^k \tag{2.12}$$

Remark 2.3.8. For a one-form ω we have the following identity:

$$\nabla_j \omega_i = \partial_j \omega_i - \Gamma^k_{ji} \omega_k \tag{2.13}$$

This arises from the duality pairing by $\langle \cdot, \cdot \rangle$ as follows:

$$\langle \nabla_j(\omega_p dx^p), \partial_i \rangle = \langle (\partial_j \omega_p) dx^p - \omega_p \Gamma^p_{jm} dx^m, \partial_i \rangle = (\partial_j \omega_p) \underbrace{dx^p(\partial_i)}_{\delta^p_i} - \omega_p \Gamma^p_{jm} \underbrace{dx^m(\partial_i)}_{\delta^m_i}$$

Definition 2.3.9. The *torsion tensor* of a connection ∇ on TM is defined by:

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \qquad \forall X,Y \in \Gamma(TM) \tag{2.14}$$

Definition 2.3.10. The connection ∇ on the tangent bundle TM is called *torsion free* if its torsion tensor vanishes identically, i.e.

$$T \equiv 0$$

Lemma 2.3.11. The connection ∇ on TM is torsion free if and only if

$$\Gamma^k_{ij} = \Gamma^k_{ji} \qquad \forall i,j,k \tag{2.15}$$

Definition 2.3.12. A connection ∇ on TM is called *flat* if each point in M possesses a neighborhood U with local coordinates for which all the coordinate vector fields ∂_i are parallel, that is

$$\nabla_j \partial_i = 0 \qquad \forall i,j$$

Theorem 2.3.13. A connection ∇ on TM is flat if and only if its Riemann curvature tensor and torsion tensor both vanish identically.

2.4 Metric connection, Levi-Civita connection

Definition 2.4.1. Let E be a vector bundle on a differentiable manifold M with bundle metric $\langle \cdot, \cdot \rangle$. A connection D on E is called *metric*, if

$$d\langle \mu, \nu \rangle = \langle D\mu, \nu \rangle + \langle \mu, D\nu \rangle \qquad \forall \mu, \nu \in \Gamma(E)$$

Remark 2.4.2. On a tangent bundle, the previous definition is equivalent to the fact that:

$$\nabla g = 0$$

Lemma 2.4.3. Let D be a metric connection on the vector bundle E with bundle metric $\langle \cdot, \cdot \rangle$. Assume that with respect to a metric bundle chart we have the decomposition

$$D = d + A$$

Then for any $X \in TM$, the matrix $A(X)$ is skew symmetric, i.e.

$$A(X) \in \mathfrak{o}(n)$$

where n is the rank of E (=dimension of the fiber of E) and $\mathfrak{o}(n)$ is the Lie algebra of $O(n)$.

Remark 2.4.4. By $\Omega^p(\mathrm{Ad}(E))$ we denote the space of those elements of $\Omega^p(\mathrm{End}(E))$ for which the endomorphism of each fiber is skew symmetric. Thus, if $D = d + A$ is a metric connection, we have $A \in \Omega^1(\mathrm{Ad}(E))$.

Theorem 2.4.5 (Fundamental theorem of Riemannian geometry)**.** On a (pseudo-)Riemannian manifold (M, g) there exists a unique (exactly one) metric and torsion-free connection ∇ (on TM).

Definition 2.4.6. The metric and torsion-free connection ∇ on TM is called the *Levi-Civita connection.*

Theorem 2.4.7. For the Levi-Civita connection we have:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}) \qquad \forall i, j, k \tag{2.16}$$

Remark 2.4.8. Let us define:

$$\Gamma_{mij} := g_{mk} \Gamma^k_{ij} = \frac{1}{2} (\partial_j g_{im} + \partial_i g_{jm} - \partial_m g_{ij}) \qquad \forall i, j, m$$

thus for the Levi-Civita connection, the Riemann curvature tensor takes form of:

$$g_{hm} R^m_{ijk} =: R_{hijk} = \frac{1}{2} (\partial^2_{ij} g_{hk} + \partial^2_{hk} g_{ij} - \partial^2_{ik} g_{hj} - \partial^2_{hj} g_{ik}) + g^{lm} (\Gamma_{mij} \Gamma_{lhk} - \Gamma_{mik} \Gamma_{lhj}) \tag{2.17}$$

Lemma 2.4.9. For the Levi-Civita connection, the coordinate expression of R satisfies:

$$R_{klij} = -R_{klji} \tag{2.18}$$

$$R_{klij} = -R_{lkij} \tag{2.19}$$

$$R_{klij} = R_{ijkl} = R_{jilk} \tag{2.20}$$

2.5 Tensors constructed from Riemann tensor

Definition 2.5.1. *Ricci form* ϱ is a tensor field of a type $(0,2)$, which is defined at every point $p \in M$ by:

$$\varrho(Y,Z) := \sum_{i=1}^{n} \langle R(\partial_i, Y)Z, dx^i\rangle \qquad \forall Y, Z \in T_pM \tag{2.21}$$

where $\{\partial_i\}$ is an arbitrary orthonormal basis of T_pM and $\{dx^i\}$ of T_p^*M .

Remark 2.5.2. The Ricci form is symmetric and its definition does not depend on the choice of basis of T_pM and T_p^*M. In coordinates, we have:

$$\varrho_{jl} = \varrho(\partial_j, \partial_l) := \langle R(\partial_i, \partial_j)\partial_l, dx^i\rangle = R^i_{lij} \tag{2.22}$$

In other words the Ricci form is a contraction of the Riemann curvature tensor.

Definition 2.5.3. The trace of the Ricci form:

$$\mathcal{R} := \sum_{i=1}^{n} \varrho(\partial_i, \partial_i)$$

is called the *Ricci scalar (curvature)* of (M, g), where $\{\partial_i\}$ is an arbitrary orthonormal basis of T_pM.

Remark 2.5.4. This fact is expressed in coordinates as follows:

$$\mathcal{R} = g^{lj}\varrho_{jl} = \varrho^j_j \tag{2.23}$$

Definition 2.5.5. Let M be a (pseudo-)Riemannian manifold with $\dim M = n$. The $(0,2)$ tensor defined by:

$$S(X,Y) := \frac{1}{n-2}\left(\varrho(X,Y) - \frac{\mathcal{R}}{2(n-1)}g(X,Y)\right) \quad \forall X, Y \in \Gamma(TM)$$

is called the *Schouten tensor* of M.

Remark 2.5.6. In coordinates this becomes

$$S_{ij} = \frac{1}{n-2}\left(\varrho_{ij} - \frac{\mathcal{R}}{2(n-1)}g_{ij}\right)$$

Proposition 2.5.7. The Schouten tensor is symmetric in its arguments:

$$S(X,Y) = S(Y,X)$$

Theorem 2.5.8 (Second Bianchi Identity)**.** Let R be the Riemann tensor 2.3.6 defined with respect to Levi-Civita connection. Then R satisfies the following identity:

$$(\nabla_X R)(Y,Z)V + (\nabla_Z R)(X,Y)V + (\nabla_Y R)(Z,X)V = 0 \qquad \forall X, Y, Z, V \in \Gamma(TM)$$

Remark 2.5.9. The equation in the theorem above is a special case of 2.2.6 and is known under the same name. In coordinates, that becomes:

$$\nabla_l R^h_{ijk} + \nabla_j R^h_{ikl} + \nabla_k R^h_{ilj} = 0 \qquad \text{or equiv.} \qquad \nabla_l R_{hijk} + \nabla_j R_{hikl} + \nabla_k R_{hilj} = 0 \tag{2.24}$$

2.6 Ricci identities

Lemma 2.6.1. [4] Let $f \in C^\infty(M)$, then the following identity holds:

$$\nabla_j \nabla_i f - \nabla_i \nabla_j f = \partial_j \partial_i f - \partial_i \partial_j f = 0$$

Theorem 2.6.2 (Ricci identity 1)**.** [4] Coordinate expression of the Riemann curvature tensor satisfies:

$$R^l_{ijk} \omega_l = \nabla_k \nabla_j \omega_i - \nabla_j \nabla_k \omega_i \qquad \forall i, j, k \quad \omega \in \Omega^1(M)$$

Remark 2.6.3. [4] In the same way as commuting of partial derivatives arises as an integrability condition for a solution of a certain overdetermined system of partial differential equations (see Frobenius theorem in e.g. [5] - chapter 6), Ricci identity must be used as an integrability condition when one treats the same overdetermined system in terms of covariant derivatives instead of normal partial differentiation.

Theorem 2.6.4 (Ricci identity 2)**.** [4] Let M be a $(0,2)$ tensor, then its components satisfy the following identity[2]:

$$\nabla_l \nabla_k M_{ij} - \nabla_k \nabla_l M_{ij} = M_{ih} R^h_{jkl} + M_{hj} R^h_{ikl} \qquad \forall i, j, k, l$$

[2]Naturally there is a general formula for a tensor of an arbitrary rank due to Ricci. However in this text we shall not need other than these special cases.

Chapter 3

Conformal transformations

In this chapter we shall define a conformal transformation and determine how known quantities (Riemann curvature tensor, Ricci form, Ricci scalar) behave under such transformation. As a result of our endeavors we shall find a quantity that is invariant under conformal transformations - the Weyl tensor.

3.1 Conformal transformations, conformal equivalence

Remark 3.1.1. Henceforth (until specified otherwise) we shall make use of a connection on a tangent bundle $\nabla : \Gamma(TM) \otimes \Gamma(TM) \to \Gamma(TM)$ defined in 2.3.5, specifically of the Levi-Civita connection defined in 2.4.6.

Definition 3.1.2. Let (M, g) be a (pseudo-)Riemannian manifold. A diffeomorphism $\phi : M \to M$ is called *a conformal transformation* if it satisfies:

$$\phi^* g_{\phi(p)} = e^{2\sigma(p)} g_p \qquad p \in M, \sigma \in C^\infty(M) \equiv \Omega^0(M) \tag{3.1}$$

Remark 3.1.3. In other words a diffeomorphism $\phi : M \to M$ is a conformal transformation if and only if it preserves metric up to a scale.

Remark 3.1.4. The expression 3.1 takes the form of:

$$g_{\phi(p)}(\phi_* X, \phi_* Y) = e^{2\sigma(p)} g_p(X, Y) \tag{3.2}$$

when a pair of tangent vectors $X, Y \in T_pM$ is inserted.

Remark 3.1.5. In coordinates we have $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ hence the defining equation for the inverse denoted by $\tilde{g}^{ij}$ yields:

$$\tilde{g}^{ij} = e^{-2\sigma} g^{ij} \tag{3.3}$$

Obviously:

$$\tilde{g}_{ij}\tilde{g}^{jk} = e^{2\sigma} g_{ij} e^{-2\sigma} g^{jk} = \delta_i^k$$

Remark 3.1.6. The set of conformal transformations on M denoted by $\mathrm{Conf}(M)$ is a group, called the conformal group. Obviously for $\phi, \psi \in \mathrm{Conf}(M)$, we have:

$$\psi^* \quad \text{acting from the left on} \quad \phi^* g_{\phi(p)} = e^{2\sigma(p)} g_p \qquad \to \qquad \psi^* \phi^* g_{\phi(p)} = \underbrace{e^{2\varrho(p)} e^{2\sigma(p)} g_p}_{(e^{2\varrho(p)} + e^{2\sigma(p)}) g_p}$$

therefore $\psi \circ \phi \in \mathrm{Conf}(M)$. Associativity for any three elements of $\mathrm{Conf}(M)$ is an obvious consequence of a straightforward equation that is an analogy to the one above and the fact that the product of exponentials of real functions is associative. Identity element ε is determined by zero function $o(p)$:

$$\varepsilon^* g_{\varepsilon(p)} = e^{2o(p)} g_p = e^0 g_p = g_p$$

Inverse element $\forall \phi \in \mathrm{Conf}(M)$ is ϕ^{-1} acting with the scaling of $-2\sigma(p)$, we have:

$$(\phi^{-1})^* \phi^* g_{\phi^{-1}\phi(p)} = e^{-2\sigma(p)} e^{2\sigma(p)} g_p = g_p = \varepsilon^* g_{\varepsilon(p)}$$

Definition 3.1.7. *Angle* θ between two tangent vectors $X, Y \in T_pM$ is defined by:

$$\cos\theta = \frac{g_p(X,Y)}{\sqrt{g_p(X,X) g_p(Y,Y)}}$$

Proposition 3.1.8. A conformal transformation ϕ preserves the angle.

Proof. Let $X, Y \in T_pM$, then we have:

$$\cos\theta' = \frac{g_p(\phi_* X, \phi_* Y)}{\sqrt{g_p(\phi_* X, \phi_* X) g_p(\phi_* Y, \phi_* Y)}} = \frac{e^{2\sigma(p)} g_p(X,Y)}{\sqrt{e^{2\sigma(p)} g_p(X,X) e^{2\sigma(p)} g_p(Y,Y)}} = \cos\theta$$

□

Definition 3.1.9. Let $g, \tilde{g}$ be a pair of metric tensors on a manifold M. The metric $\tilde{g}$ is said to be *conformally equivalent* to g if there exists a conformal transformation between the two metrics.

Remark 3.1.10. An explicit relation for the two metrics is:

$$\tilde{g}_p = e^{2\sigma(p)} g_p \tag{3.4}$$

This is really an equivalence relation among the set of metrics on M. Thanks to the group properties of $\mathrm{Conf}(M)$, we have that the relation is symmetric (with scaling $e^{-2\sigma(p)}$ corresponding to $e^{2\sigma(p)}$), reflexive (with the scaling coefficient $e^{2o(p)} = 1$, which, as we already know from remark 3.1.6, exists) and transitive (compositions of conformal transformations belong to $\mathrm{Conf}(M)$ as well).

3.2 Transformation of Levi-Civita connection

Definition 3.2.1. Let K be the difference of the Levi-Civita connections $\tilde{\nabla}$ (that is metric and torsion free) with respect to $\tilde{g}$ and ∇ with respect to g:

$$K(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y \qquad \forall X, Y \in \Gamma(TM) \tag{3.5}$$

Lemma 3.2.2. [2] Let $\sigma \in \Omega^0$ and U be the vector field which corresponds to the one-form $d\sigma$, so that $Z(\sigma) = d\sigma(Z) = g(U, Z)$. Then:

$$K(X,Y) = X(\sigma)Y + Y(\sigma)X - g(X,Y)U \qquad \forall X, Y \in \Gamma(TM) \tag{3.6}$$

Proof. First, let us prove, that K is symmetric $K(X,Y) = K(Y,X)$. It follows from the torsion free property of Levi-Civita connection. For all $A, B \in \Gamma(TM)$, we have:

$$0 = T(A,B) := \nabla_A B - \nabla_B A - [A,B] \quad \rightarrow \quad \nabla_A B = \nabla_B A + [A,B]$$
$$0 = \tilde{T}(B,A) := \tilde{\nabla}_B A - \tilde{\nabla}_A B - [B,A] \quad \rightarrow \quad \tilde{\nabla}_B A = \tilde{\nabla}_A B - [A,B]$$

Now we add these two equations together:

$$\nabla_A B + \tilde{\nabla}_B A = \nabla_B A + \tilde{\nabla}_A B \quad \rightarrow \quad \underbrace{\tilde{\nabla}_B A - \nabla_B A}_{=:K(B,A)} = \underbrace{\tilde{\nabla}_A B - \nabla_A B}_{=:K(A,B)}$$

From the fact, that Levi-Civita connection is metric, it follows that:

$$X(\tilde{g}(Y,Z)) = \tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}_X Z) \tag{3.7}$$

and also:

$$X(e^{2\sigma(p)} g(Y,Z)) = 2X(\sigma) e^{2\sigma(p)} g(Y,Z) + e^{2\sigma(p)} [g(\nabla_X Y, Z) + g(Y, \nabla_X Z)] \tag{3.8}$$

Now we subtract the first equation from the second (3.8-3.7) and obtain:

$$0 = 2X(\sigma)e^{2\sigma(p)} g(Y,Z) + e^{2\sigma(p)}[g(\nabla_X Y, Z) + g(Y, \nabla_X Z)] - \underbrace{\tilde{g}(\tilde{\nabla}_X Y, Z)}_{e^{2\sigma(p)} g(\tilde{\nabla}_X Y, Z)} - \underbrace{\tilde{g}(Y, \tilde{\nabla}_X Z)}_{e^{2\sigma(p)} g(Y, \tilde{\nabla}_X Z)}$$

and finally after dividing by a non-zero factor $e^{2\sigma(p)}$:

$$0 = 2X(\sigma)g(Y,Z) - g(K(X,Y),Z) - g(Y, K(X,Z)) \tag{A}$$

Permutations $(X \rightarrow Y \rightarrow Z)$ yield:

$$0 = 2Y(\sigma)g(Z,X) - g(K(Y,Z),X) - g(Z, K(Y,X)) \tag{B}$$

$$0 = 2Z(\sigma)g(X,Y) - g(K(Z,X),Y) - g(X, K(Z,Y)) \tag{C}$$

The combination of $(A) + (B) - (C)$ leads to:

$$X(\sigma)g(Y,Z) + Y(\sigma)g(Z,X) - Z(\sigma)g(X,Y) - g(K(X,Y),Z) = 0$$

and if we make use of the equality $Z(\sigma) = g(U, Z)$, we get:

$$g(X(\sigma)Y, Z) + g(Y(\sigma)X, Z) - g(g(X,Y)U, Z) - g(K(X,Y), Z) = 0$$

This leads to:

$$g((X(\sigma)Y + Y(\sigma)X - g(X,Y)U - K(X,Y), Z) = 0$$

Since the equality must hold for every Z, clearly:

$$K(X,Y) = X(\sigma)Y + Y(\sigma)X - g(X,Y)U$$

ergo the proof of lemma 3.2.2 is complete.

□

3.3 Transformation of Riemann tensor and its traces

Lemma 3.3.1. Let $\sigma \in \Omega^0$ and U be the vector field which corresponds to the one-form $d\sigma$, so that $Z(\sigma) = d\sigma(Z) = g(U, Z)$. Then the difference of Riemann tensors defined by the equality 2.3.6 (with respect to $\tilde{\nabla}, \nabla$ corresponding to the two conformally equivalent metric tensors $\tilde{g}, g$) takes form of:

$$\tilde{R}(X,Y)Z - R(X,Y)Z = -[g(Y,Z)BX - g(BX,Z)Y + g(BY,Z)X - g(X,Z)BY] \quad (3.9)$$

where B is a type (1,1) tensor field defined by:

$$B(X) := -X(\sigma)U + \nabla_X U + \frac{1}{2}U(\sigma)X \quad (3.10)$$

Proof. Throughout the whole calculation the term $\Sigma(X,Y)$ will denote the symmetric part of the expression positioned before it (sometimes called symmetrizer), in other words, the same expression only with interchanged X and Y. We shall make use of 3.5 and now, from the definition:

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z =$$

$$= \left\{\tilde{\nabla}_X[K(Y,Z) + \nabla_Y Z]\right\} - \Sigma(X,Y) - K([X,Y],Z) - \nabla_{[X,Y]}Z =$$

$$= \left\{K(X, K(Y,Z) + \nabla_Y Z) + \nabla_X[K(Y,Z) + \underbrace{\nabla_Y Z}]\right\} - \Sigma(X,Y) - K([X,Y],Z) - \underbrace{\nabla_{[X,Y]}Z}$$

The underbraced terms (along with the part in the symmetrizer) are equal to the untilded tensor $R(X,Y)Z$. Let us therefore rewrite the equality as follows:

$$\tilde{R}(X,Y)Z - R(X,Y)Z = K(X,K(Y,Z)) + K(X,\nabla_Y Z) + \nabla_X K(Y,Z) - \Sigma(X,Y) - K([X,Y],Z) =$$

Using the equality 3.6 we get:

$$\tilde{R}(X,Y)Z - R(X,Y)Z = \{X(\sigma)[\nabla_Y Z + K(X,Y)] + (\nabla_Y Z)(\sigma)X + K(Y,Z)(\sigma)X -$$

$$-g(X, \nabla_Y Z + K(Y,Z))U + \nabla_X[Y(\sigma)Z + Z(\sigma)Y - g(Y,Z)U]\} - \Sigma(X,Y) -$$

$$-[X,Y](\sigma)Z - Z(\sigma)[X,Y] + g(Z,[X,Y])U =$$

$$= \Bigg\{\underbrace{X(\sigma)\nabla_Y Z}_{\beta} + X(\sigma)[\underbrace{Y(\sigma)Z}_{\gamma} + \underbrace{Z(\sigma)Y}_{\delta} - g(Y,Z)U] +$$

$$+(\nabla_Y Z)(\sigma)X + [\underbrace{Y(\sigma)Z}_{\delta} + Z(\sigma)Y - g(Y,Z)U](\sigma)X -$$

$$-\underbrace{g(X,\nabla_Y Z)U}_{\varepsilon} - g(X, Y(\sigma)Z + Z(\sigma)Y - g(Y,Z)U)U +$$

$$+\underbrace{X(Y(\sigma))Z}_{\alpha}+\underbrace{Y(\sigma)\nabla_X Z}_{\beta}+X(Z(\sigma))Y+\underbrace{Z(\sigma)\nabla_X Y}_{T}-$$

$$\left.-g(\nabla_X Y,Z)U-\underbrace{g(Y,\nabla_X Z)U}_{\varepsilon}-g(Y,Z)\nabla_X U\right\}-$$

$$-\underbrace{\Sigma(X,Y)}_{\alpha,\beta,\gamma,\delta,\varepsilon,T}-\underbrace{[X(Y(\sigma))-Y(X(\sigma))]Z}_{\alpha}-\underbrace{Z(\sigma)[X,Y]}_{T}+g(Z,[X,Y])U$$

Here the terms underbraced with the same greek letter subtract from each other. Terms underbraced T do not annihilate this way, but they are still zero together, because they form a zero torsion tensor (under Levi-Civita connection). We can follow with:

$$\tilde{R}(X,Y)Z-R(X,Y)Z=\left\{-\underbrace{X(\sigma)g(Y,Z)U}_{\zeta}+(\nabla_Y Z)(\sigma)X+Z(\sigma)Y(\sigma)X-\right.$$

$$-g(Y,Z)U(\sigma)X-\underbrace{Y(\sigma)g(X,Z)U}_{\zeta}-\underbrace{Z(\sigma)g(X,Y)U}_{\eta}+g(Y,Z)g(X,U)U+X(Z(\sigma))Y-$$

$$\left.-g(\underbrace{\nabla_X Y}_{T},Z)U-g(Y,Z)\nabla_X U\right\}-\underbrace{\Sigma(X,Y)}_{\zeta,\eta,T}+g(Z,\underbrace{[X,Y]}_{T})U=$$

$$=\left\{\underbrace{(\nabla_Y Z)(\sigma)X}+Z(\sigma)Y(\sigma)X-g(Y,Z)U(\sigma)X+\right.$$

$$\left.+g(Y,Z)g(X,U)U+\underbrace{X(Z(\sigma))Y}-g(Y,Z)\nabla_X U\right\}-\Sigma(X,Y)$$

Due to the duality pairing between U and σ via $X(\sigma)=g(U,X)\quad\forall X\in\Gamma(TM)$, underbraced terms together can be rewritten as:

$$\{(\nabla_Y Z)(\sigma)X+(X(Z(\sigma)))Y\}-\Sigma(X,Y)=$$

$$=g(U,\nabla_Y Z)X+X[g(U,Z)]Y-g(U,\nabla_X Z)Y-Y[g(U,Z)]X=$$

$$=\underbrace{g(U,\nabla_Y Z)X}_{\vartheta}+g(\nabla_X U,Z)Y+\underbrace{g(U,\nabla_X Z)Y}_{\iota}-$$

$$-\underbrace{g(U,\nabla_X Z)Y}_{\iota}-g(\nabla_Y U,Z)X-\underbrace{g(U,\nabla_Y Z)X}_{\vartheta}$$

Now in this last step, where we insert $B(X):=-X(\sigma)U+\nabla_X U+\frac{1}{2}U(\sigma)X$ into our equation, we near the end of the proof. Let us rearrange:

$$\tilde{R}(X,Y)Z-R(X,Y)Z=\{Z(\sigma)Y(\sigma)X-g(Y,Z)U(\sigma)X+$$

$$+g(Y,Z)X(\sigma)U-g(Y,Z)\nabla_X U-g(\nabla_Y U,Z)X\}-\Sigma(X,Y)=$$

$$= \{-g(Y,Z)[+U(\sigma)X - X(\sigma)U + \nabla_X U] + \underbrace{Z(\sigma)Y(\sigma)X}_{Y(\sigma)g(U,Z)X} - g(\nabla_Y U, Z)X\} - \Sigma(X,Y) =$$

$$= \{-B(X)g(Y,Z) - \frac{1}{2}U(\sigma)g(Y,Z)X + g(Y(\sigma)U,Z)X - g(\nabla_Y U,Z)X\} - \Sigma(X,Y) =$$
$$= \{-B(X)g(Y,Z) - g(B(Y),Z)X\} - \Sigma(X,Y)$$

The fact that B is a tensor (multilinearity with respect to $f \in C^\infty(M)$) is obvious from the superseding equation:

$$B(fX) = -(fX)(\sigma)U + \nabla_{fX}U + \frac{1}{2}U(\sigma)(fX) = -fX(\sigma)U + f\nabla_X U + \frac{1}{2}fU(\sigma)X = fB(X)$$

where we have made use of the definition of a connection. The proof is now complete. □

Remark 3.3.2. Equation 3.9 obtained in the previous lemma becomes by explicit calculation in coordinate notation (in line with cordinate convention established by equation 2.8):

$$\tilde{R}^l_{ijk} = R^l_{ijk} + \langle dx^l, -g(\partial_k, \partial_i)B(\partial_j) + g(B(\partial_j), \partial_i)\partial_k - g(B(\partial_k), \partial_i)\partial_j + g(\partial_j, \partial_i)B(\partial_k)\rangle =$$

$$= R^l_{ijk} + \langle dx^l, -g_{ik}[B(\partial_j)]^h\partial_h + g([B(\partial_j)]^h\partial_h, \partial_i)\partial_k - g([B(\partial_k)]^h\partial_h, \partial_i)\partial_j + g_{ij}[B(\partial_k)]^h\partial_h\rangle =$$
$$= R^l_{ijk} - g_{ik}B^h_j\delta^l_h + B^h_j g_{hi}\delta^l_j - B^h_k g_{hi}\delta^l_j + g_{ij}B^h_k\delta^l_h$$

And therefore we have:

$$\tilde{R}^l_{ijk} = R^l_{ijk} - g_{ik}B^l_j + B^h_j g_{hi}\delta^l_k - B^h_k g_{hi}\delta^l_j + g_{ij}B^l_k \tag{3.11}$$

In a very similar manner we find coordinate expression of 3.10:

$$B^l_j = \langle dx^l, B(\partial_j)\rangle = \langle dx^l, -\partial_j(\sigma)U^h\partial_h + \nabla_{\partial_j}(U^h\partial_h) + \frac{1}{2}U^h\partial_h(\sigma)\partial_j\rangle =$$

$$= -\partial_j(\sigma)U^h\delta^l_h + (\partial_j U)^l + \frac{1}{2}U^h\partial_h(\sigma)\delta^l_j =$$

$$= -\partial_j(\sigma)g^{nl}\partial_n(\sigma) + g^{lk}(\partial_k\partial_j(\sigma) - \partial_i(\sigma)\Gamma^i_{kj}) + \frac{1}{2}g^{nh}\partial_n(\sigma)\partial_h(\sigma)\delta^l_j$$

Now we have made use of the fact that $Z(\sigma) = d\sigma(Z) = g(U,Z)$ for each $Z \in T_pM$:

$$\partial_h(\sigma) := g(U^n\partial_n, \partial_h) = g_{nh}U^n = U_h \quad \rightarrow \quad g^{hl}\partial_h\sigma = U^l$$

In a rather simpler notation:

$$B^l_j = -\partial_j\sigma g^{nl}\partial_n\sigma + g^{lk}(\partial_k\partial_j\sigma - \partial_a\sigma\Gamma^a_{kj}) + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma\delta^l_j \tag{3.12}$$

Now we lower the upper index:

$$B_{ij} = -\partial_i\sigma\partial_j\sigma + \partial_i\partial_j\sigma - \partial_a\sigma\Gamma^a_{ij} + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{ij} \tag{3.13}$$

From this equation it can be readily seen, that $B_{ij} := g_{il}B^l_j = B_{ji}$ is symmetric.

Lemma 3.3.3. The relationship between $\tilde{R}$ and R viewed as a $(0,4)$ tensors can be expressed in a coordinate-free way through the Kulkarni-Nomizu product:

$$\tilde{R} = e^{2\sigma(p)}[R - g \owedge B] \tag{3.14}$$

where B is a $(0,2)$ tensor defined by equation 3.13.

Proof. It is an immediate consequence of lowering the index on 3.11:

$$e^{-2\sigma}\tilde{R}_{lijk} = R_{lijk} + g_{ij}B_{lk} + B_{ij}g_{lk} - B_{ik}g_{lj} - g_{ik}B_{lj}$$

and recalling the Kulkarni-Nomizu product in coordinates (definition 1.4.8 and equality 1.6). □

Remark 3.3.4. If we are to ask how do other tensors or quantities change under conformal transformation of the metric, the answer is at hand. By contracting the expression 3.11 for l and j we arrive at the following equation:

$$\tilde{R}^l_{ilk} =: \tilde{\varrho}_{ik} = \varrho_{ik} - g_{ik}B^l_l + B^h_l g_{hi}\delta^l_k - B^h_k g_{hi}\delta^l_l + g_{il}B^l_k = \varrho_{ik} - g_{ik}B^l_l - (n-2)B_{ik} \tag{3.15}$$

Here $\varrho_{ik}, \tilde{\varrho}_{ik}$ are (from equality 2.22) the Ricci forms and $n \in \mathbb{N}$ is the dimension of the manifold. Yet another contraction in this expression leads (because of equality 3.3) to:

$$\tilde{g}^{ik}\tilde{\varrho}_{ik} =: \tilde{\mathcal{R}} = e^{-2\sigma}g^{ik}[\varrho_{ik} - g_{ik}B^l_l - (n-2)B_{ik}] = e^{-2\sigma}[\mathcal{R} - nB^l_l - (n-2)B^l_l] = e^{-2\sigma}[\mathcal{R} - 2(n-1)B^l_l]$$

where $\mathcal{R}, \tilde{\mathcal{R}}$ are (by equality 2.23) the Ricci scalar curvatures. We shall rewrite this for the future convenience as follows:

$$\tilde{g}_{ik}\tilde{\mathcal{R}} = g_{ik}[\mathcal{R} - 2(n-1)B^l_l] \tag{3.16}$$

Now we shall be able to factor out expressions for B^l_l and B_{ik}. From the equation 3.16 we have:

$$B^l_l = -\frac{\tilde{g}_{ik}\tilde{\mathcal{R}}}{2(n-1)g_{ik}} + \frac{\mathcal{R}}{2(n-1)}$$

and 3.15 yields:

$$B_{ik} = \frac{\varrho_{ik} - \tilde{\varrho}_{ik} - g_{ik}B^l_l}{n-2} = \frac{1}{n-2}(\varrho_{ik} - \tilde{\varrho}_{ik}) - \frac{1}{2(n-1)(n-2)}(\mathcal{R}g_{ik} - \tilde{\mathcal{R}}\tilde{g}_{ik})$$

3.4 Weyl tensor

Lemma 3.4.1. Coordinate expression 3.11 can be substituted in and rearranged so that it is possible to separate tilded and untilded coordinate expressions of the same quantity on the two sides of the resulting equation.

Proof. Let us insert equations for B_{ik} and B^j_j (found above) into 3.11 after lowering the indices at all coordinate expressions of B:

$$\tilde{R}^l_{ijk} = R^l_{ijk} - g_{ik}g^{nl}B_{nj} + B_{ij}\delta^l_k - B_{ik}\delta^l_j + g_{ij}g^{nl}B_{nk} =$$

$$= R^l_{ijk} - g_{ik}g^{nl}\left[\frac{1}{n-2}(\varrho_{nj} - \tilde{\varrho}_{nj}) - \frac{1}{2(n-1)(n-2)}(\mathcal{R}g_{nj} - \tilde{\mathcal{R}}\tilde{g}_{nj})\right] +$$
$$+ \left[\frac{1}{n-2}(\varrho_{ij} - \tilde{\varrho}_{ij}) - \frac{1}{2(n-1)(n-2)}(\mathcal{R}g_{ij} - \tilde{\mathcal{R}}\tilde{g}_{ij})\right]\delta^l_k -$$
$$- \left[\frac{1}{n-2}(\varrho_{ik} - \tilde{\varrho}_{ik}) - \frac{1}{2(n-1)(n-2)}(\mathcal{R}g_{ik} - \tilde{\mathcal{R}}\tilde{g}_{ik})\right]\delta^l_j +$$
$$+ g_{ij}g^{nl}\left[\frac{1}{n-2}(\varrho_{nk} - \tilde{\varrho}_{nk}) - \frac{1}{2(n-1)(n-2)}(\mathcal{R}g_{nk} - \tilde{\mathcal{R}}\tilde{g}_{nk})\right]$$

After separating tilded and untilded terms on different sides of the equation, we arrive at:

$$\text{LHS} = \tilde{R}^l_{ijk} + \frac{1}{(n-2)}[\tilde{\varrho}_{ij}\delta^l_k - \tilde{\varrho}_{ik}\delta^l_j + g_{ij}g^{nl}\tilde{\varrho}_{nk} - g_{ik}g^{nl}\tilde{\varrho}_{nj}] + \frac{\tilde{\mathcal{R}}}{(n-1)(n-2)}[\tilde{g}_{ik}\delta^l_j - \tilde{g}_{ij}\delta^l_k]$$

$$\text{RHS} = R^l_{ijk} + \frac{1}{(n-2)}[\varrho_{ij}\delta^l_k - \varrho_{ik}\delta^l_j + g_{ij}g^{nl}\varrho_{nk} - g_{ik}g^{nl}\varrho_{nj}] + \frac{\mathcal{R}}{(n-1)(n-2)}[g_{ik}\delta^l_j - g_{ij}\delta^l_k]$$

Because $g_{ij}g^{nl} = g_{ij}e^{2\sigma}e^{-2\sigma}g^{nl} = \tilde{g}_{ij}\tilde{g}^{nl}$, it is obvious that on both sides of the equation is the same quantity expressed in the means of $\tilde{g}$ and g respectively. □

Definition 3.4.2. The coordinate expression

$$W^l_{ijk} := R^l_{ijk} + \frac{1}{(n-2)}[\varrho_{ij}\delta^l_k - \varrho_{ik}\delta^l_j + g_{ij}g^{nl}\varrho_{nk} - g_{ik}g^{nl}\varrho_{nj}] + \frac{\mathcal{R}}{(n-1)(n-2)}[g_{ik}\delta^l_j - g_{ij}\delta^l_k] \tag{3.17}$$

defines a tensor on (M, g), called the *Weyl tensor (conformal curvature tensor*[1]*).*

Proposition 3.4.3. Weyl tensor is indeed a tensor.

Proof. Follows from the fact that on the right hand side of equation 3.17 figure only tensors or products of tensors. □

Theorem 3.4.4. Weyl tensor W is an invariant of conformal transformations of the metric tensor.

Proof. Follows immediately as a corollary of remark 3.3.4 and the result of lemma 3.4.4, that:

$$\tilde{W}^l_{ijk} = W^l_{ijk}$$

□

[1]The reason why the Weyl tensor is sometimes reffered to as the conformal curvature tensor is obvious from some of its properties that we are to prove in chapter 4.

Chapter 4

Weyl and Cotton tensors

In the previous chapter we found a quantity invariant under conformal transformations, namely the Weyl tensor. Here we shall probe deeper into its properties as well as come across another quantity that bears a strong resemblance to it - the Cotton tensor.

4.1 Properties of Weyl tensor, Riemann curvature tensor decompostition

Lemma 4.1.1. The Weyl tensor of (M, g) satisfies

$$W^l_{ijk} = -W^l_{ikj} \qquad \forall i, j, k, l \tag{4.1}$$

Proof. As follows from corollary 2.10 the Riemann curvature tensor is anti-symmetrical in the last two indices. Let us now explicitly write the right hand side of equation 4.1 that we are to prove:

$$-W^l_{ikj} = \underbrace{-R^l_{ikj}}_{=R^l_{ijk}} - \frac{1}{(n-2)}[\varrho_{ik}\delta^l_j - \varrho_{ij}\delta^l_k + g_{ik}g^{nl}\varrho_{nj} - g_{ij}g^{nl}\varrho_{nk}] - \frac{\mathcal{R}}{(n-1)(n-2)}[g_{ij}\delta^l_k - g_{ik}\delta^l_j]$$

In comparison with the definition 3.4.2 this right hand side is equal to W^l_{ijk}. $\square$

Remark 4.1.2. Let us compute the coordinate expression for the $(0, 4)$ Weyl tensor. From the definition:

$$\begin{aligned} W_{lijk} &:= g_{lm}W^m_{ijk} = \\ &= g_{lm}\left\{R^m_{ijk} + \frac{1}{n-2}\left[(\varrho_{ij}\delta^m_k - \varrho_{ik}\delta^m_j + g_{ij}\varrho^m_k - g_{ik}\varrho^m_j) + \frac{\mathcal{R}}{n-1}(g_{ik}\delta^m_j - g_{ij}\delta^m_k)\right]\right\} = \\ &= R_{lijk} + \frac{1}{n-2}\left[(\varrho_{ij}g_{kl} - \varrho_{ik}g_{jl} + g_{ij}\delta^n_l\varrho_{nk} - g_{ik}\delta^n_l\varrho_{nj}) + \frac{\mathcal{R}}{n-1}(g_{ik}g_{lj} - g_{ij}g_{lk})\right] \end{aligned}$$

and finally

$$W_{lijk} = R_{lijk} + \frac{1}{n-2}(\varrho_{ij}g_{kl} - \varrho_{ik}g_{jl} + g_{ij}\varrho_{lk} - g_{ik}\varrho_{lj}) + \frac{\mathcal{R}}{(n-1)(n-2)}(g_{ik}g_{lj} - g_{ij}g_{lk}) \tag{4.2}$$

Lemma 4.1.3. Coordinate expression of the $(0,4)$ Weyl tensor satisfies:

$$W_{klij} = -W_{klji} \tag{4.3}$$

$$W_{klij} = -W_{lkij} \tag{4.4}$$

$$W_{klij} = W_{ijkl} = W_{jilk} \tag{4.5}$$

Proof. Starting from the Weyl tensor in the form of 4.2 the proof is being lead in a very simmilar manner to that of lemma 4.1. Using the fact, that for the Levi-Civita connection[1] the Riemann tensor has the symmetries we are lookig for (see lemma 2.4.9), it is trivial to check from the coordinate expression that the same holds true for W. □

Lemma 4.1.4. The Weyl tensor W is trace-free.

Proof. By a straightforward calculation, we have from defining equation 3.17 where we contract indices l and j:

$$\begin{aligned} W^l_{ilk} &= R^l_{ilk} + \frac{1}{(n-2)}[\varrho_{il}\delta^l_k - \varrho_{ik}\delta^l_l + g_{il}g^{nl}\varrho_{nk} - g_{ik}g^{nl}\varrho_{nl}] + \frac{\mathcal{R}}{(n-1)(n-2)}[g_{ik}\delta^l_l - g_{il}\delta^l_k] = \\ &= \varrho_{ik} + \frac{1}{(n-2)}[\varrho_{il}\delta^l_k - n\varrho_{ik} + \delta^n_i\varrho_{nk} - g_{ik}\mathcal{R}] + \frac{\mathcal{R}}{(n-1)(n-2)}\underbrace{[ng_{ik} - g_{ik}]}_{=(n-1)g_{ik}} = \\ &= \varrho_{ik} + \frac{1}{(n-2)}\underbrace{[\varrho_{ik} - n\varrho_{ik} + \varrho_{ik}]}_{=-(n-2)\varrho_{ik}} = 0 \end{aligned}$$

The same is true for setting the indices l and k in definition 3.4.2 equal by the previous lemma 4.1.1. The last remaining non-trivial trace would be $l = i$ however under this contraction the Weyl tensor vanishes as well due to its antisymmetry in the first two indices (equation 4.4). □

Theorem 4.1.5. The $(0,4)$ Riemann curvature tensor of (M,g) can be decomposed as follows:

$$R = W + g \owedge S \tag{4.6}$$

where W is the $(0,4)$ Weyl tensor and S is the $(0,2)$ Schouten tensor defined in 2.5.5 as

$$S(V,Z) = \frac{1}{n-2}\left[\varrho(V,Z) - \frac{\mathcal{R}}{2(n-1)}g(V,Z)\right]$$

Proof. It is an obvious fact, once we take into account the equation 4.2 and the definition of the Kulkarni-Nomizu product 1.4.8. □

[1] Starting from remark 3.1.1 we have been working with the Levi-Civita connection.

Lemma 4.1.6. Under the conformal transformation of the metric tensor $g \to e^{2\sigma(p)} g = \tilde{g}$ the $(0,4)$ Weyl tensor transforms as follows:

$$e^{-2\sigma(p)} \tilde{W}(V,X,Y,Z) = W(V,X,Y,Z) \qquad \forall V,X,Y,Z \in \Gamma(TM)$$

Proof. In theorem 3.4.4 we found that the (1,3) Weyl tensor is an invariant of conformal transformations of the metric. Now if we apply $\tilde{g}_{lh} = e^{2\sigma} g_{lh}$ to the equation

$$\tilde{W}^h_{ijk} = W^h_{ijk}$$

we obtain

$$\tilde{g}_{lh} \tilde{W}^h_{ijk} = \tilde{W}_{lijk} = e^{2\sigma} W_{lijk}$$

This is an equality in tensors, thus it does not depend on the choice of the basis and we can rewrite it in the coordinate-free manner. □

Remark 4.1.7. The decomposition of

$$R(V,X,Y,Z) = W(V,X,Y,Z) + (g \owedge S)(V,X,Y,Z)$$

and the trace-free property of the Weyl tensor (lemma 4.1.4) is the reason why the Weyl tensor is sometimes reffered to as the *trace-free part of the Riemann tensor.*

Theorem 4.1.8. [7] Let (M,g) be a (pseudo-)Riemannian manifold with $\dim M = 3$ then the Riemann tensor of M can be expressed as follows:

$$R^l_{ijk} = -\varrho_{ij}\delta^l_k + \varrho_{ik}\delta^l_j - g_{ij}g^{nl}\varrho_{nk} + g_{ik}g^{nl}\varrho_{nj} + \frac{\mathcal{R}}{2}(g_{ik}\delta^l_j - g_{ij}\delta^l_k)$$

Proof. First, let us raise the second index on the expression that we are to prove. We arrive at an equivalent expression:

$$\underbrace{R^{lm}{}_{jk}}_{\text{LHS}} = \underbrace{-\varrho^m_j\delta^l_k + \varrho^m_k\delta^l_j - \delta^m_j g^{nl}\varrho_{nk} + \delta^m_k g^{nl}\varrho_{nj} - \frac{\mathcal{R}}{2}(\delta^m_k\delta^l_j - \delta^m_j\delta^l_k)}_{\text{RHS}} \tag{R}$$

By our assumption (M,g) is a three-dimensional manifold and thus at least two of the indices l,m,j,k has to be equal. We shall verify, that the equality (R) holds for the following cases:

- Let $l = m$, then:

$$\text{RHS} = -\varrho^l_j\delta^l_k + \varrho^l_k\delta^l_j - \delta^l_j g^{nl}\varrho_{nk} + \delta^l_k g^{nl}\varrho_{nj} - \frac{\mathcal{R}}{2}\underbrace{(\delta^l_k\delta^l_j - \delta^l_j\delta^l_k)}_{=0} =$$
$$= -\varrho^l_j\delta^l_k + \varrho^l_k\delta^l_j - \delta^l_j\varrho^l_k + \delta^l_k\varrho^l_j = 0$$

  The LHS of (R) is zero as well, because of the antisymmetry of the Riemann tensor (equality 2.19) which implies $R_{lljk} = -R_{lljk} = 0$. This remains true after raising the first two indices.

- Let $l = j$, then:

$$\begin{aligned}\text{RHS} &= -\varrho_l^m \delta_k^l + \varrho_k^m \delta_l^l - \delta_l^m g^{nl} \varrho_{nk} + \delta_k^m g^{nl} \varrho_{nl} - \frac{\mathcal{R}}{2}(\delta_k^m \delta_l^l - \delta_l^m \delta_k^l) = \\ &= -\varrho_k^m + 3\varrho_k^m - \delta_l^m \varrho_k^l + \delta_k^m \mathcal{R} - \frac{\mathcal{R}}{2}(3\delta_k^m - \delta_k^m) = \\ &= \varrho_k^m + \delta_k^m \mathcal{R} - \frac{\mathcal{R}}{2}(2\delta_k^m) = \varrho_k^m\end{aligned}$$

For the LHS we have $R^{lm}{}_{lk} = g^{mn} R^l_{nlk} = g^{mn} \varrho_{nk} = \varrho_k^m$ and therefore the equality (R) holds.

- Let $l = k$, then

$$\begin{aligned}\text{RHS} &= -\varrho_j^m \delta_l^l + \varrho_l^m \delta_j^l - \delta_j^m g^{nl} \varrho_{nl} + \delta_l^m g^{nl} \varrho_{nj} - \frac{\mathcal{R}}{2}(\delta_l^m \delta_j^l - \delta_j^m \delta_l^l) = \\ &= -3\varrho_j^m + \varrho_j^m - \delta_j^m \mathcal{R} + \varrho_j^m - \frac{\mathcal{R}}{2}(\delta_j^m - 3\delta_j^m) = -\varrho_j^m\end{aligned}$$

and the LHS $= R^{lm}{}_{jl} = -g^{mh} R^l_{hlj} = -\varrho_j^m$.

- Let $m = j$, then

$$\begin{aligned}\text{RHS} &= -\varrho_m^m \delta_k^l + \varrho_k^m \delta_m^l - \delta_m^m g^{nl} \varrho_{nk} + \delta_k^m g^{nl} \varrho_{nm} - \frac{\mathcal{R}}{2}(\delta_k^m \delta_m^l - \delta_m^m \delta_k^l) = \\ &= -\mathcal{R}\delta_k^l + \varrho_k^l - 3\varrho_k^l + \varrho_k^l - \frac{\mathcal{R}}{2}(\delta_k^l - 3\delta_k^l) = -\varrho_k^l\end{aligned}$$

For the LHS $= R^{lm}{}_{mk} = -g^{lh} R^m_{hmk} = -g^{lh} \varrho_{hk} = -\varrho_k^l$ the equality (R) holds.

- Let $m = k$, then

$$\begin{aligned}\text{RHS} &= -\varrho_j^k \delta_k^l + \varrho_k^k \delta_j^l - \delta_j^k g^{nl} \varrho_{nk} + \delta_k^k g^{nl} \varrho_{nj} - \frac{\mathcal{R}}{2}(\delta_k^k \delta_j^l - \delta_j^k \delta_k^l) = \\ &= -\varrho_j^l + \delta_j^l \mathcal{R} - \varrho_j^l + 3\varrho_j^l - \frac{\mathcal{R}}{2}(3\delta_j^l - \delta_j^l) = \varrho_j^l\end{aligned}$$

and the LHS $= R^{lk}{}_{jk} = g^{kh} g^{il} R_{ihjk} = g^{kh} g^{il} R_{hikj} = g^{il} \varrho_{ij} = \varrho_j^l$.

- Let $j = k$, then

$$\text{RHS} = -\varrho_j^m \delta_j^l + \varrho_j^m \delta_j^l - \delta_j^m g^{nl} \varrho_{nj} + \delta_j^m g^{nl} \varrho_{nj} - \frac{\mathcal{R}}{2}(\delta_j^m \delta_j^l - \delta_j^m \delta_j^l) = 0$$

For the LHS we have $R^{lm}{}_{jj} = -R^{lm}{}_{jj} = 0$.

Now let us consider the cases where three of the four indices l, m, j, k are equal. The left-hand side of the Riemann tensor is always zero in these cases because of the antisymmetries (lemma 2.4.9) $R_{lllk} = -R_{lllk}$ and similarly for the Riemann tensor with the first two indices raised. We shall only verify the vanishing of the right-hand sides. We have:

- Let $l = m = j$, then the RHS is zero because of the case $l = m$ above.
- Let $l = m = k$, then the RHS is zero because of the case $l = m$ above.
- Let $l = k = j$, then the RHS is zero because of the case $j = k$ above.
- Let $m = k = j$, then the RHS is zero because of the case $j = k$ above.

We have verified the equality (R) ergo the proof of 4.1.8 is finished. □

Lemma 4.1.9. Let (M, g) be a (pseudo-)Riemannian manifold with a $\dim M = 3$. Then the Weyl tensor vanishes identically.

Proof. This lemma is equivalent to the previous theorem, for the Riemann tensor decompostition (theorem 4.1.5) on a threedimensional manifold takes exactly the form from the theorem 4.1.8 if and only if the Weyl tensor is identically zero. □

Theorem 4.1.10. The $(0, 4)$ Weyl tensor satisfies the first Bianchi identity:

$$W(V,X,Y,Z) + W(V,Z,X,Y) + W(V,Y,Z,X) = 0 \qquad \forall V,X,Y,Z \in \Gamma(TM) \tag{4.7}$$

Proof. Because of the decomposition $W = R - g \owedge S$ (theorem 4.1.5) and the fact, that the Riemann curvature tensor satisfies the first Bianchi identity it clearly suffices to prove that also the Kulkarni-Nomizu product (definition 1.4.8) of two symmetric $(0, 2)$ tensors satisfies the first Bianchi identity. By direct computation (for all $V, X, Y, Z \in \Gamma(TM)$):

$$H \owedge Q(V,X,Y,Z) + H \owedge Q(V,Z,X,Y) + H \owedge Q(V,Y,Z,X) =$$

$$= \underbrace{H(V,Z)Q(X,Y) + H(X,Y)Q(V,Z)}_{\alpha} \underbrace{-H(V,Y)Q(X,Z) - H(X,Z)Q(V,Y)}_{\beta} +$$

$$+ \underbrace{H(V,Y)Q(Z,X) + H(Z,X)Q(V,Y)}_{\beta} \underbrace{-H(V,X)Q(Z,Y) - H(Z,Y)Q(V,X)}_{\gamma} +$$

$$+ \underbrace{H(V,X)Q(Y,Z) + H(Y,Z)Q(V,X)}_{\gamma} \underbrace{-H(V,Z)Q(Y,X) - H(Y,X)Q(V,Z)}_{\alpha} = 0$$

where the terms underbraced by the same greek letter subtract with each other. □

Proposition 4.1.11. [4] Coordinate expression of the $(1, 3)$ Weyl tensor satisfies:

$$\nabla_h W^l_{ijk} + \nabla_j W^l_{ikh} + \nabla_k W^l_{ihj} = \frac{1}{n-2}\left(\delta^l_h C_{ijk} + \delta^l_j C_{ikh} + \delta^l_k C_{ihj} + g_{ik} C^l_{jh} + g_{ih} C^l_{kj} + g_{ij} C^l_{hk}\right)$$

where

$$C_{ijk} = \nabla_k \varrho_{ij} - \nabla_j \varrho_{ik} + \frac{1}{2(n-1)}\left(g_{ik}\nabla_j \mathcal{R} - g_{ij}\nabla_k \mathcal{R}\right)$$

Proof. We start by explicitly calculating the action of the connection on components of the Weyl tensor:

$$\nabla_h W^l_{ijk} = \nabla_h R^l_{ijk} + \frac{\delta^l_j \nabla_h \varrho_{ik} - \delta^l_k \nabla_h \varrho_{ij} + g_{ik}\nabla_h \varrho^l_j - g_{ij}\nabla_h \varrho^l_k}{n-2} + \frac{\delta^l_k g_{ij} - \delta^l_j g_{ik}}{(n-1)(n-2)}\nabla_h \mathcal{R}$$

$$\nabla_j W^l_{ikh} = \nabla_j R^l_{ikh} + \frac{\delta^l_k \nabla_j \varrho_{ih} - \delta^l_h \nabla_j \varrho_{ik} + g_{ih}\nabla_j \varrho^l_k - g_{ik}\nabla_j \varrho^l_h}{n-2} + \frac{\delta^l_h g_{ik} - \delta^l_k g_{ih}}{(n-1)(n-2)}\nabla_j \mathcal{R}$$

$$\nabla_k W^l_{ihj} = \nabla_k R^l_{ihj} + \frac{\delta^l_h \nabla_k \varrho_{ij} - \delta^l_k \nabla_k \varrho_{ih} + g_{ij}\nabla_k \varrho^l_h - g_{ih}\nabla_k \varrho^l_j}{n-2} + \frac{\delta^l_j g_{ih} - \delta^l_h g_{ij}}{(n-1)(n-2)}\nabla_k \mathcal{R}$$

By summing these three equations and making use of 2.24, we obtain:

$$\nabla_h W^l_{ijk} + \nabla_j W^l_{ikh} + \nabla_k W^l_{ihj} =$$

$$= \frac{1}{n-2}\Bigg[\Bigg(\delta^l_j \underbrace{\nabla_h \varrho_{ik}}_{C_{ikh}} - \delta^l_k \underbrace{\nabla_h \varrho_{ij}}_{C_{ihj}} + g_{ik}\nabla_h \varrho^l_j - g_{ij}\nabla_h \varrho^l_k +$$

$$+ \delta^l_k \underbrace{\nabla_j \varrho_{ih}}_{C_{ihj}} - \delta^l_h \underbrace{\nabla_j \varrho_{ik}}_{C_{ijk}} + g_{ih}\nabla_j \varrho^l_k - g_{ik}\nabla_j \varrho^l_h +$$

$$+ \delta^l_h \underbrace{\nabla_k \varrho_{ij}}_{C_{ijk}} - \delta^l_k \underbrace{\nabla_k \varrho_{ih}}_{C_{ikh}} + g_{ij}\nabla_k \varrho^l_h - g_{ih}\nabla_k \varrho^l_j\Bigg) +$$

$$+ \frac{2}{2(n-1)}\Bigg(\delta^l_k \underbrace{g_{ij}\nabla_h \mathcal{R}}_{C_{ihj}} - \delta^l_j \underbrace{g_{ik}\nabla_h \mathcal{R}}_{C_{ikh}} + \delta^l_h \underbrace{g_{ik}\nabla_j \mathcal{R}}_{C_{ijk}} - \delta^l_k \underbrace{g_{ih}\nabla_j \mathcal{R}}_{C_{ihj}} + \delta^l_j \underbrace{g_{ih}\nabla_k \mathcal{R}}_{C_{ikh}} - \delta^l_h \underbrace{g_{ij}\nabla_k \mathcal{R}}_{C_{ijk}}\Bigg)\Bigg] =$$

$$= \frac{1}{n-2}\left(\delta^l_h C_{ijk} + \delta^l_j C_{ikh} + \delta^l_k C_{ihj}\right) +$$

$$+ \frac{1}{n-2}\Bigg[\Bigg(g_{ik} \underbrace{\nabla_h \varrho^l_j}_{C^l_{jh}} - g_{ij} \underbrace{\nabla_h \varrho^l_k}_{C^l_{hk}} + g_{ih} \underbrace{\nabla_j \varrho^l_k}_{C^l_{kj}} - g_{ik} \underbrace{\nabla_j \varrho^l_h}_{C^l_{jh}} + g_{ij} \underbrace{\nabla_k \varrho^l_h}_{C^l_{hk}} - g_{ih} \underbrace{\nabla_k \varrho^l_j}_{C^l_{kj}}\Bigg) +$$

$$\frac{1}{2(n-1)}\Bigg(g_{ij} \underbrace{\delta^l_k \nabla_h \mathcal{R}}_{C^l_{hk}} - g_{ik} \underbrace{\delta^l_j \nabla_h \mathcal{R}}_{C^l_{jh}} + g_{ik} \underbrace{\delta^l_h \nabla_j \mathcal{R}}_{C^l_{jh}} - g_{ih} \underbrace{\delta^l_k \nabla_j \mathcal{R}}_{C^l_{kj}} + g_{ih} \underbrace{\delta^l_j \nabla_k \mathcal{R}}_{C^l_{kj}} - g_{ij} \underbrace{\delta^l_h \nabla_k \mathcal{R}}_{C^l_{hk}}\Bigg)\Bigg]$$

We made use of the defining equation for C_{ijk} as well as

$$C^i_{jk} = g^{is} C_{sjk} = \nabla_k \varrho^i_j - \nabla_j \varrho^i_k + \frac{1}{2(n-1)}(\delta^i_k \nabla_j \mathcal{R} - \delta^i_j \nabla_k \mathcal{R})$$

□

4.2 Properties of Cotton tensor

Definition 4.2.1. The coordinate expression

$$C_{ijk} = \nabla_k \varrho_{ij} - \nabla_j \varrho_{ik} + \frac{1}{2(n-1)} \left(g_{ik} \nabla_j \mathcal{R} - g_{ij} \nabla_k \mathcal{R} \right) \tag{4.8}$$

defines a $(0,3)$ tensor on (M,g), called the *Cotton tensor*.

Remark 4.2.2. The Cotton tensor of M with $\dim M = n$ can be expressed in terms of the Schouten tensor (definition 2.5.5):

$$C_{ijk} = (n-2)(\nabla_k S_{ij} - \nabla_j S_{ik})$$

Lemma 4.2.3. The Cotton tensor C is anti-symmetric in the last two indices, i.e.

$$C_{ijk} = -C_{ikj} \qquad \forall i,j,k$$

Proof. Follows immediately from the defining equation 4.14. □

Proposition 4.2.4. The Cotton tensor satisfies the following identity:

$$C_{ijk} + C_{kij} + C_{jki} = 0$$

Proof. Using remark 4.2.2, we can write:

$$\frac{1}{n-2}(C_{ijk} + C_{kij} + C_{jki}) =$$
$$= \underbrace{\nabla_k S_{ij}}_{\alpha} - \underbrace{\nabla_j S_{ik}}_{\beta} + \underbrace{\nabla_j S_{ki}}_{\beta} - \underbrace{\nabla_i S_{kj}}_{\gamma} + \underbrace{\nabla_i S_{jk}}_{\gamma} - \underbrace{\nabla_k S_{ji}}_{\alpha} = 0$$

where we have employed the symmetry of the Schouten tensor.[2] □

Lemma 4.2.5. [4] The Ricci form and the Ricci scalar are related by:

$$\nabla_l \varrho^l_j = \frac{1}{2} \nabla_j \mathcal{R} \tag{4.9}$$

Proof. Let us start from the Second Bianchi Identity 2.24:

$$\nabla_l \underbrace{R^h_{ijk}}_{=-R^h_{ikj}} + \nabla_j R^h_{ikl} + \underbrace{\nabla_k R^h_{ilj}}_{=g^{hm}\nabla_k R_{milj}} = 0$$

where we have made use of the equality 2.10. Now we shall contract the obtained equation for h and k and use equation 2.19:

$$-\nabla_l \varrho_{ij} + \nabla_j \varrho_{il} + g^{km} \nabla_k \underbrace{R_{milj}}_{=-R_{imlj}} = 0$$

[2] Later on we shall prove that $C_{ijk} = 0$ identically on a manifold with $\dim M < 3$ therefore this proposition holds even for $n = 2$ where the previous line of proof does not make sense.

In order to complete the proof, we multiply the equation by g^{il} and sum for i and l:

$$-\nabla_l \varrho^l_j + \nabla_j \mathcal{R} - \underbrace{g^{km}\nabla_k \varrho_{mj}}_{=\nabla_k \varrho^k_j} = 0 \qquad \to \qquad 2\nabla_k \varrho^k_j = \nabla_j \mathcal{R}$$

After dividing by 2 this results in the sought-after identity. □

Lemma 4.2.6. The Cotton tensor is trace-free.

Proof. We have after raising the index and contracting

$$\begin{aligned} C^i_{ik} = g^{is}C_{sik} &= \nabla_k \varrho^i_i - \nabla_i \varrho^i_k + \frac{1}{2(n-1)}(\delta^i_k \nabla_i \mathcal{R} - \delta^i_i \nabla_k \mathcal{R}) = \\ &= \nabla_k \mathcal{R} - \underbrace{\frac{1}{2}\nabla_k \mathcal{R}}_{\text{by lemma 4.2.5}} + \frac{1}{2(n-1)}(\nabla_k \mathcal{R} - n\nabla_k \mathcal{R}) = \\ &= \frac{1}{2}\nabla_k \mathcal{R} - \frac{n-1}{2(n-1)}\nabla_k \mathcal{R} = 0 \end{aligned}$$

The same would be true for setting the indices i and k equal $C^k_{jk} = g^{ik}C_{ijk}$. □

Proposition 4.2.7. Let (M, g) be a (pseudo-)Riemannian manifold with $n = \dim M > 3$. The coordinate expressions of the Weyl tensor and the Cotton tensor are related by:

$$\nabla_h W^h_{ijk} = \frac{n-3}{n-2} C_{ijk} \tag{4.10}$$

Proof. We start by contracting the equation obtained in proposition 4.1.11 for h and l:

$$\nabla_h W^h_{ijk} + \nabla_j W^h_{ikh} + \nabla_k W^h_{ihj} = \frac{1}{n-2}\left(\delta^h_h C_{ijk} + \delta^h_j C_{ikh} + \delta^h_k C_{ihj} + g_{ik}C^h_{jh} + g_{ih}C^h_{kj} + g_{ij}C^h_{hk}\right)$$

Now using the trace-free properties of the Weyl tensor and the Cotton tensor from lemmas 4.1.4 and 4.2.6 we have:

$$\nabla_h W^h_{ijk} + \underbrace{\nabla_j W^h_{ikh}}_{=0} + \underbrace{\nabla_k W^h_{ihj}}_{=0} = \frac{1}{n-2}(nC_{ijk} + C_{ikj} + C_{ikj} + g_{ik}\underbrace{C^h_{jh}}_{=0} + \underbrace{g_{ih}C^h_{kj}}_{=C_{ikj}} + g_{ij}\underbrace{C^h_{hk}}_{=0})$$

If we employ the anti-symmetry from lemma 4.2.3 we arrive at:

$$\nabla_h W^h_{ijk} = \frac{1}{n-2}(nC_{ijk} - C_{ijk} - C_{ijk} - C_{ijk}) = \frac{n-3}{n-2}C_{ijk}$$

□

Corollary 4.2.8. Let (M, g) be a (pseudo-)Riemannian manifold with $n = \dim M > 3$. If the Weyl tensor is a zero tensor, then the Cotton tensor is a zero tensor.

Proposition 4.2.9. Let (M, g) be a (pseudo-)Riemannian manifold with $\dim M = 2$. Then the Cotton tensor of M vanishes identically.

Proof. On a two-dimensional manifold, the Cotton tensor (after raising the first index) takes form of:

$$C^i_{jk} = g^{is}C_{sjk} = \nabla_k \varrho^i_j - \nabla_j \varrho^i_k + \frac{1}{2}(\delta^i_k \nabla_j \mathcal{R} - \delta^i_j \nabla_k \mathcal{R}) \qquad i,j,k \in \{1,2\}$$

As an obvious consequence at least two indices must be equal. Components with i equal to j or k vanish because of the trace-free property of the Cotton tensor (lemma 4.2.6). If $j = k$ then we have by the anti-symmetry of C (lemma 4.2.3) the equality $C^i_{jj} = -C^i_{jj}$ ergo components for which is this satisfied vanish as well. All its components vanish, hence the Cotton tensor is a zero tensor if $\dim M = 2$. □

4.3 Cotton tensor under conformal transformations

Theorem 4.3.1. Let (M, g) be a (pseudo-)Riemannian manifold with $\dim M = n \geq 3$. Then under a conformal transformation $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ of the metric tensor the Cotton tensor of M transforms as follows:

$$\tilde{C}_{ijk} = C_{ijk} - (n-2)(\partial_a \sigma) W^a_{ijk}$$

Proof. We start from the equation from remark 4.2.2:

$$\tilde{C}_{ijk} = (n-2)(\tilde{\nabla}_k \tilde{S}_{ij} - \tilde{\nabla}_j \tilde{S}_{ik}) = (n-2)(\partial_k \tilde{S}_{ij} - \tilde{\Gamma}^a_{ki}\tilde{S}_{aj} - \tilde{\Gamma}^a_{kj}\tilde{S}_{ai}) - \Sigma(j,k) \qquad (4.11)$$

where we have made use of the symmetrizer.[3] In order to advance further, we need to find the transformational properties of $\tilde{S}_{aj}$ and $\tilde{\Gamma}^a_{ki}$ first. Shall we do that, we are to prove a couple of lemmas.

Lemma 4.3.2. Let (M, g) be a (pseudo-)Riemannian manifold. Then under a conformal transformation $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ of the metric tensor Christoffell symbols transform as follows:

$$\tilde{\Gamma}^a_{ki} = \Gamma^a_{ki} + \partial_k \sigma \delta^a_i + \partial_i \sigma \delta^a_k - g_{ik} g^{ah} \partial_h \sigma \qquad \forall a, i, k \qquad \text{(A)}$$

Proof. From the defining equation 3.5 and lemma 3.2.2 we already know, how the Levi-Civita connection changes under a conformal transformation of the metric tensor. Now it is necessary to express the equality

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)U \qquad \forall X, Y \in \Gamma(TM), U^l = g^{hl}\partial_h \sigma$$

in coordinates. We do that by using the duality pairing of TM and T^*M with their respective bases $\{\partial_i\}$ and $\{dx^j\}$:

$$\begin{aligned} \tilde{\Gamma}^a_{ki} = \tilde{\Gamma}^c_{ki}\delta^a_c = \langle \tilde{\Gamma}^c_{ki}\partial_c, dx^a\rangle = \langle \tilde{\nabla}_k \partial_i, dx^a \rangle = \\ = \langle \nabla_k \partial_i + \partial_k(\sigma)\partial_i + \partial_i(\sigma)\partial_k - g(\partial_k, \partial_i)U^l\partial_l, dx^a\rangle = \\ = \langle \Gamma^c_{ki}\partial_c + \partial_k(\sigma)\partial_i + \partial_i(\sigma)\partial_k - g_{ki}g^{hl}\partial_h\sigma\partial_l, dx^a\rangle = \\ = \Gamma^a_{ki} + \partial_k\sigma\delta^a_i + \partial_i\sigma\delta^a_k - g_{ik}g^{ah}\partial_h\sigma \end{aligned}$$

We have proved the desired equality (A). □

[3]The symbol $\Sigma(j,k)$ represents the symmetric part of the whole expression (from the last equality =) positioned in front of it; i.e. the same expression only with interchanged j and k.

Lemma 4.3.3. Let (M,g) be a (pseudo-)Riemannian manifold. Then under a conformal transformation $\tilde{g}_{ij} = e^{2\sigma} g_{ij}$ of the metric tensor the $(0,2)$ Schouten tensor S transforms as follows:

$$\tilde{S}_{aj} = S_{aj} - B_{aj} \tag{B}$$

where B is the $(0,2)$ tensor whose components are defined by equation 3.13 followingly:

$$B_{aj} = -\partial_a\sigma\partial_j\sigma + \partial_a\partial_j\sigma - \partial_c\sigma\Gamma^c_{aj} + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{aj}$$

Proof. Let us start from the definition 2.5.5. Using known transformational behaviors of the Ricci form and the Ricci scalar (equalities 3.15 and 3.16 respectively), we can write:

$$\begin{aligned}\tilde{S}_{aj} &= \frac{1}{n-2}\left(\tilde{\varrho}_{aj} - \frac{\tilde{\mathcal{R}}}{2(n-1)}\tilde{g}_{aj}\right) = \\ &= \frac{1}{n-2}\left(\varrho_{aj} - \underbrace{g_{aj}B^l_l} - (n-2)B_{aj} - \frac{1}{2(n-1)}e^{-2\sigma}[\mathcal{R} - \underbrace{2(n-1)B^l_l}]e^{2\sigma}g_{aj}\right) = \\ &= \frac{1}{n-2}\left(\varrho_{aj} - (n-2)B_{aj} - \frac{\mathcal{R}}{2(n-1)}g_{aj}\right) = S_{aj} - B_{aj}\end{aligned}$$

The proof of the equality (B) is now complete. □

Now we shall be able to continue in our endeavors to prove the theorem 4.3.1. Let us substitute from (A) and (B) into the equation 4.11.

$$\begin{aligned}\tilde{C}_{ijk} =&(n-2)(\tilde{\nabla}_k\tilde{S}_{ij} - \tilde{\nabla}_j\tilde{S}_{ik}) = (n-2)(\partial_k\tilde{S}_{ij} - \tilde{\Gamma}^a_{ki}\tilde{S}_{aj} - \tilde{\Gamma}^a_{kj}\tilde{S}_{ai}) - \Sigma(j,k) = \\ =&(n-2)\Bigg[\underbrace{\partial_k S_{ij}}_{C} - \partial_k B_{ij} - \\ &- (\underbrace{\Gamma^a_{ki}}_{C} + \partial_k\sigma\delta^a_i + \partial_i\sigma\delta^a_k - g_{ik}g^{ah}\partial_h\sigma)(\underbrace{S_{aj}}_{C} - B_{aj}) - \\ &- (\underbrace{\Gamma^a_{kj}}_{C} + \partial_k\sigma\delta^a_j + \partial_j\sigma\delta^a_k - g_{jk}g^{ah}\partial_h\sigma)(\underbrace{S_{ai}}_{C} - B_{ai})\Bigg] - \Sigma(j,k)\end{aligned}$$

Here the underbraced terms form together the Cotton tensor C_{ijk} therefore we can rewrite (any term symmetric in j and k cancels out with respective term in the symmetrizer):

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) =$$

$$= -(\partial_k\sigma\delta_i^a + \partial_i\sigma\delta_k^a - g_{ik}g^{ah}\partial_h\sigma)S_{aj} + (\Gamma^a_{ki} + \partial_k\sigma\delta_i^a + \partial_i\sigma\delta_k^a - g_{ik}g^{ah}\partial_h\sigma)B_{aj} -$$
$$-(\partial_k\sigma\delta_j^a + \partial_j\sigma\delta_k^a - g_{jk}g^{ah}\partial_h\sigma)S_{ai} + \underbrace{(\Gamma^a_{kj} + \partial_k\sigma\delta_j^a + \partial_j\sigma\delta_k^a - g_{jk}g^{ah}\partial_h\sigma)}_{\text{symmetric } j\leftrightarrow k}B_{ai} -$$
$$-\partial_k B_{ij} - \Sigma(j,k) =$$
$$= -\partial_k\sigma S_{ij} - \underbrace{\partial_i\sigma S_{kj}}_{\alpha} + g_{ik}S^h_j\partial_h\sigma - \underbrace{\partial_k\sigma S_{ji}}_{\beta} - \underbrace{\partial_j\sigma S_{ki}}_{\beta} + \underbrace{g_{jk}S^h_i\partial_h\sigma}_{\gamma} +$$
$$+(\Gamma^a_{ki} + \partial_k\sigma\delta_i^a + \partial_i\sigma\delta_k^a - g_{ik}g^{ah}\partial_h\sigma)(-\partial_a\sigma\partial_j\sigma + \partial_a\partial_j\sigma - \partial_c\sigma\Gamma^c_{aj} + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{aj}) +$$
$$+\partial_k\partial_i\sigma\partial_j\sigma + \underbrace{\partial_i\sigma\partial_k\partial_j\sigma}_{\delta} - \underbrace{\partial_k\partial_i\partial_j\sigma}_{\varepsilon} + \partial_k\partial_c\sigma\Gamma^c_{ij} + \partial_c\sigma\partial_k\Gamma^c_{ij} -$$
$$-\frac{1}{2}(\partial_k g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma g_{ij} + g^{\alpha\beta}\partial_k\partial_\alpha\sigma\partial_\beta\sigma g_{ij} + g^{\alpha\beta}\partial_\alpha\sigma\partial_k\partial_\beta\sigma g_{ij} + g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma\partial_k g_{ij}) - \underbrace{\Sigma(j,k)}_{\alpha,\beta,\gamma,\delta,\varepsilon}$$

We have substituted for the tensor B and canceled the terms symmetric in j and k against the symmetrizer. Further manipulations yield[4]:

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) =$$
$$= -\partial_k\sigma S_{ij} + g_{ik}S^h_j\partial_h\sigma - \partial_a\sigma\partial_j\sigma\Gamma^a_{ki} + \underbrace{\partial_a\partial_j\sigma\Gamma^a_{ki}}_{\mu} - \partial_c\sigma\underbrace{\Gamma^c_{aj}\Gamma^a_{ki}}_{R} + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{aj}\Gamma^a_{ki} -$$
$$-\underbrace{\partial_i\sigma\partial_j\sigma\partial_k\sigma}_{\zeta} + \underbrace{\partial_i\partial_j\sigma\partial_k\sigma}_{\lambda} - \partial_c\sigma\Gamma^c_{ij}\partial_k\sigma + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{ij}\partial_k\sigma - \underbrace{\partial_k\sigma\partial_j\sigma\partial_i\sigma}_{\eta} + \underbrace{\partial_k\partial_j\sigma\partial_i\sigma}_{\vartheta} -$$
$$-\underbrace{\partial_c\sigma\Gamma^c_{kj}\partial_i\sigma}_{\iota} + \underbrace{\frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{kj}\partial_i\sigma}_{\kappa} + \partial_a\sigma\partial_j\sigma g_{ik}g^{ah}\partial_h\sigma - \partial_a\partial_j\sigma g_{ik}g^{ah}\partial_h\sigma +$$
$$+\partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}(\partial\sigma)^2 g_{aj}g_{ik}g^{ah}\partial_h\sigma + \underbrace{\partial_k\partial_i\sigma\partial_j\sigma}_{\lambda} + \underbrace{\partial_k\partial_a\sigma\Gamma^a_{ij}}_{\mu} + \partial_c\sigma\underbrace{\partial_k\Gamma^c_{ij}}_{R} -$$
$$-\frac{1}{2}(\partial_k g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma g_{ij} + \underbrace{g^{\alpha\beta}\partial_k\partial_\alpha\sigma\partial_\beta\sigma g_{ij} + g^{\alpha\beta}\partial_\alpha\sigma\partial_k\partial_\beta\sigma g_{ij}}_{=2g^{\alpha\beta}\partial_k\partial_\alpha\sigma\partial_\beta\sigma g_{ij}} + (\partial\sigma)^2\partial_k g_{ij}) - \underbrace{\Sigma(j,k)}_{\zeta,\eta,\vartheta,\iota,\kappa,\lambda,\mu,R}$$

Here the terms denoted by R do not cancel the way the terms denoted by greek letters do, rather they represent (along with the terms in the symmetrizer) the Riemann tensor:

$$-\partial_c\sigma R^c_{ijk} = +\partial_c\sigma\partial_k\Gamma^c_{ji} - \partial_c\sigma\Gamma^c_{ja}\Gamma^a_{ki} - \Sigma(j,k)$$

Using this fact, we have:

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) + \partial_c\sigma R^c_{ijk} + \underbrace{\partial_k\sigma S_{ij}}_{=\partial_c\sigma S_{ij}\delta^c_k} - g_{ik}S^c_j\partial_c\sigma - \underbrace{\partial_j\sigma S_{ik}}_{=\partial_c\sigma S_{ik}\delta^c_j} + g_{ij}S^c_k\partial_c\sigma =$$

[4]Henceforth we will somewhat loosely use the notation $(\partial\sigma)^2 := g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma$ in order to simplify the expression.

$$= -\underbrace{\partial_a\sigma\partial_j\sigma\Gamma^a_{ki}}_{\varrho} + \frac{1}{2}(\partial\sigma)^2 g_{aj}\Gamma^a_{ki} - \underbrace{\partial_c\sigma\Gamma^c_{ij}\partial_k\sigma}_{\varrho} + \underbrace{\frac{1}{2}(\partial\sigma)^2 g_{ij}\partial_k\sigma}_{\xi} +$$

$$+ \underbrace{\partial_j\sigma g_{ik}(\partial\sigma)^2}_{\xi} - \underbrace{\partial_a\partial_j\sigma g_{ik}g^{ah}\partial_h\sigma}_{\nu} + \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \underbrace{\frac{1}{2}(\partial\sigma)^2 g_{ik}\partial_j\sigma}_{\xi} -$$

$$- \frac{1}{2}\partial_k g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma g_{ij} - \underbrace{g^{\alpha\beta}\partial_k\partial_\alpha\sigma\partial_\beta\sigma g_{ij}}_{\nu} - \frac{1}{2}(\partial\sigma)^2\partial_k g_{ij} - \underbrace{\Sigma(j,k)}_{\varrho,\xi,\nu}$$

Now because:

$$\partial_c\sigma R^c_{ijk} - S_{ik}\delta^c_j\partial_c\sigma - S^c_j g_{ik}\partial_c\sigma + S^c_k g_{ij}\partial_c\sigma + S_{ij}\delta^c_k\partial_c\sigma = \partial_c\sigma W^c_{ijk}$$

where W^c_{ijk} is the (1,3) Weyl tensor, we can rewrite our equality:

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) + \partial_c\sigma W^c_{ijk} =$$

$$= \frac{1}{2}(\partial\sigma)^2 g_{aj}\Gamma^a_{ki} - \frac{1}{2}(\partial\sigma)^2\partial_k g_{ij} + \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}\partial_k g^{ch}\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k)$$

For the Levi-Civita connection the equality 2.16 holds and thus we obtain:

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) + \partial_c\sigma W^c_{ijk} =$$

$$= \frac{1}{2}(\partial\sigma)^2 g_{aj}\frac{1}{2}g^{ah}(\partial_k g_{hi} - \partial_i g_{hk} - \partial_h g_{ki}) - \frac{1}{2}(\partial\sigma)^2\partial_k g_{ij} +$$
$$+ \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}\partial_k g^{ch}\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k) =$$
$$= \frac{1}{4}(\partial\sigma)^2(\partial_k g_{ji} - \underbrace{\partial_i g_{jk}}_{\tau} - \partial_j g_{ki}) - \frac{1}{2}(\partial\sigma)^2\partial_k g_{ij} +$$
$$+ \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}\partial_k g^{ch}\partial_c\sigma\partial_h\sigma g_{ij} - \underbrace{\Sigma(j,k)}_{\tau} =$$
$$= \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}\partial_k g^{ch}\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k) +$$
$$+ \frac{1}{4}(\partial\sigma)^2(\partial_k g_{ji} - \partial_j g_{ki} - \partial_j g_{ki} + \partial_k g_{ji}) - \frac{1}{2}(\partial\sigma)^2\partial_k g_{ij} + \frac{1}{2}(\partial\sigma)^2\partial_j g_{ik} =$$
$$= \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma - \frac{1}{2}\partial_k g^{ch}\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k)$$

In order to finish the proof of the theorem 4.3.1 we shall prove one last lemma.

Lemma 4.3.4. [4] For the Levi-Civita connection the following identity holds:

$$\partial_l g^{im} = -\left(g^{jm}\Gamma^i_{lj} + g^{ij}\Gamma^m_{lj}\right) \qquad \forall i, l, m \tag{C}$$

Proof. Let us calculate:

$$\Gamma_{jkl} + \Gamma_{kjl} = \frac{1}{2}(\partial_k g_{jl} + \partial_l g_{jk} - \partial_j g_{kl} + \partial_j g_{kl} + \partial_l g_{jk} - \partial_k g_{jl}) = \partial_l g_{jk} \tag{E}$$

Using the partial derivative ∂_l on the equality $g^{ij}g_{kj} = \delta^i_k$ we arrive at:

$$(\partial_l g^{ij})g_{kj} + g^{ij}(\partial_l g_{kj}) = 0$$

Now we multiply this equation by the inverse metric g^{km}:

$$\partial_l g^{ij}\delta^m_j + g^{ij}g^{km}\partial_l g_{kj}$$

Substituting from (E) and rearranging we have:

$$\partial_l g^{im} = -g^{ij}g^{km}\partial_l g_{kj} = -g^{ij}g^{km}(\Gamma_{jkl} + \Gamma_{kjl}) = -(g^{km}\Gamma^i_{kl} + g^{ij}\Gamma^m_{jl})$$

Therefore lemma 4.3.4 has been proved. □

If we make use of the identity (C) in our transformation formula, we arrive at the following:

$$\frac{1}{n-2}(\tilde{C}_{ijk} - C_{ijk}) + \partial_c\sigma W^c_{ijk} =$$
$$= \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma + \frac{1}{2}\left(g^{ca}\Gamma^h_{ka} + g^{ah}\Gamma^c_{ka}\right)\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k) =$$
$$= \partial_c\sigma\Gamma^c_{aj}g_{ik}g^{ah}\partial_h\sigma + \frac{1}{2}\underbrace{g^{ca}\Gamma^h_{ka}\partial_c\sigma\partial_h\sigma g_{ij}}_{g^{ha}\Gamma^c_{ka}\partial_h\sigma\partial_c\sigma g_{ij}} + \frac{1}{2}g^{ah}\Gamma^c_{ka}\partial_c\sigma\partial_h\sigma g_{ij} - \Sigma(j,k) =$$
$$= \underbrace{\left(\Gamma^c_{aj}g_{ik} + \Gamma^c_{ak}g_{ij}\right)}_{\text{symmetric } j \leftrightarrow k} g^{ha}\partial_c\sigma\partial_h\sigma - \Sigma(j,k) = 0$$

This can be finally rewritten as:

$$\tilde{C}_{ijk} = C_{ijk} - (n-2)\partial_c\sigma W^c_{ijk}$$

hence the proof of the theorem 4.3.1 is now complete.

□

Theorem 4.3.5. Let (M, g) be a (pseudo-)Riemannian manifold with $\dim M = 3$ then the $(0, 3)$ Cotton tensor is an invariant of conformal transformations of the metric tensor.

Proof. Follows as an immediate consequence of the transformation formula from the previous theorem:

$$\tilde{C}_{ijk} = C_{ijk} - (n-2)\partial_c\sigma W^c_{ijk}$$

and the fact that the Weyl tensor is a zero tensor (see lemma 4.1.9) on a three-dimensional (pseudo-)Riemannian manifold. All together we have:

$$\tilde{C}_{ijk} = C_{ijk}$$

□

4.4 Obstructions to local conformal flatness

Definition 4.4.1. [3] A (pseudo-)Riemannian manifold (M, g) is *locally conformally flat* if for any $p \in M$, there exists a neighborhood V of p and a $C^\infty(V)$ function σ such that $(V, \tilde{g} = e^{2\sigma} g)$ is flat.

Theorem 4.4.2 (Obstructions to conformal flatness)**.** A (pseudo-)Riemannian manifold (M, g) with $\dim M = n$ is locally conformally flat if and only if

- for $n \geq 4$ the Weyl tensor of M vanishes
- for $n = 3$ the Cotton tensor of M vanishes

Proof. We shall prove this theorem in two implications.

$(\rightarrow)$ We assume the conformal flatness of (M, g), therefore we have that for any $p \in M$, there exists a neighborhood V of p and a $C^\infty(V)$ function σ such that $(V, \tilde{g} = e^{2\sigma} g)$ is flat. Therefore the Riemann curvature tensor $\tilde{R}$ vanishes. Using the decomposition of the Riemann tensor (theorem 4.1.5) and the transformation formula for the $(0, 4)$ Weyl tensor from lemma 4.1.6:

$$\tilde{R} - \tilde{g} \owedge \tilde{S} = \tilde{W} = e^{2\sigma(p)} W$$

Since $\tilde{R} = 0$ the Schouten tensor $\tilde{S}$ must be a zero tensor as well because it constitutes of the traces of $\tilde{R}$, namely the $\tilde{\varrho}$ Ricci form and the $\tilde{\mathcal{R}}$ Ricci scalar curvature. These traces are zero. Therefore

$$0 = e^{-2\sigma(p)} \tilde{W} = W$$

If $\dim M = 3$ then using remark 4.2.2 and the fact that in three dimensions the Cotton tensor is conformally invariant (theorem 4.3.5) we have:

$$\tilde{\nabla}_k \tilde{S}_{ij} - \tilde{\nabla}_j \tilde{S}_{ik} = \tilde{C}_{ijk} = C_{ijk}$$

Now by the same reasoning as in the previous case the Schouten tensors $\tilde{S}$ vanish and thus:

$$0 = \tilde{C}_{ijk} = C_{ijk}$$

This implication is now proved.

$(\leftarrow)$ Let the Weyl tensor be a zero tensor. The condition for $(V, \tilde{g} = e^{2\sigma} g)$ (where we have to show that such function $\sigma \in C^\infty(V) = \Omega^0(V)$ exists) to be flat is the vanishing of the Riemann tensor

$$\tilde{R}(V, X, Y, Z) = 0 \qquad \forall V, X, Y, Z \in \Gamma(TM)$$

Using the knowledge of the behavior of the Riemann curvature tensor under conformal transformations of the metric tensor (lemma 3.3.3), this is equivalent to:

$$0 = \tilde{R} = R - B \owedge g$$

where B is the $(0, 2)$ tensor whose components are defined in lemma 3.13 as

$$B_{ij} = -\partial_i\sigma\partial_j\sigma + \partial_i\partial_j\sigma - \partial_a\sigma\Gamma^a_{ij} + \frac{1}{2}g^{nh}\partial_n\sigma\partial_h\sigma g_{ij}$$

Because of the decomposition $R = W + S \owedge g$ (theorem 4.1.5) and our initial assumption that W is a zero tensor this can be equivalently rewritten yet again:

$$0 = \underbrace{W}_{=0} + S \owedge g - B \owedge g \quad \to \quad (S - B) \owedge g = 0$$

By proposition 1.4.11 the Kuklarni-Nomizu product is possible to be viewed as a map from $\Gamma(TM \otimes TM)$ to $\Gamma(\bigotimes^4 TM)$ that is injective. We have by definition that the metric is non-degenerate and therefore thanks to the injectivity of $g \owedge \cdot$ the tensor $(S - B)$ has to be a $(0, 2)$ zero tensor. Ergo the condition for $(V, \tilde{g})$ to be flat is now $S = B$. Here S is the Schouten tensor (definition 2.5.5) and B is defined by B_{ij} above. That is additionally possible to be rewritten as

$$B_{ij} = -\nabla_i\sigma\nabla_j\sigma + \nabla_i\nabla_j\sigma + \frac{1}{2}g^{nh}\nabla_n\sigma\nabla_h\sigma g_{ij}$$

All together we have:

$$S_{ij} = B_{ij} = -\nabla_i\sigma\nabla_j\sigma + \nabla_i\nabla_j\sigma + \frac{1}{2}g^{nh}\nabla_n\sigma\nabla_h\sigma g_{ij}$$

We have to prove that there is indeed such function $\sigma \in \Omega^0(V)$ that satisfies the last equality. That suffices for $(V, \tilde{g})$ to be flat and therefore for the manifold M to be locally conformally flat. Before we shall proceed with the proof of the theorem 4.4.2 we are to find an equivalent condition.

Lemma 4.4.3. A function $\sigma \in \Omega^0(V)$ being a solution to

$$S_{ij} = -\nabla_i\sigma\nabla_j\sigma + \nabla_i\nabla_j\sigma + \frac{1}{2}g^{nh}\nabla_n\sigma\nabla_h\sigma g_{ij} \tag{A}$$

is equivalent to $\omega \in \Omega^1(V)$ being a solution to

$$S_{ij} = -\omega_i\omega_j + \nabla_i\omega_j + \frac{1}{2}g^{nh}\omega_n\omega_h g_{ij} \tag{B}$$

Proof. If σ is a solution of (A) then certainly $\omega := d\sigma$ solves (B). Conversely, let $\omega \in \Omega^1(V)$ be a solution of (B). We can rewrite that as

$$\nabla_i\omega_j = S_{ij} + \omega_i\omega_j - \frac{1}{2}g^{nh}\omega_n\omega_h g_{ij}$$

Now by the symmetry of the Schouten tensor, we see that the right-hand side of this equation is symmetric. Hence the left-hand side must be symmetric as well and we have

$$\nabla_i\omega_j = \nabla_j\omega_i \qquad \forall i, j \tag{C}$$

We are to find $d\omega$ using the Cartan identity 1.3.2 on the coordinate vector fields.

$$\begin{aligned} d\omega(\partial_i, \partial_j) &= \partial_i(\omega(\partial_j)) - \partial_j(\omega(\partial_i)) - \omega(\underbrace{\partial_i\partial_j - \partial_j\partial_i}_{=0}) = \\ &= \nabla_i\omega_j + \omega_a\Gamma^a_{ij} - \nabla_j\omega_i - \omega_a\Gamma^a_{ji} - \omega(0) = \\ &= \nabla_i\omega_j - \nabla_j\omega_i \end{aligned}$$

We have made use of the symmetry of Christoffel symbols in the two lower indices for the Levi-Civita connection. Because (C) holds, we have $d\omega = 0$. Ergo ω is a closed one-form and by Poincaré lemma 1.2.19 it is locally exact. More specifically around an arbitrary point $p \in M$ there exists a neighborhood V such that ω is exact. This implies that there exists $\sigma \in \Omega^0(V) = C^\infty(V)$ such that $\omega = d\sigma$. If we substitute $\omega = d\sigma$ into (B) we obtain (A). Proof of the lemma 4.4.3 is now complete. □

We can continue in our efforts to prove the theorem 4.4.2. In order to prove the local conformal flatness of M, we have to find a one-form ω such that

$$\nabla_i\omega_j = \omega_i\omega_j + S_{ij} - \frac{1}{2}g^{nh}\omega_n\omega_h g_{ij} \tag{4.12}$$

This is a differential equation in terms of the Levi-Civita connection (its extension as covariant derivative). For a solution ω to exist the integrability condition of the Ricci identity 2.6.2 must be satisfied (see remark 2.6.3). Shall we compute the integrability condition, we are to find $\nabla_k\nabla_j\omega_i$ first. Substituting from the equation 4.12:

$$\begin{aligned} \nabla_k\nabla_j\omega_i &= \nabla_k(\omega_j\omega_i + S_{ji} - \frac{1}{2}g^{nh}\omega_n\omega_h g_{ji}) = \\ &= (\nabla_k\omega_j)\omega_i + \omega_j(\nabla_k\omega_i) + \nabla_k S_{ij} - \frac{1}{2}g^{\alpha\beta}g_{ij}[(\nabla_k\omega_\alpha)\omega_\beta + \omega_\alpha(\nabla_k\omega_\beta)] = \\ &= (\nabla_k\omega_j)\omega_i + \omega_j(\nabla_k\omega_i) + \nabla_k S_{ij} - g^{\alpha\beta}g_{ij}(\nabla_k\omega_\alpha)\omega_\beta \end{aligned}$$

Now we substitute from 4.12 yet again:

$$\begin{aligned} \nabla_k\nabla_j\omega_i = (S_{kj} + \omega_k\omega_j - \frac{1}{2}\omega^2 g_{kj})\omega_i + \omega_j(S_{ki} + \omega_k\omega_i - \frac{1}{2}\omega^2 g_{ki}) + \\ + \nabla_k S_{ij} - g^{\alpha\beta}g_{ij}(S_{k\alpha} + \omega_k\omega_\alpha - \frac{1}{2}\omega^2 g_{k\alpha})\omega_\beta \end{aligned}$$

where we denote $g^{\alpha\beta}\omega_\alpha\omega_\beta \equiv \omega^2$. Expanding all terms, we shall substitute this into the Ricci identity, exploiting the properties of the symmetrizer[5] in a very simmilar manner to

[5] The symbol $\Sigma(j,k)$ represents the symmetric part of the whole expression (from the last equality =) positioned in front of it; i.e. the same expression only with interchanged j and k.

that included in the proof of lemma 3.3.1.

$$\begin{aligned}
R^l_{ijk}\omega_l = {} & \underbrace{S_{kj}\omega_i}_{\kappa} + \underbrace{\omega_k\omega_j\omega_i}_{\lambda} - \frac{1}{2}\underbrace{\omega^2 g_{kj}\omega_i}_{\mu} + S_{ki}\omega_j + \underbrace{\omega_k\omega_i\omega_j}_{\nu} - \frac{1}{2}\omega^2 g_{ki}\omega_j - \\
& - g^{\alpha\beta}g_{ij}\omega_\beta S_{k\alpha} - g^{\alpha\beta}g_{ij}\omega_\beta\omega_k\omega_\alpha + \frac{1}{2}g^{\alpha\beta}g_{ij}\omega_\beta\omega^2 g_{k\alpha} + \nabla_k S_{ij} - \underbrace{\Sigma(j,k)}_{\kappa,\lambda,\mu,\nu} = \\
& = S_{ki}\omega_j - \frac{1}{2}\omega^2 g_{ki}\omega_j - g^{\alpha\beta}g_{ij}\omega_\beta S_{k\alpha} - \underbrace{\omega^2 g_{ij}\omega_k}_{\iota} + \underbrace{\frac{1}{2}g_{ij}\omega_k\omega^2}_{\iota} + \nabla_k S_{ij} - \Sigma(j,k) = \\
& = S_{ki}\omega_j - \underbrace{\frac{1}{2}\omega^2 g_{ki}\omega_j}_{\vartheta} - g^{\alpha\beta}g_{ij}\omega_\beta S_{k\alpha} + \nabla_k S_{ij} - \underbrace{\frac{1}{2}g_{ij}\omega_k\omega^2}_{\vartheta} - \underbrace{\Sigma(j,k)}_{\vartheta} = \\
& = S_{ki}\omega_j - g_{ij}\omega_\beta S^\beta_k + \nabla_k S_{ij} - \Sigma(j,k)
\end{aligned} \tag{4.13}$$

Let us now expand the left-hand side of the equation using the decomposition of the Riemann tensor 4.1.5 with the first index raised:

$$\begin{aligned}
R^l_{ijk}\omega_l &= (W^l_{ijk} + -S_{ij}\delta^l_k + S_{ik}\delta^l_k - S^l_k g_{ij} + S^l_j g_{ik})\omega_l = \\
&= W^l_{ijk}\omega_l - S_{ij}\omega_k + S_{ik}\omega_j - S^l_k g_{ij}\omega_l + S^l_j g_{ik}\omega_l
\end{aligned}$$

Now we substitute this into the left-hand side of the left-hand side of our integrability condition and after all the terms cancel out and we employ the initial assumption of vanishing of the Weyl tensor, what remains is:

$$\nabla_k S_{ij} - \nabla_j S_{ik} = 0$$

Comparing this with remark 4.2.2 it is obvious that after multiplying the equation by $(n-2)$ the left-hand side becomes exactly the Cotton tensor of M. We arrived at the vanishing of the Cotton tensor as an integrability condition for the manifold to be locally conformally flat. In dimensions $\dim M = n \geq 4$ this is ensured by the corollary 4.2.8 because there when the Weyl tensor vanishes, the Cotton tensor vanishes as well (and the vanishing of the Weyl tensor was our initial assumption). In dimension $n = 3$ vanishing of the Cotton tensor is the integrability condition itself and therefore the proof of vanishing of W and C being the sufficient condition for M to be locally conformally flat in respective dimensions is now complete. □

4.5 Cotton-York tensor

Remark 4.5.1. Let (M, g) be a Riemannian manifold with $\dim M = 3$. It is possible to percieve C_{ijk} as a vector-valued 2-form thanks to its anti-symmetry in the last two indices (see lemma 4.2.3). Symbolically:

$$C_i = \frac{1}{2}C_{ijk}dx^j \wedge dx^k \qquad i, j, k = 1, 2, 3$$

Now we are to use the Hodge star $\star$ on this 2-form on a three-dimensional Riemannian manifold. Using the equality 1.5:

$$Y_i := \star(C_i) = \frac{\sqrt{|\det g|}}{2} C_{ijk}\varepsilon^{jk}{}_h dx^h = \frac{1}{2} C_{ijk} g^{\alpha j} g^{\beta k} \underbrace{\sqrt{|\det g|}\varepsilon_{\alpha\beta h}}_{\epsilon_{\alpha\beta h}} dx^h = \frac{1}{2} C_{ijk}\epsilon^{jkl} g_{lh} dx^h$$

where we have employed the definition (see equality 1.3) of the Levi-Civita tensor as well. The value of Y_i on a basis element $\partial_n \in T_pM$:

$$Y_{in} := Y_i(\partial_n) = \frac{1}{2} C_{ijk}\epsilon^{jkl} g_{lh} \underbrace{dx^h(\partial_n)}_{\delta^h_n} = \frac{1}{2} C_{ijk}\epsilon^{jkl} g_{ln}$$

is a $(0,2)$ tensor. Finally we rewrite (by the anti-symmetry of ϵ):

$$\begin{aligned} Y_{in} &= \frac{1}{2}\left[\nabla_k \varrho_{ij} - \nabla_j \varrho_{ik} + \frac{1}{4}\left(g_{ik}\nabla_j \mathcal{R} - g_{ij}\nabla_k \mathcal{R}\right)\right]\epsilon^{jkl} g_{ln} = \\ &= \frac{1}{2}\left[(-1)\nabla_k\left(\varrho_{ij} - \frac{1}{4} g_{ij}\mathcal{R}\right)\epsilon^{kjl} - \nabla_j\left(\varrho_{ik} - \frac{1}{4} g_{ik}\mathcal{R}\right)\epsilon^{jkl}\right] g_{ln} \end{aligned}$$

By renaming the summing indices $j \leftrightarrow k$ in the first part of the expression Y_{in} we obtain the following:

$$Y_{in} = -\epsilon^{jkl} g_{ln}\nabla_j\left(\varrho_{ik} - \frac{1}{4} g_{ik}\mathcal{R}\right)$$

Now we raise both indices of Y_{in}:

$$Y^{\alpha\beta} = -\epsilon^{jkl}\delta^\beta_l \nabla_j\left(\varrho^\alpha_k - \frac{1}{4}\delta^\alpha_k \mathcal{R}\right) = -\epsilon^{jk\beta}\nabla_j\left(\varrho^\alpha_k - \frac{1}{4}\delta^\alpha_k \mathcal{R}\right)$$

Definition 4.5.2. [8] The coordinate expression

$$Y^{ij} = \epsilon^{ikl}\nabla_k\left(\varrho^j_l - \frac{1}{4}\mathcal{R}\delta^j_l\right) \tag{4.14}$$

defines a $(2,0)$ tensor on (M,g) with $\dim M = 3$, called the *Cotton-York tensor* (sometimes called the Cotton form).

Proposition 4.5.3. [6] The $(2,0)$ Cotton-York tensor is symmetric, i.e.

$$Y^{ij} = Y^{ji}$$

Proof. First let us prove the following lemma:

Lemma 4.5.4. Let A_{ijk} be a $(0,3)$ tensor antisymmetric in i and j and M^{ij} be a general $(2,0)$ tensor. Then

$$A_{ijk}M^{ij} = 0$$

if and only if M^{ij} is symmetric in i and j.

Proof. We shall prove this lemma in two implications
($\rightarrow$) Let $A_{ijk}M^{ij} = 0$. We can then rewrite this condition by the antisymmetry of A_{ijk} followingly:

$$\frac{1}{2}A_{ijk}M^{ij} - \frac{1}{2}A_{jik}M^{ij} = 0$$

Now we rename the summing indices on the second term $i \leftrightarrow j$ and therefore obtain:

$$A_{ijk}(M^{ij} - M^{ji}) = 0 \qquad \rightarrow \qquad M^{ji} = M^{ij}$$

($\leftarrow$) Let M^{ij} be a symmetric $(2,0)$ tensor. We have:

$$A_{ijk}M^{ij} = \frac{1}{2}A_{ijk}M^{ij} - \frac{1}{2}A_{jik}M^{ji} = 0$$

which proves the lemma. □

Now we are to apply the obtained lemma in the proof of proposition 4.5.3. We shall multiply the $(2,0)$ tensor with the totally antisymmetric $(0,3)$ Levi-Civita tensor ϵ_{ijk}:

$$\begin{aligned}\epsilon_{ijk}Y^{ij} &= \epsilon_{ijk}\epsilon^{isl}\nabla_s\left(\varrho^j_l - \frac{1}{4}\mathcal{R}\delta^j_l\right) = (\delta^s_j\delta^l_k - \delta^l_j\delta^s_k)\nabla_s\left(\varrho^j_l - \frac{1}{4}\mathcal{R}\delta^j_l\right) = \\ &= \nabla_j\varrho^j_k - \frac{1}{4}\nabla_k\mathcal{R} - \nabla_k\varrho^j_j + \frac{1}{4}\underbrace{\delta^j_l\delta^l_j}_{=\dim M=3}\nabla_k\mathcal{R}\end{aligned}$$

Making use of the fact that $\nabla_j\varrho^j_k = \frac{1}{2}\nabla_k\mathcal{R}$ (lemma 4.2.5) we find:

$$\epsilon_{ijk}Y^{ij} = \frac{1}{2}\nabla_k\mathcal{R} - \frac{1}{4}\nabla_k\mathcal{R} - \nabla_k\mathcal{R} + \frac{3}{4}\nabla_k\mathcal{R} = \left(\frac{1}{4} - 1 + \frac{3}{4}\right)\nabla_k\mathcal{R} = 0$$

Using the lemma 4.5.4 vanishing of $\epsilon_{ijk}Y^{ij}$ is equal to the fact that Y^{ij} is a symmetric $(2,0)$ tensor. □

Proposition 4.5.5. [6] The $(2,0)$ Cotton-York tensor is trace-less, i.e.

$$g_{ij}Y^{ij} = 0$$

Proof. By direct calculation (making the use of the total antisymmetry of ϵ^{ikl} and the symmetry of the Ricci form and the metric tensor)

$$\begin{aligned}g_{ij}Y^{ij} &= g_{ij}\epsilon^{ikl}\nabla_k\left(\varrho^j_l - \frac{1}{4}\mathcal{R}\delta^j_l\right) = \epsilon^{ikl}\nabla_k\left(\varrho_{il} - \frac{1}{4}\mathcal{R}g_{il}\right) = \\ &= \underbrace{\epsilon^{lik}}_{-\epsilon^{lki}}\nabla_k\left(\varrho_{li} - \frac{1}{4}\mathcal{R}g_{li}\right) = -\epsilon^{ikl}\nabla_k\left(\varrho_{il} - \frac{1}{4}\mathcal{R}g_{il}\right)\end{aligned}$$

In the last step, we have renamed the indices $i \leftrightarrow l$. From this it can be readily seen that

$$g_{ij}Y^{ij} = -g_{ij}Y^{ij}$$

and that is only possible when $g_{ij}Y^{ij} = 0$. □

Proposition 4.5.6. The $(2,0)$ Cotton-York tensor satisfies:

$$\nabla_j Y^{ij} = 0$$

Proof. Computing, we have:

$$\begin{aligned}\nabla_j Y^{ij} &= \epsilon^{ikl}\nabla_j\nabla_k\left(\varrho_l^j - \frac{1}{4}\mathcal{R}\delta_l^j\right) = \epsilon^{ikl}g^{\beta j}\nabla_j\nabla_k\varrho_{\beta l} - \frac{1}{4}\epsilon^{ikl}\nabla_l\nabla_k\mathcal{R} = \\ &= \epsilon^{ikl}g^{\beta j}\nabla_j\nabla_k\varrho_{\beta l} - \frac{1}{8}\epsilon^{ikl}\underbrace{(\nabla_l\nabla_k - \nabla_k\nabla_l)\mathcal{R}}_{=0}\end{aligned}$$

We have made use of the anti-symmetry of ϵ^{ikl} and renamed the indices $k \leftrightarrow l$. The second term must vanish because of lemma 2.6.1. By using the Ricci identity for the $(0,2)$ Ricci form (theorem 2.6.4), our expression becomes:

$$\begin{aligned}\nabla_j Y^{ij} &= \epsilon^{ikl}g^{\beta j}\nabla_j\nabla_k\varrho_{\beta l} = \\ &= \epsilon^{ikl}g^{\beta j}(\nabla_k\nabla_j\varrho_{\beta l} + \varrho_{\beta h}R^h_{lkj} + \varrho_{hl}R^h_{\beta kj}) = \\ &= \epsilon^{ikl}\nabla_k\underbrace{\nabla_j\varrho_l^j}_{=\frac{1}{2}\nabla_l\mathcal{R}} + \epsilon^{ikl}g^{\beta j}(\varrho_{\beta h}R^h_{lkj} + \varrho_{hl}R^h_{\beta kj})\end{aligned}$$

where the equality from lemma 4.2.5 has been employed. The first term in this last equality can be therefore rewritten (making use of the anti-symmetry of ϵ^{ikl} and renaming indices) as:

$$\frac{1}{2}\epsilon^{ikl}\nabla_k\nabla_l\mathcal{R} = \frac{1}{4}\epsilon^{ikl}(\nabla_k\nabla_l - \nabla_l\nabla_k)\mathcal{R} = 0$$

which is again zero by lemma 2.6.1. Now, on a three-dimensional manifold the Riemann curvature tensor is possible to be expressed in terms of Ricci forms and Ricci scalars (see theorem 4.1.8), hence the equality becomes:

$$\begin{aligned}\nabla_j Y^{ij} &= \epsilon^{ikl}g^{\beta j}(\varrho_{\beta h}R^h_{lkj} + \varrho_{hl}R^h_{\beta kj}) = \\ = \epsilon^{ikl}g^{\beta j}&\left\{\varrho_{\beta h}\left[\varrho_{lj}\delta^h_k - \underbrace{\varrho_{lk}\delta^h_j} + g_{lj}\varrho^h_k - \underbrace{g_{lk}\varrho^h_j} + \frac{\mathcal{R}}{2}\left(\underbrace{g_{lk}\delta^h_j} - g_{lj}\delta^h_k\right)\right] + \right. \\ &\left. + \varrho_{hl}\left[\varrho_{\beta j}\delta^h_k - \varrho_{\beta k}\delta^h_j + g_{\beta j}\varrho^h_k - g_{\beta k}\varrho^h_j + \frac{\mathcal{R}}{2}\left(g_{\beta k}\delta^h_j - g_{\beta j}\delta^h_k\right)\right]\right\}\end{aligned}$$

Here any term symmetric in l and k vanishes, for it is multiplied by the totally anti-symmetric Levi-Civita tensor. Expanding all terms we obtain:

$$\begin{aligned}\nabla_j Y^{ij} &= \epsilon^{ikl}g^{\beta j}(\varrho_{\beta h}R^h_{lkj} + \varrho_{hl}R^h_{\beta kj}) = \\ =\epsilon^{ikl}&\left\{g^{\beta j}\varrho_{\beta h}\varrho_{lj}\delta^h_k + g^{\beta j}\varrho_{\beta h}g_{lj}\varrho^h_k - \frac{\mathcal{R}}{2}g^{\beta j}\varrho_{\beta h}g_{lj}\delta^h_k + \right. \\ &+ g^{\beta j}\varrho_{hl}\varrho_{\beta j}\delta^h_k - g^{\beta j}\varrho_{hl}\varrho_{\beta k}\delta^h_j + g^{\beta j}\varrho_{hl}g_{\beta j}\varrho^h_k - \\ &\left. - g^{\beta j}\varrho_{hl}g_{\beta k}\varrho^h_j + \frac{\mathcal{R}}{2}g^{\beta j}\varrho_{hl}g_{\beta k}\delta^h_j - \frac{\mathcal{R}}{2}g^{\beta j}\varrho_{hl}g_{\beta j}\delta^h_k\right\} =\end{aligned}$$

After simplifications:

$$\nabla_j Y^{ij} = \epsilon^{ikl}\left\{\varrho^j_k\varrho_{lj} + \varrho_{lh}\varrho^h_k - \underbrace{\frac{\mathcal{R}}{2}\varrho_{lk}} + \underbrace{\mathcal{R}\varrho_{kl}} - \varrho_{jl}\varrho^j_k + 3\varrho_{hl}\varrho^h_k - \varrho_{hl}\varrho^h_k + \underbrace{\frac{\mathcal{R}}{2}\varrho_{lk}} - \underbrace{\frac{3}{2}\mathcal{R}\varrho_{kl}}\right\} =$$

$$= 4\epsilon^{ikl}\varrho_{lh}\varrho^h_k = 4\epsilon^{ikl}g^{\alpha h}\varrho_{lh}\varrho_{\alpha k} = 2\epsilon^{ikl}g^{\alpha h}\varrho_{lh}\varrho_{\alpha k} - \underbrace{2\epsilon^{ilk}g^{h\alpha}\varrho_{l\alpha}\varrho_{hk}}_{2\epsilon^{ikl}g^{h\alpha}\varrho_{k\alpha}\varrho_{hl}} = 0$$

We have proved the desired equality. □

Conclusion

We have studied conformal transformations of the metric tensor on a pseudo-Riemannian manifold. As a consequence of finding transformation formulas for the Levi-Civita connection, the Riemann curvature tensor and its traces we encountered and defined the Weyl tensor. We have described and proved all fundamental properties of the Weyl tensor and as a result of searching for further symmetries in its derivatives we have found the Cotton tensor. We have shown that this tensor is closely tied by its properties to the Weyl tensor and the overall conformal geometry of the manifold. The proof of the essential theorem regarding the Weyl and Cotton tensors acting as an obstruction to local conformal flatness of the manifold was given using the integrability condition argument. We have used the Hodge star to convert the Cotton tensor to an equivalent tensor of lower order on a three-dimensional manifold and studied its agebraic properties.

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