Effective action and homological perturbation lemma

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Abstract

Loop homotopy Lie algebras, which appear in closed string field theory, are a generalization of homotopy Lie algebras. For a loop homotopy Lie algebra, we transfer its structure on its homology and prove that the transferred structure is again a loop homotopy algebra. Moreover, we show that the homological perturbation lemma can be regarded as a path integral, integrating out the degrees of freedom which are not in the homology. The transferred action then can be interpreted as an effective action in the Batalin-Vilkovisky formalism.

A review of necessary results from Batalin-Vilkovisky formalism and homotopy algebras is included as well.

Contents

Introduction

In this work, we describe a connection between *BV formalism*, *homological perturbation lemma* and *loop homotopy algebras*. The Batalin-Vilkovisky formalism [1] is a powerful method in quantum field theory for quantizing gauge theories. A central identity to BV formalism is the *quantum master equation*

$$2\hbar\Delta S + \{S, S\} = 0$$

for an action S, which is a condition on the gauge invariance of the theory. Among other situations, a BV action was constructed for closed string field theory in [2] by Zwiebach.

It was already known that the tree level of a closed string field theory has a structure (its interaction vertices, to be precise) of a L_∞ algebra, an example of homotopy algebra. In the general case, the algebraic structure was named *loop homotopy Lie algebra* by Markl in [3]. The BV formalism, which can be defined for a loop homotopy algebra, then encodes its main axioms in the quantum master equation constructed from the algebra operations.

A homological perturbation lemma is a computational tool for transferring differentials along homotopy equivalences of chain complexes. For homotopy algebras, it was used for construction of *minimal models* (see references in [4]), new homotopy algebras defined on a homology of the original algebra.

The aim of this work was to repeat a similar transfer of algebra structure on the homology for a loop homotopy Lie algebra. However, because these algebras are intricately connected with physics, the homological perturbation lemma should have a physical counterpart. This turns out to be a construction of *effective S-matrix*, using the path integral to integrate out the degrees of freedom not in the homology. This gives a striking interpretation of the various formulas that appear in homological perturbation lemma – they are just summarizing the Feynman diagram expansion of path integral.

This physical interpretation of homological perturbation theory appears e.g. in works of P. Mnev [5], K. Costello and O. Gwilliam [6, 7] and H. Kajiura [4] and notably in a presentation C. Albert's [8] from Cargèse conference 2009. The general case for algebras over modular operads was considered by Chuang and Lazarev [9].

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Notation

Notation

$\triangle$ and $\diamond$	denote the end of definition and example, respectively.
$V^{\#}$	is a linear dual of V, for graded vector spaces see appendix A. All vector spaces are assumed to be finite-dimensional, all graded vector spaces are degree-wise finite-dimensional.
d, Q	usually denote differentials. We will work in the cohomological convention, that is with differentials of degree $+1$.
$\mathbb{S}_n$	is the permutation group of n elements.
$\mathbb{1}$	is used to denote identity morphisms.
$\mathbb{k}$	is a field of characteristic 0, we always have real or complex numbers on mind.
$\circ, \circ_i$	$\circ$ is a composition of maps, $f \circ_j g$ is defined as $$f \circ \left(\mathbb{1}^{\otimes(j-1)} \otimes g \otimes \mathbb{1}^{\otimes(n-j)}\right)$$ for f with n arguments.
$\frac{\partial_R}{\partial \phi}, \frac{\partial_L}{\partial \phi}$	are the right and left derivative.
$a_i \phi^i$	means $\sum_i a_i \phi^i$, i.e. we use the Einstein summation convention.

Abbreviations

RHS and LHS	are short for *right hand side* and *left hand side*.
BV	stands for Batalin-Vilkovisky.
BRST	stands for Becchi, Rouet, Stora and Tyutin.

1 Batalin-Vilkovisky formalism

The Batalin-Vilkovisky formalism (BV henceforth) will allow us to connect quantum field theory and homotopy algebras. This chapter aims to explain its origin from physics, describe its geometrical interpretation and collect necessary results. The connection to the homotopy algebras is then made clear in chapter 2.

The quantum field part of the explanation follows closely the first chapter of textbook [10] by S. Weinberg. See also [11] for a thorough physical treatment.

1.1 BV formalism and path integral

A basic physical object we will be studying is the path integral. As is well known, the path integral itself is not well defined, but allows a diagrammatical description for its expansion in powers of coupling constants. Naive application of these *Feynman rules* doesn't give a right answer in the case of gauge theories (i.e. the computed S-matrix is not unitary).

Faddeev–Poppov procedure and BRST incorporate the gauge invariance by adding more fields to the theory: ghosts c, antighosts b (with *ghost numbers* equal to 1 and -1, respectively) and also Lautrup–Nakanishi field h. We will assume that the statistics of a field is equal to its ghost number $\bmod 2$. This allows us to mention only one grading, the ghost grading.

Intuitively, the additional ghosts and antighosts cancel the superficial integration over the gauge redundant degrees of freedom; technically, they are used to "lift" a determinant factor to the exponential, so that we can treat it using the standard diagrammatic techniques. They are called ghosts because they don't obey the spin-statistics theorem, i.e. usually they are anticommuting (that is fermionic) scalars.

The BRST formalism, a predecessor of BV, introduces a nilpotent BRST transformation s. One then shows that the actions and observables belong to the cohomology classes of s. Choosing a gauge then reduces to choosing a representative of a cohomology class of the action.

BV formalism introduces new field $\chi^\ddagger$ for every field χ of BRST, called *antifield*. These antifields have opposite statistics and ghost number equal such that $\operatorname{gh}\chi + \operatorname{gh}\chi^\ddagger = -1$. In contrast to BRST, we do not integrate over these new fields, but rather fix a subspace of fields and antifields to integrate over.

Let us illustrate it on the case of Yang Mills theory. After BRST, one arrives at an action that can be written in the form

$$S_{\mathrm{BRST}}[\phi, b, c, h] = S_{\mathrm{YM}}[\phi] + s\Psi[\phi, b, c, h]\,.$$

Here, ϕ is just the original YM gauge field, b and c are ghosts and antighosts and h is the Launtrup-Nakanishi field. The operator s is the nilpotent BRST differential. The fermionic functional Ψ with $\operatorname{gh}\Psi = -1$ is called a gauge fixing fermion. It can be shown that the physical matrix elements (i.e. matrix elements of gauge invariant observables between physical states) are independent of the choice of Ψ. Moreover, thanks to gauge invariance of S_{YM}, $sS_{\mathrm{BRST}} = 0$. The path integral, computing the expectation value of observable $X[\phi]$ is in this case

$$\langle X \rangle = \int \mathcal{D}\phi \mathcal{D}b \mathcal{D}c \mathcal{D}h \; X[\phi] e^{S_{\mathrm{YM}}[\phi] + s\Psi[\phi,b,c,h]}\,.$$

Now, in BV formalism, we add the antifields $\chi^\ddagger$ (note that antighost b and antifield for ghost $c^\ddagger$ are two different fields) and extend the Yang Mills action as

$$S[\chi, \chi^\ddagger] = S_{\mathrm{YM}}[\phi] + (s\chi)^n \chi^\ddagger_n\,,$$

where the index n represents the discrete (representation vector space indices) and the continuous (spacetime) degrees of freedom. Integration and sum is implied by the Einstein summation convention, e.g.

$$(s\chi)^n \chi^\ddagger_n \equiv \sum_i \int \mathrm{d}x^4 (s\chi)^i(x)\chi^\ddagger_i(x)\,.$$

This action now satisfies a *classical master equation*

$$\frac{\delta_R S}{\delta \chi^\ddagger_n}\frac{\delta_L S}{\delta \chi^n} = 0\,,$$

because

$$\frac{\delta_R S}{\delta \chi^\ddagger_n} = (s\chi)^n$$

and the master equation gives

$$(s\phi)^n \frac{\delta_L S_{\mathrm{YM}}}{\delta \phi^n} + (s\chi)^m \frac{\delta_L (s\chi)^n}{\delta \chi^m}\chi^\ddagger_m = 0\,.$$

The term without antifields is just sS_{YM}, while the term with antifields gives $s^2\chi^m$. The first term is zero by gauge invariance of S_{YM} and the second by nilpotency of s

When integrating, we prescribe a definite value to the antifields by the formula[1]

$$\chi^\ddagger_n = \frac{\delta \Psi[\chi]}{\delta \chi^n}\,,$$

which gives an action

$$S\left[\chi, \frac{\delta \Psi[\chi]}{\delta \chi^n}\right] = S_{\mathrm{YM}}[\phi] + (s\chi^n)\frac{\delta \Psi[\chi]}{\delta \chi^n} = S_{\mathrm{YM}}[\phi] + s\Psi[\chi]\,,$$

exactly as the BRST procedure.

This action can be generalized to theories where the action is now no longer linear in antifields. We construct the general action such that it is bosonic and has ghost number 0. Again, we integrate by setting $\chi^\ddagger_n = \delta\Psi[\chi]/\delta\chi^n$ and we want the result to be independent of the gauge fixing fermion $\Psi[\chi]$ (with ghost number -1), which generalizes the gauge invariance of the Yang Mills theory.

How do we ensure this gauge invariance? Choosing a general functional $H[\chi, \chi^\ddagger]$, we want the path integral

$$\int \mathcal{D}\chi\, H\left[\chi, \frac{\delta \Psi[\chi]}{\delta \chi}\right] \tag{1}$$

to be independent of $\Psi[\chi]$. If we deform $\Psi[\chi]$ by a small functional $\epsilon[\chi]$, the above integral 1 changes (to the first order) by

$$\int \mathcal{D}\chi\, \frac{\delta_R H}{\delta \chi^\ddagger_n}\frac{\delta_L \epsilon}{\delta \chi^n}\,,$$

where we write the right derivative of H because the other term from the chain rule comes to the right of it. The derivative of ϵ can be left or right, since ϵ is fermionic. Using integration by parts, we obtain

$$\int \mathcal{D}\chi\, (-1)^{|H||\chi^n|+1}\frac{\delta_L}{\delta \chi^n}\frac{\delta_R H}{\delta \chi^\ddagger_n}\epsilon\,.$$

[1] We do not specify if the functional derivative is left or right, since both cases are the same:

$$\frac{\delta_R \Psi}{\delta \chi^n} = (-1)^{(|\Psi|-|\chi^n|)|\chi^n|}\frac{\delta_L \Psi}{\delta \chi^n}\,,$$

where the sign factor is +1 because $|\Psi| = 1$.

Since ϵ is arbitrary, the invariance of the integral 1 is equivalent to the condition

$$\Delta H \equiv (-1)^{|H||\chi^n|+1} \frac{\delta_L}{\delta \chi^n} \frac{\delta_R H}{\delta \chi^\ddagger_n} = 0\,. \tag{2}$$

Note that this Δ is *different* from operator Δ_{W} of [10, equation 15.9.34], for bosonic H, this Δ_{W} is minus the Δ from equation 2.[2]

Usually, we deal with H in the form $e^{\alpha S[\chi,\chi^\ddagger]}$, where the factor α is usually $\pm i$, $i/\hbar$ or $-1/\hbar$. A simple calculation shows that (for bosonic action)

$$\Delta e^{\alpha S[\chi,\chi^\ddagger]} = \alpha^2 e^{\alpha S[\chi,\chi^\ddagger]} \left(\{S, S\} + \frac{2}{\alpha} \Delta S \right) \stackrel{!}{=} 0\,,$$

where we used *antibracket*

$$\{F, G\} \equiv \frac{\delta_R F}{\delta \chi^n} \frac{\delta_L G}{\delta \chi^\ddagger_n} - \frac{\delta_R F}{\delta \chi^\ddagger_n} \frac{\delta_L G}{\delta \chi^n}\,. \tag{3}$$

Thus, if we want a gauge invariant theory, the action has to satisfy the *quantum master equation*

$$\{S, S\} - 2\hbar \Delta S = 0\,, \tag{4}$$

where we have chosen $\alpha = -1/\hbar$.

Moreover, if we take an bosonic observable $\mathcal{O}[\chi, \chi^\ddagger]$, the expectation value is gauge independent if

$$0 = \Delta(\mathcal{O}[\chi, \chi^\ddagger] e^{\alpha S[\chi,\chi^\ddagger]}) = (\Delta \mathcal{O} + \alpha \{\mathcal{O}, S\})\, e^{\alpha S[\chi,\chi^\ddagger]}\,, \tag{5}$$

where we used the quantum master equation $\Delta e^{\alpha S} = 0$. Usually, the observables do not depend on antifields and we get a condition

$$\{\mathcal{O}, S\} = 0\,.$$

1.2 Geometrical interpretation

The geometric interpretation of BV formalism was started by a short paper by E. Witten [13], where he interpreted the antifields as a multivector fields on the manifold of fields. More complete treatment in the context of supermanifolds was given by A. Schwarz in well-known [14] and later

[2] The Δ_{W} from [10], defined as

$$\Delta_{\mathrm{W}} \equiv \frac{\delta_R}{\delta \chi^\ddagger_n} \frac{\delta_L}{\delta \chi^n} = \frac{\delta_L}{\delta \chi^n} \frac{\delta_R}{\delta \chi^\ddagger_n}\,,$$

is problematic in few aspects, mostly in the overall minus sign: the master equation [10, equation 15.9.35] should actually read $(S, S) + 2i\Delta S = 0$. This is, however, nonstandard, and authors usually choose a different Δ, which is S. Weinberg's $-\Delta_{\mathrm{W}}$ on bosonic functionals: we e.g. Henneaux, Teitelboim [11, section 15.5.3].

In the original reference by Batalin, Vilkovisky [1, equation 16], the Δ is defined with left derivative for the antifield and right derivative for the field. See also physics SE discussion at [12], where the difference between S. Weinberg's and the original Δ is discussed.

Generally, when working with BV algebras, one needs additional sign factor $(-1)^{|H||\chi_n|}$ to make Δ and the bracket compatible: see section 1.3. Henneaux and Teitelboim have this sign, but their Δ_{HT} behaves like a right derivative [11, equation 18.5a]:

$$\Delta_{\mathrm{HT}}(FG) = F(\Delta_{\mathrm{HT}} G) + (-1)^{|G|} (\Delta_{\mathrm{HT}} F) G + (-1)^{|G|} \{F, G\}\,.$$

In the section 1.3 will take our Δ_{our} to be

$$\Delta_{\mathrm{our}} H = (-1)^{|H|} \Delta_{\mathrm{HT}} H\,,$$

so that Δ_{our} acts "from the left".

One of the possible sources of confusion might be the derivation of the master equation: there are multiple choices one can make for the left/right derivatives and at the end arrives at $\Delta H = 0$, which admits additional sign factors like -1 or $(-1)^{|H|}$. Furthermore, H is usually e^S, which is bosonic.

by his student H. Khudaverdian [15], who used semidensities as natural integration objects on Lagrangian submanifolds.

The basic idea presented in [13] is that although for finite dimensional manifolds, the differential forms and multivector (i.e. antisymmetric vector) fields on a manifold are (non-canonically) isomorphic, this is no longer true for infinite dimension. Specifically, one does not have a volume form to integrate against, but in the multivector picture, the volume form corresponds to functions, a more manageable object.

Back in the finite dimension, the isomorphism between forms and multivectors can be defined by taking a volume form α and giving the isomorphism by i_α, which takes k-dimensional multivector fields to $(n-k)$-dimensional forms, where n is the manifold dimension. Under this isomorphism, the exterior derivative becomes

$$\Delta = \sum_i \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial x^i} ,$$

where ξ_i are vector fields $\partial/\partial x^i$. This is (up to signs) the BV operator Δ from previous section.

The condition $\Delta e^S = 0$ then says that the volume form $e^S \alpha$ is closed, which is trivial for ordinary manifolds. However, in field theory, some of the fields are fermionic and the relevant manifold is supermanifold. This is the situation considered in the Schwarz's paper [14], whose results for BV geometry we will now present.

An SP-manifold M is a (n,n)-dimensional manifold with compatible P and S structures. The P structure can be specified by a closed odd nondegenerate form ω – an odd symplectic form. The S structure is given by a volume element ρ, meaning that it transforms with a *Berezinian*. These two structures are compatible if one can find Darboux coordinates for the form where ρ is equal to 1.

The odd symplectic form allow us to define a bracket in coordinates z^i

$$\{F, G\} = \frac{\partial_R F}{\partial z^i} \omega^{ij} \frac{\partial_L G}{\partial z^j} ,$$

where ω^{ij} is the inverse matrix to the coordinates expression of the odd symplectic form ω. Furthermore, we can define operator Δ on functions as

$$\Delta H = \mathrm{div}\, K_H ,$$

where div is the divergence associated with the density ρ and K_H is the hamiltonian vector field $K^i_H = \omega^{ij} \frac{\partial_L H}{\partial z^j}$. In the Darboux coordinates with x^i even and ξ_i odd and with $\rho = 1$, we get[3]

$$\Delta = \sum_i \frac{\partial_R}{\partial x^a} \frac{\partial_L}{\partial \xi_a} .$$

Here, $\Delta^2 = 0$ and for general (i.e. not necessarily compatible) ω and ρ, the condition $\Delta^2 = 0$ implies the compatibility of ρ and ω (see [14, theorem 5]).

This supermanifold M should be viewed as finite dimensional version of the space of fields and antifields with ω giving their pairing. Thus, we don't want to integrate over the whole manifold, but over its *Lagrangian* submanifolds. These are manifolds L with $\omega_L = 0$, e.g. specified by $\xi_i = \cdots = \xi_k = 0$ and $x^{k+1} = \cdots = x^n = 0$ in Darboux coordinates. On such submanifold, we can integrate over a volume form $\mathrm{d}x^1 \dots \mathrm{d}x^k \mathrm{d}\xi_{k+1} \dots \mathrm{d}\xi_n$, in general coordinates the volume form on L would be proportional to $\sqrt{\rho}$.

Schwarz proves two important theorems which explain the results known from the field theory.

[3] This operator is defined without the signs, as in Weinberg [10], and suffers from the same problems as Δ_{W} from [10] – see previous section.

Theorem 1.1 ([14, theorems 1, 2]). *Let L_0 and L_1 be closed oriented Lagrangian submanifolds of an orientable SP-manifold M. If the two manifolds can be connected with smooth family of Lagrangian submanifolds L_t for $0 \leq t \leq 1$, then*

$$\int_{L_0} H \mathrm{d}\lambda_0 = \int_{L_1} H \mathrm{d}\lambda_1 \tag{6}$$

for every function H satisfying $\Delta H = 0$. Moreover, if $H = \Delta K$, then for every closed Lagrangian manifold L

$$\int_L H \mathrm{d}\lambda = 0\,. \tag{7}$$

Proof. See [14], we will only sketch the main ideas in the following. □

1.3 BV algebras

We have seen multiple version of an operator Δ similar to

$$\Delta H = (-1)^{|H||\chi^n|+1} \frac{\delta_L}{\delta \chi^n} \frac{\delta_R H}{\delta \chi_n^\ddagger}\,, \tag{8}$$

and brackets

$$\{F, G\} = \frac{\delta_R F}{\delta \chi^n} \frac{\delta_L G}{\delta \chi_n^\ddagger} - \frac{\delta_R F}{\delta \chi_n^\ddagger} \frac{\delta_L G}{\delta \chi^n}\,. \tag{9}$$

The Δ operator is a second order differential operator, which means it does not satisfy the Leibniz rule. The failure of the Leibniz rule is precisely the bracket, as can be easily checked

$$\Delta(FG) = F\Delta G + (-1)^{|G|}(\Delta F)G + (-1)^{|G|}\{F, G\}\,. \tag{10}$$

Furthermore, Δ squares to 0, the bracket has a graded symmetry

$$\{F, G\} = -(-1)^{(|F|+1)(|G|+1)}\{G, F\}$$

and is a graded derivative in each of its arguments. These are a basis of definition *BV algebra*, but the signs in the equation 10 are usually taken to be different, so that Δ is a left, not right derivative. This corresponds to redefinition

$$\Delta H \to \Delta_{\text{our}} H = (-1)^{|H|} \Delta H\,.$$

This gives

$$\Delta_{\text{our}} H = (-1)^{|H||\chi^n|+1+|H|} \frac{\delta_L}{\delta \chi^n} \frac{\delta_R H}{\delta \chi_n^\ddagger} = (-1)^{|H||\chi^n|} \frac{\delta_R}{\delta \chi^n} \frac{\delta_L H}{\delta \chi_n^\ddagger}\,. \tag{11}$$

With these signs, we can define BV algebras (see e.g. [16, section 4]).

Definition 1.2. A *BV algebra* is a graded commutative associative algebra on graded vector space V with a bracket $\{,\} : V^{\otimes 2} \to V$ of degree 1 that satisfies

$$\{F, G\} = -(-1)^{(|F|+1)(|G|+1)}\{G, F\}\,, \tag{12}$$

$$\{F, \{G, H\}\} = \{\{F, G\}, H\} + (-1)^{(|F|+1)(|G|+1)}\{G, \{F, H\}\}\,, \tag{13}$$

$$\{F, GH\} = \{F, G\}H + (-1)^{(|F|+1)|G|}G\{F, H\} \tag{14}$$

and a square zero operator $\Delta : V \to V$ of degree one such that

$$\Delta(FG) = (\Delta F)G + (-1)^{|F|}F\Delta G + (-1)^{|F|}\{F, G\}\,. \tag{15}$$

△

If the vector space V comes with a differential d, we usually also postulate its compatibility with Δ as in $\Delta \mathrm{d} + \mathrm{d}\Delta = 0$. For algebras with unit 1, we will take $\Delta(1) = 0$.

Since the bracket can be defined using Δ, one can define a BV algebra without mentioning it. The Poisson and Jacobi identities of the bracket are then encoded in so-called seven-term identity for Δ

$$\begin{aligned}\Delta(FGH) =& \Delta(FG)H + (-1)^{|G||H|}\Delta(FH)G + (-1)^{|F|(|G|+|H|)}\Delta(GH)F \\ &- \Delta(F)GH - (-1)^{|F|}F\Delta(G)H - (-1)^{|F|+|G|}FG\Delta(H)\,.\end{aligned} \tag{16}$$

In the following, we will also use a compatibility between Δ and $\{,\}$ which can be derived from $\Delta^2(FG) = 0$

$$\Delta\{F, G\} = \{\Delta F, G\} + (-1)^{|F|+1}\{F, \Delta G\}\,. \tag{17}$$

1.4 Flat BV geometry on $\mathrm{Sym}\, V^{\#}$

For us, a most important example of a BV algebra will be a BV algebra associated with a graded vector space with odd symplectic form,

Definition 1.3. An *odd symplectic structure* on a vector space V is a nondegenerate graded skewsymmetric bilinear map $V \otimes V \to \Bbbk$ of degree -1. This degree makes it honestly skewsymmetric, i.e. $\omega(v_1, v_2) = -\omega(v_2, v_1)$. △

The vector space V should be thought of as combined space of fields and antifields, the symplectic form being their pairing.

This is in fact a flat variant of BV algebra on a SP supermanifold of A. Schwarz: the P structure is given by the odd symplectic form and the S structure is just the constant Lebesgue measure on the vector space.

The definition of the BV Δ operator will be different from the one from [14] by signs, see also discussions in previous sections. Note that we now work with graded vector spaces, not super vector spaces.

The following construction appears for example in P. Mnev's [5, section 4.1] and in [17]. We start by recalling some notions useful for discussion of symplectic vector spaces from the book by McDuff and Salamon [18, chapter 2].

Definition 1.4. For ω a symplectic form on V, we define a *symplectic complement* of a subspace $W \subset V$ to be the set of vectors

$$W^{\omega} \equiv \{v \in V \mid \omega(v, w) = 0,\ \forall w \in W\}.$$

This complements satisfies $\dim W + \dim W^{\omega} = \dim V$ and $W^{\omega\omega} = W$, see [18, lemma 2.2].

We can classify subspaces by the behaviour of their complement: we call $W \subset V$
isotropic if $W \subset W^{\omega}$,
coisotropic if $W^{\omega} \subset W$,
symplectic if $W \cap W^{\omega} = \{\mathbf{0}\}$,
Lagrangian or *maximally isotropic* if $W = W^{\omega}$.

△

Let us take a graded vector space V and a symplectic form ω of degree -1. At first, we want to define ω on the dual vector space. We can view ω as a map $\natural : V \to V^{\#}$. The $\omega^{\#}$ on the dual vector space can be then defined as

$$\omega^{\#}(\alpha, \beta) = \omega(\natural^{-1}(\alpha), \natural^{-1}(\beta))\,. \tag{18}$$

We will describe everything in coordinates: choose a basis $\mathbf{e}_i$ of V and a dual basis ϕ^i of $V^{\#}$. The odd symplectic form in this coordinates is an antisymmetric matrix

$$\omega_{ij} \equiv \omega(\mathbf{e}_i, \mathbf{e}_j)$$

with block structure: ω_{ij} is nonzero only if $|\mathbf{e}_i| + |\mathbf{e}_j| = -|\omega| = 1$. The matrix of $\omega^\#$ is exactly the inverse matrix of ω_{ij}, which we will denote ω^{ij}. It is again antisymmetric and has a similar block structure: ω^{ij} is nonzero only for $|\mathbf{e}_i| + |\mathbf{e}_j| = 1$, which can be seen from

$$\sum_j \omega_{ij}\omega^{jk} = \delta_i^k \,,$$

or from equation 18.

In the dual basis, this means that ω^{ij} is nonzero only for $|\phi^i| + |\phi^j| = -1$. We define a BV operator $\Delta : \mathcal{F}(V) \to \mathcal{F}(V)$ as

$$\Delta(F) = \frac{1}{2}\sum_{i,j}(-1)^{|\phi^i|}\omega^{ij}\frac{\partial_L}{\partial\phi^i}\frac{\partial_L}{\partial\phi^j}F \,. \tag{19}$$

Since $|\phi^i| + |\phi^j| = -1$ for nonzero ω^{ij}, this operator removes covectors of total degree -1 and therefore $|\Delta| = 1$. The factor $(-1)^{|\phi^i|}$ counters the skew-symmetry of ω^{ij}, because without it we would contract it with symmetric $\frac{\partial_L}{\partial\phi^i}\frac{\partial_L}{\partial\phi^j}$. The factor $\frac{1}{2}$ is there exactly because of this symmetry. twice.

Corresponding BV bracket is defined by

$$\Delta(FG) = (\Delta F)G + (-1)^{|F|}F\Delta G + (-1)^{|F|}\{F, G\}$$

and can be written as

$$\{F, G\} = \sum_{i,j}\omega^{ij}\frac{\partial_R F}{\partial\phi^i}\frac{\partial_L G}{\partial\phi^j} \,.$$

These two operations together form a BV algebra in the sense of definition 1.2. This can be checked by direct computation, or by comparing with previous examples.

1.4.1 Master equation and $\hbar$

To write down master equation with explicit $\hbar$, we add $\hbar$ as a formal parameter. This means enlarging our functions $\mathcal{F}(V)$ to $\mathcal{F}(V)[[\hbar]] \equiv \mathcal{F}(V) \otimes \Bbbk[[\hbar]]$, i.e. tensoring with formal power series in $\hbar$. We will, however, abuse the notation slightly and write $\hbar$ as if it were another commutative variable.

Then, we can write the master equation

$$2\hbar\Delta S + \{S, S\} = 0 \,.$$

Note that S_{class} the classical part of S, i.e. sum of terms with $\hbar^0$, satisfies the *classical master equation*

$$\{S_{\text{class}}, S_{\text{class}}\} = 0 \,.$$

Moreover, taking only the quadratic part of this equation and assuming that S has no linear terms, we get that S_0, the quadratic part of S_{class} (which corresponds to the kinetic term) also satisfies classical master equation

$$\{S_0, S_0\} = 0 \,.$$

Solutions of master equation give us yet another differential, given by twisting Δ by $e^{S/\hbar}$, i.e. taking a map $F \mapsto e^{-S/\hbar}\Delta(Fe^{S/\hbar})$. This always is a square zero map, but it has no constant component only for S which is a solution of master equation. Direct calculation gives

$$e^{-S/\hbar}\Delta(Fe^{S/\hbar}) = \Delta F + \frac{1}{\hbar}\{S, F\} \,.$$

We will move the constant $\hbar$ to the Δ term and define

$$T_S F \equiv \hbar\Delta F + \{S, F\} \,.$$

We can also compute T_S^2 directly, which gives

$$T_S^2(F) = \hbar\Delta(\hbar\Delta F + \{S, F\}) + \{S, \hbar\Delta F + \{S, F\}\} = \{\hbar\Delta S + \frac{1}{2}\{S, S\}, F\}\,.$$

We see that T_S^2 is zero iff $\hbar\Delta S + \frac{1}{2}\{S, S\}$ is constant. However, because Δ and $\{\}$ have degree 1, $\hbar\Delta S + \frac{1}{2}\{S, S\}$ would have to be degree 1 constant, so it has to be 0. Thus, the condition T_S^2 is equivalent to the quantum master equation for S.

Expressing the BV algebra operations with antifields, we can write the master equation as

$$\frac{\partial_R S}{\partial \chi^i}\frac{\partial_L S}{\partial \chi_i^\ddagger} + \hbar\frac{\partial_R}{\partial \chi^i}\frac{\partial_L}{\partial \chi_i^\ddagger} S = 0\,, \tag{20}$$

which is useful when comparing with [2].

1.4.2 Differential on V

In our application, the graded vector space V will usually be equipped with a differential, which we will now denote Q. This differential is also compatible with ω, in a sense that

$$\omega(Q \otimes \mathbb{1} + \mathbb{1} \otimes Q) = 0\,,$$

i.e. Q is self-adjoint with respect to ω. We are now going to consider a cohomology with respect to this differential Q, which we denote H_Q. With the projection operator $[\,] : \operatorname{Ker} Q \to H_Q$, we can define a symplectic form on this cohomology as

$$\tilde{\omega}([v], [w]) \equiv \omega(v, w)\,. \tag{21}$$

This does not depend on the representants v and w: adding exact terms gives

$$\tilde{\omega}([v + Qx], [w + Qy]) = \omega(v + Qx, w + Qy) = \omega(v, w) + \omega(v, Qy) + \omega(Qx, w) + \omega(Qx, Qy)\,.$$

Here, the last three terms are zero, since v and w are closed and we can move Q between the arguments. Thus, we get

$$\tilde{\omega}([v + Qx], [w + Qy]) = \omega(v, w) = \tilde{\omega}([v], [w])\,.$$

This $\tilde{\omega}$ is again antisymmetric and has degree -1, but is it non-degenerate? To prove that it is, we will use the fact that homology is in fact a *symplectic reduction*, i.e. a quotient of coisotropic subspace by its symplectic complement. We cite the following from [18].

Lemma 1.5 ([18, lemma 2.7.i])**.** *Let (V, ω) be a symplectic vector space and $W \subset V$ be a coisotropic subspace. Then the quotient $V' = W/W^\omega$ carries a natural symplectic structure $\tilde{\omega}$, defined as $\tilde{\omega}([v], [w]) = \omega(v, w)$, where $[\,]$ is the quotient map.*

Note that this lemma also applies for our case, we can just forget about the grading on our vector space.

Starting with $\operatorname{Ker} Q$, let's look at its symplectic complement. For all $v \in \operatorname{Ker} Q$, the expression $\omega(v, x)$ is zero for all $x \in \operatorname{Im} Q$. Thus $\operatorname{Im} Q$ is a subspace of $(\operatorname{Ker} Q)^\omega$, but since it has the right dimension, i.e. $\dim \operatorname{Ker} Q + \dim \operatorname{Im} Q = \dim V$, it has to be the symplectic complement. Thus, we see that $\operatorname{Ker} Q$ is a coisotropic subspace and

$$\operatorname{Ker} Q/(\operatorname{Ker} Q)^\omega = \operatorname{Ker} Q/\operatorname{Im} Q = H_Q$$

inherits a nondegenerate symplectic form, defined exactly as we did in equation 21.

With $\tilde{\omega}$, we can define a BV algebra on the functions on homology $\mathcal{F}(H_Q)$, which we will denote Δ' and $\{\}'$.

Similarly, if we decompose the vector space V as $V' \oplus V''$ such that $(V')^\omega = V''$, we get two *symplectic* subspaces of V, both with non-degenerate symplectic forms. This means we can define a BV algebra on both of this spaces, with operations Δ', $\{\}'$ and Δ'', $\{\}''$ respectively. Moreover, the operator Δ on the total vector space is a sum of these two operators

$$\Delta = \Delta' + \Delta''$$

and similarly for brackets

$$\{\} = \{\}' + \{\}'' .$$

This is because ω does not pair vectors (or covectors) from V' and V'', and therefore the sums in definitions of Δ and $\{\}$ split into two parts, giving BV Laplacians on V' and V''.

If we choose this decomposition such that V' will be isomorphic to H_Q, with the isomorphism respecting the structures, [4] then we can view the algebra of functions $\mathcal{F}(H_Q)$ as a part of the algebra $\mathcal{F}(V)$, with a decomposition of the BV algebra. This will be a starting point for a construction of *special deformation retract* between $\mathcal{F}(V)$ and $\mathcal{F}(H_Q)$ in the following chapters.

[4] If we choose a injective map $i : H_Q \to V$ such that its a symplectomorphism and a chain map, we immediately have $V = \operatorname{Im} i \oplus (\operatorname{Im} i)^\omega$. This is because the symplectic form ω on $\operatorname{Im} i$ is non-degenerate, and thus no vector from $(\operatorname{Im} i)^\omega$ can also be in $\operatorname{Im} i$. It is a general fact that $\dim V = \dim \operatorname{Im} i + \dim(\operatorname{Im} i)^\omega$, so the disjointedness of these two subspaces gives the direct sum decomposition.

2 Loop homotopy algebras

Loop homotopy algebras are a generalization of cyclic homotopy algebras, admitting maps with *genus* higher than 0. They were first defined by M. Markl in [3] by axiomatizing the algebraic properties of string products of closed string field theory [2]. We will only describe the generalization of cyclic L_∞ algebra, the loop homotopy Lie algebra.

We will be following the two original sources we mentioned, [2] and [3].

2.1 Closed string field theory

In [2], B. Zwiebach constructed *string products* and *string functions*. These operations take multiple elements of a Hilbert space $\mathcal{H}_{\mathrm{rel}}$ (a subspace of the Hilbert space $\mathcal{H}$ of the whole theory) and output another state in $\mathcal{H}_{\mathrm{rel}}$ and an complex number, respectively.

The string function, or a *string field vertex* is defined first, as an integral over a space of surfaces associated to an interaction vertex. Such vertex is specified by the n, the number of incoming strings and the genus g. For states $B_1, \dots, B_n$, and a surface of genus g, Zwiebach constructs a form (for details, see [2, sections 7.3, 7.4]) and integrates it over the mentioned space of surfaces, which gives a number denoted by

$$\{B_1, \dots, B_n\}_g \,. \tag{22}$$

This product is graded commutative and nonzero only on fields with total degree [5] $-2n$.

The Hilbert space is also equipped with a bilinear inner product, denoted

$$\langle B_1, B_2 \rangle \,.$$

On the subspace $\mathcal{H}_{\mathrm{rel}}$ of fields B, this product is nondegenerate, symmetric and has degree -5. The nondegeneracy means it can be used to “lift one of the indices” of the multilinear string function. This is done by choosing a two bases $\{\Phi_s\}$ and $\{\Phi^r\}$ of $\mathcal{H}_{\mathrm{rel}}$ that satisfy

$$\langle \Phi_s, \Phi^r \rangle = (-1)^{|\Phi_r|} \delta_s^r \,.$$

Then we can define the string products $[\dots]_g$ as

$$[B_1, \dots, B_n]_g = \sum_t (-1)^{|\Phi_t|} \Phi^t \{\Phi_t, B_1, \dots, B_n\}_g \,.$$

The inverse of this relation is given by

$$\{B_0, B_1, \dots, B_n\}_g = \langle B_0, [B_1, \dots, B_n]_g \rangle \,. \tag{23}$$

These brackets then have a degree $3 - 2n$ and are graded symmetric. Moreover, as a consequence of the the symmetry of the string functions, the brackets satisfy additional property: M. Markl expresses this by saying that element [3, eq. 7]

$$\sum_s \Phi_s \otimes [\Phi^s, B_1, \dots, B_{n-1}]_g \in \mathcal{H}_{rel}^{\otimes 2}$$

is antisymmetric with respect to the switching morphism.

This is, however, not all the structure these operations have. Because the string vertices have to generate the complete moduli space of Riemann surfaces, Zwiebach uses geometrical recursion

[5] Regarding the degrees, the Hilbert space has double grading, $\mathbb{Z}$-grading by *ghost number* and $\mathbb{Z}_2$-grading by *Grassmanality*. Because the strings are bosonic, the Grassmanality is equal to ghost number modulo 2 (see [ZwiebachBV, see the paragraph *Grassmanality* in section 3.1])

relations for the decomposition of surfaces to prove the *main identity* for the string products [2, eq. 4.13]

$$0 = \sum_{\substack{g_1+g_2=g\\ k+l=n}} \sum_{\sigma\in \mathrm{Unsh}(l,k)} \varepsilon(\sigma)[B_{\sigma(1)},\ldots,B_{\sigma(l)},[B_{\sigma(l+1)},\ldots,B_{\sigma(l+k)}]_{g_2}]_{g_1} + \frac{1}{2}\sum_s (-1)^{|\Psi_s|}[\Psi_s,\Psi^s,B_1,\ldots,B_n]_{g-1}\,, \tag{24}$$

which should be satisfied for all $n \geq 0$ and $g \geq 0$, for which we set $[\ldots]_{-1}$ equal to 0.

The string functions can be used to define an action S. We take a string field Ψ and define

$$S(\Psi) = \frac{1}{\kappa^2}\sum_{g\geq 0}(\hbar\kappa^2)^g \sum_{n\geq 0}\frac{\kappa^n}{n!}\{\Psi,\ldots,\Psi\}_g\,. \tag{25}$$

where the Ψ is inserted n times into the string function $\{\ldots\}_g$ with n arguments. We see that the parameter $\hbar$ just tracks the genus of the string function and κ is equal to $2(g-1)+n$. This κ is the *closed string field coupling constant*, which we will ignore. As a consequence of the main identity, this action then satisfies a *quantum master equation* [2, equation 4.69]

$$\frac{\partial_R S}{\partial\psi^s}\frac{\partial_L S}{\partial\psi^*_s} + \hbar\frac{\partial_R}{\partial\psi^s}\frac{\partial_L S}{\partial\psi^*_s} = 0\,,$$

where the *fields* ψ^s and *antifields* ψ^*_s correspond to regraded elements of dual Hilbert space (see [2, equation 3.33] and also [4, definition 3.2]).

2.2 Loop homotopy Lie algebras

The genus zero products of this theory have a structure L_∞, as was already know before Zwiebach (see [2, section 4.5]). Furthermore, with the bilinear form, we get a cyclic L_∞ algebra.

In [3], Markl coined a name *loop homotopy Lie algebra* for the full structure with $g \neq 0$ operations. To relate it to the usual definition of L_∞ algebra, which in turn is usually defined to agree with the dg Lie algebra, one has to shift the degree: instead of looking at $\mathcal{H}_{\mathrm{rel}}$, we consider $U \equiv \mathbf{r}(\downarrow \mathcal{H}_{\mathrm{rel}})$. On this vector space, operations $l^g_n : U^{\otimes n} \to U$ are defined with a Koszul sign coming from commuting graded elements and the operator $\downarrow$, which has degree 1:

$$l^g_n \equiv (-1)^{(n-1)|v_1|+(n-2)|v_2|+\cdots+v_{n-1}}\mathrm{r}\downarrow[\uparrow\mathbf{r}(v_1),\ldots,\uparrow\mathbf{r}(v_n)]_g \tag{26}$$

for $v_i \in U$. This shift and sign then makes these l^g_n antisymmetric, with degree $n-2$.

On U, we define a graded symmetric (which is, due to its degree, symmetric) form B, which comes from the inner product $\langle -,-\rangle$. Here, Markl does not use a "Koszul" sign, but defines

$$B(u,v) \equiv \langle\uparrow\mathbf{r}(u),\uparrow\mathbf{r}(v)\rangle\,.$$

The following definition is almost verbatim [3, definition 2.1].

Definition 2.1. A loop homotopy Lie algebra consists of

1. A $\mathbb{Z}$-graded vector space U,
2. a graded symmetric nondegenerate bilinear degree 3 form $B : U \otimes U \to \mathbb{k}$
3. a set $\{l^g_n\}_{n,g\geq 0}$ of degree $n-2$ multilinear graded antisymmetric operations $l^g_n : U^{\otimes n} \to U$,

satisfying these two conditions:

1. For any $n, g \geq 0$, for vectors $v_1, \ldots, v_n \in U$, we have the *main identity*

$$0 = \sum_{\substack{k,l,g_1,g_2 \geq 0 \\ k+l=n+1 \\ g_1+g_2=g}} \sum_{\sigma \in \mathrm{unsh}(l,n-l)} \chi(\sigma)(-1)^{l(k-1)} l_k^{g_1}(l_l^{g_2}(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)})$$

$$+\frac{1}{2}\sum_s (-1)^{h_s+n} l_{n+2}^{g-1}(h_s, h^s, v_1, \ldots, v_n)\,.$$

The map l_n^g with $g = -1$ is taken to be 0. Vectors $\{h_s\}$ and $\{h^s\}$ form two bases of the vector space dual in the sense of the form B

$$B(h^s, h_t) = \delta_t^s\,.$$

2. For any $n, g \geq 0$ and $v_1, \ldots, v_{n-1} \in U$, the element

$$\sum_s (-1)^{n+1} h_s \otimes l_n^g(h^s, v_1, \ldots, v_{n-1}) \in U \otimes U \tag{27}$$

is symmetric (with respect to the action of the switching map τ on $U \otimes U$).

$\triangle$

The $g = 0$ operations determine a so-called L_∞ algebra (see e.g. [4]) Moreover, the bilinear form makes this into a cyclic L_∞ algebra.

A special case of the main identity, for $n = 1$ and $g = 0$, is a condition $(l_{n=1}^{g=0})^2 = 0$, which means that the vector space is naturally endowed with a differential. Moreover, the second condition tells us that this differential is compatible with the bilinear form; see the next section.

2.3 Loop homotopy Lie algebras and master equation

We have seen that on $V \equiv \downdownarrows \mathcal{H}_{\mathrm{rel}}$, the string products have degree 1. Moreover, the string functions $\{\ \}_g$ have degree 0 and the symmetric bilinear form $\langle , \rangle$ has product -1. If we, however, define

$$\omega(v, w) \equiv (-1)^{|v|} \langle \upuparrows v, \upuparrows w \rangle\,, \tag{28}$$

we have

$$\omega(w, v) = (-1)^{|w|} \langle \upuparrows w, \upuparrows v \rangle = (-1)^{|w|} \langle \upuparrows v, \upuparrows w \rangle = (-1)^{|w|+|v|} \omega(v, w) = -\omega(v, w)\,,$$

i.e. an odd symplectic form on V.

We also denote

$$s_n^g(v_1, \ldots, v_n) \equiv \{\upuparrows v_1, \ldots, \upuparrows v_n\}_g$$

and

$$\lambda_n^g(v_1, \ldots, v_n) \equiv \downdownarrows [\upuparrows v_1, \ldots, \upuparrows v_n]_g\,.$$

the shifted versions of string functions and string products. They are related by

$$s_{n+1}^g(v_0, \ldots v_n) = (-1)^{|v_0|} \omega(v_0, \lambda_n^g(v_1, \ldots, v_n))\,,$$

with the sign $(-1)^{|v_0|}$ making the functions s graded symmetric in all of its arguments.

The shifted string functions s_n^g can now be considered as degree 0 elements of $\mathcal{F}(V)$. Adding the $\hbar$ as a formal parameter (see section 1.4.1), we can also consider the action

$$S(v) \equiv \sum_{n,g \geq 0} \frac{\hbar^g}{n!} \{\upuparrows v, \ldots, \upuparrows v\}_g = \sum_{n,g \geq 0} \frac{\hbar^g}{n!} s_n^g(v, \ldots, v)$$

as a degree 0 element of $\mathcal{F}(V)[[\hbar]]$.

The master equation for this action

$$2\hbar\Delta S + \{S, S\} = 0$$

is in fact equivalent to the main identity 24. Zwiebach proves this, see [2, equation 4.69]. Note that the factorial factors from the action S correspond to the factorials one gets in the main identity 24 from the sum over unshuffles, because all arguments are the same.

Because s^g_n are graded symmetric and have degree 0, we have

$$s^g_2(v, w) = (-1)^{|w|} s^g_2(w, v)\,,$$

which implies

$$(-1)^{|v|}\omega(v, \lambda^g_1(w)) = \omega(w, \lambda^g_1(v))\,.$$

If we denote Q the differential λ^0_1, we can write this as

$$\omega(Qv, w) + (-1)^{|v|}\omega(v, Qw) = \omega \circ (Q \otimes \mathbb{1} + \mathbb{1} \otimes Q)(v, w) = 0\,.$$

Another useful identity is the relation between the action and the multilinear operations λ^g_n via the bracket. We can view $\{S, -\}$ as a left derivative of degree 1, which means it is completely specified by its action on covectors ϕ^k

$$\{S, \phi^k\} = \sum_{g,n} \frac{\hbar^g}{n!}\{s^g_n, \phi_k\}\,.$$

Looking at just the component s^g_n, we have (see A for the convention regarding duals)

$$\begin{aligned}
\{s^g_n, \phi^k\} = \frac{\partial_R s^g_n}{\partial \phi^i}\omega^{ik} &= n s^g_n(-, \ldots, -, \mathbf{e}_i)\omega^{ik} \\
&= (-1)^{|\mathbf{e}_i|} n \omega^{ik} s^{g_n}(\mathbf{e}_i, -, \ldots, -) \\
&= n\omega^{ik}\omega(e_i, \lambda^g_{n-1}(-, \ldots, -)) \\
&= n\omega^{ik}\omega_{il}\phi^l \circ \lambda^g_{n-1} \\
&= -n\phi^k \circ \lambda^g_{n-1} \\
&= (-1)^{|\phi^k|} n\,(\lambda^g_{n-1})^{\#}(\phi^k)\,.
\end{aligned}$$

This allows us to convert between the action and the loop homotopy algebra operations. A special case of this formula is for $n = 2$ and $g = 0$. This is the quadratic $\hbar^0$ component of the action, for which we have

$$\{S_0, -\} = \{\frac{1}{2}s^0_2, -\} = Q^{\#} \circ (-1)^{|-|} \equiv \hat{Q}^{\#} \tag{29}$$

If we extend the dual of Q on $\mathcal{F}(V)[[\hbar]]$ using the Leibniz rule, this equality holds also on this space.

We introduced $\hat{Q}^{\#}$

$$\hat{Q}^{\#}(\phi^k) = (-1)^{|\phi^k|} Q^{\#}(\phi^k)\,.$$

This is just for our convenience, so that we can write $\{S_0, -\} = \hat{Q}^{\#}$ on $V^{\#}$. After extending this as a derivative, we will use notation $\{S_0, -\} = \hat{Q}$.

3 Effective action and minimal model

In this chapter, we will construct a loop homotopy algebra on the homology of the vector space V. We will present two approaches to do so, heuristic path integral and a use of homological perturbation lemma. We will also (heuristically) prove their equivalence.

Underlying ideas for some of these constructions can be found in P. Mnev's [5] and H. Kajiura's [4].

3.1 Homological perturbation lemma for $\mathcal{F}(V)$

Homological perturbation lemma is a useful tool of homological algebra, giving explicit formulas for differential transferred from one chain complex to another using a *homotopy equivalence* between them. This is exactly the situation we have: we will transfer the differential given by the solution of master equation to the homology with respect to Q.

Very nice exposition to the homological perturbation lemma is a paper by Crainic [19]. We will now recall its statement to fix notation.

Definition 3.1. A *homotopy equivalence* is given by the following data

$$h \circlearrowright (V, \mathrm{d}_V) \underset{i}{\overset{p}{\rightleftarrows}} (W, \mathrm{d}_W) \quad , \tag{30}$$

where (V, d_V) and (W, d_W) are chain complexes, p, i are quasi-isomorphisms and

$$i \circ p = 1 + \mathrm{d}_V \circ h + h \circ \mathrm{d}_V \,. \tag{31}$$

Homotopy equivalence for which $p \circ i = \mathbb{1}$ on W is called *deformation retract* and a deformation retract for which the following *annihilation conditions*

$$h \circ i = 0,\ p \circ h = 0,\ h \circ h = 0 \,,$$

hold is called *special deformation retract*, or SDR.

$\triangle$

The equation 31 tells us that on homology, the maps $i \circ p$ is homotopic to identity, with homotopy given by h. The annihilation conditions give a decomposition of V into three subspaces, given by projectors $i \circ p$, $-\mathrm{d}_V \circ h$ and $-h \circ \mathrm{d}_V$.

We can now state the perturbation lemma.

Lemma 3.2. *[19, lemmas 2.4, 3.2] Given homotopy equivalence as in equation 30 and a map $\delta : V \to V$ such that $|\mathrm{d}_V| = |\delta|$ and $(\mathrm{d}_V + \delta)^2 = 0$, and such that $(1 - \delta h)$ is invertible, there is a new homotopy equivalence (with the same vector spaces)*

$$h' \circlearrowright (V, \mathrm{d}_V + \delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (W, \mathrm{d}'_W) \quad , \tag{32}$$

where the maps are given by

$$h' \equiv h + h(1 - \delta h)^{-1} \delta h \,, \tag{33}$$

$$p' \equiv p + p(1 - \delta h)^{-1} \delta h \,, \tag{34}$$

$$i' \equiv i + h(1 - \delta h)^{-1} \delta i \,, \tag{35}$$

$$\mathrm{d}'_W \equiv \mathrm{d}_W + p(1 - \delta h)^{-1} \delta i \,. \tag{36}$$

Moreover, if the initial data were a special deformation retract, the perturbed homotopy equivalence is also a special deformation retract.

Proof. Proof is done by explicitly checking the homotopy equivalence conditions for the perturbed maps, see [19]. □

3.1.1 Decomposition of odd symplectic space

In previous chapter, we have seen that associated with a loop homotopy algebra is a vector space V with odd symplectic structure ω and a solution of quantum master equation S. Specially, this gives us a differential $\mathcal{Q}$ that is compatible with ω

$$\omega \circ (\mathcal{Q} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{Q}) = 0\,.$$

We want to construct a special deformation retract between this space V and H_Q, the homology with respect to $\mathcal{Q}$. This amounts to choosing a decomposition V into subspaces as

$$V = \tilde{H}_Q \oplus \operatorname{Im} \mathcal{Q} \oplus C\,,$$

where $\tilde{H}_Q \oplus \operatorname{Im} \mathcal{Q} = \operatorname{Ker} \mathcal{Q}$. For simplicity, we will take $\tilde{H}_Q$ to be the cohomology with respect to $\mathcal{Q}$ and denote it H_Q. We can do this because $\tilde{H}_Q$ and H_Q are isomorphic by an isomorphism that is also a chain map and a symplectomorphism.

The differential $\mathcal{Q}$ is then an isomorphism of vector space $\operatorname{Im} \mathcal{Q}$ and C, so the homotopy h is minus of its inverse. The projector onto homology p is now an isomorphism between $\tilde{H}_Q$ and H_Q and i will be its inverse. The differential on H_Q is zero.

To work with this decomposition on the BV algebra on $\mathcal{F}(V)$, we want to put additional conditions to this decomposition. Specifically, we will also ask for the compatibility of h with the odd symplectic form, namely

$$\omega \circ (h \otimes \mathbb{1} - \mathbb{1} \otimes h) = 0\,.$$

Existence of such decomposition follows from [20, theorem 2.7], where it is called *harmonious Hodge decomposition*. From definition 2.1 ibid., we see that such harmonious decomposition for $(V, \mathcal{Q}, \omega)$ is given by a operator $s : V \to V$ such that

$$\begin{aligned} s^2 &= 0\,, \\ \omega(sv, w) &= (-1)^{|v|}\omega(v, sw)\,, \\ s\mathcal{Q}s &= s\,, \\ \mathcal{Q}s\mathcal{Q} &= \mathcal{Q}\,. \end{aligned}$$

Then we define (following proposition 2.5 ibid.)

$$\begin{aligned} \tilde{H}_Q &\equiv \operatorname{Im} t \equiv \operatorname{Im}(\mathbb{1} - s\mathcal{Q} - \mathcal{Q}s)\,, \\ C &\equiv \operatorname{Im} s\,, \end{aligned}$$

which gives

$$V = \tilde{H}_Q \oplus \operatorname{Im} \mathcal{Q} \oplus C$$

with $\tilde{H}_Q \simeq H_Q$ and $\tilde{H}_Q^\omega = \operatorname{Im} \mathcal{Q} \oplus C$ (this is [20, proposition 2.5]).

Finally, we discuss how the symplectic form pairs the subspaces:

- The subspace $\tilde{H}_Q$ is a symplectic subspace, meaning that the symplectic form on it is non-degenerate. Equivalently, its symplectic complement is $\operatorname{Im} \mathcal{Q} \oplus C$. Thus, we have a situation described in section 1.4.2 and the BV algebra decomposes into operations on $\mathcal{F}(\tilde{H}_Q)$ and on $\mathcal{F}(\operatorname{Im} \mathcal{Q} \oplus C)$. We denote these by Δ', $\{\}'$ and Δ'', $\{\}''$, respectively.

- The subspace $\operatorname{Im} \mathcal{Q}$ has a symplectic complement $(\operatorname{Im} \mathcal{Q})^\omega = \operatorname{Ker} \mathcal{Q} = \operatorname{Im} \mathcal{Q} \oplus \tilde{H}_Q$, thanks to the compatibility with $\mathcal{Q}$. This means that for $v \in \operatorname{Im} \mathcal{Q}$, the expression $\omega(v, w)$ is nozero only for $w \in C$.

- For C, let's choose a nonzero vector $hv \in C = \operatorname{Im} h$. Then

$$\omega(hv, w) = \pm\omega(v, hw)$$

  and we see that $\operatorname{Ker} h \subset C^\omega$. However, since it has the right dimension, we know they are equal $C^\omega = \operatorname{Ker} h = \tilde{H}_Q \oplus C$. Together with the previous point, we see that on $\operatorname{Im} Q \oplus C$, the nondegenerate form ω pairs only vectors between C and $\operatorname{Im} Q$.

To get a deformation retract, set $h = -s$, denote $p : V \to H_Q$ the projection onto a direct summand and $i : H_Q \to V$ its inverse, the inclusion. This gives $i \circ p = t = \mathbb{1} - sQ - Qs = \mathbb{1} + hQ + Qh$. The annihilation conditions are met because the vector space is decomposed as a direct sum. Thus we have a special deformation retract

$$h \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (H_Q, 0) \quad . \tag{37}$$

3.1.2 Decomposition of $\mathcal{F}(V)$

The next step is a construction of deformation retract between $\mathcal{F}(V)$ and $\mathcal{F}(H_Q)$. First, we have to dualize the SDR 37

$$h^\# \circlearrowright (V^\#, Q^\#) \underset{p^\#}{\overset{i^\#}{\rightleftarrows}} (H_Q^\#, 0) \quad . \tag{38}$$

Since the dualization switches the projection and the inclusion, all the conditions on special deformation retract remain satisfied. For example, the equation $i \circ p = \mathbb{1} + Q \circ h + h \circ Q$ becomes

$$p^\# \circ i^\# = \mathbb{1} + Q^\# \circ h^\# + h^\# \circ Q^\# \,, \tag{39}$$

which is what we want, because $p^\#$ is the new inclusion and vice versa.

Next, we replace $Q^\#$ by $\hat{Q}^\#$ (recall equation 29), so that we can write the differential as $\{S_0, -\}$. This can be done, because the new operator differs only by a sign on homogeneous elements. It means its kernel and image are the same and it the maps $p^\#$ and $i^\#$ are still chain maps. To satisfy the equation 39, we also redefine $h^\#$. Since

$$\hat{Q}^\# \equiv Q^\# \circ (-1)^{|-|} \,,$$

we define

$$\hat{h}^\# \equiv (-1)^{|-|} \circ h^\# \,.$$

Finally, we extend to symmetric power algebras. We use capital letters to denote maps in this case:

$$\hat{H} \circlearrowright (\mathcal{F}V, \hat{Q}) \underset{I}{\overset{P}{\rightleftarrows}} (\mathcal{F}(H_Q), 0) \quad . \tag{40}$$

All these operators are linear, so we will only define them on monomials of degree n. For the projection and inclusion, the extension is given by the tensor powers of the dual operators. For $\hat{Q} = \{S_0, -\}$, we use the Leibniz rule. The extension of the homology is more involved and sometimes goes by the name *tensor trick*.

- The differential $\hat{Q}$ extended with graded Leibniz rule. With Koszul sign convention, we can write this as $\hat{Q} = \sum_{j=0}^{n-1} \mathbb{1}^{\otimes j} \otimes \hat{Q}^\# \otimes \mathbb{1}^{\otimes (n-j-1)}$.

- Since i and p both have different domain and codomain, the most natural way to define them is $P = (i^\#)^{\otimes n}$ and $I = (p^\#)^{\otimes n}$.

- We will define $\hat{H}$ drawing an inspiration from [21, section XI.5], where they extend the deformation retract (*contracting homotopy* in their terminology) on the second tensor power by $h_2 = h\otimes\mathbb{1}+ip\otimes h$. Going to third tensor power, one would get $h_3 = h\otimes\mathbb{1}\otimes\mathbb{1}+ip\otimes(h\otimes\mathbb{1}+ip\otimes h)$. The pattern is that there is always one operator h, and the rest is filled with $\mathbb{1}$ and ip. Thus, we define

$$\hat{H}^{(i)} \equiv \mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)}$$

and

$$\hat{H}_{\mathrm{u}} \equiv \sum_{i=0}^{n-1} \hat{H}^{(i)} \,.$$

Since we want the result of $\hat{H}$ to be symmetric, we also have to symmetrize, so we define

$$\hat{H} \equiv \sum_{\sigma\in\mathbb{S}_n} \frac{1}{n!}\sigma^r \circ \hat{H}_{\mathrm{u}} = \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} \,.$$

Where we used the projector $\boldsymbol{\sigma}_n = \sum_{\sigma\in\mathbb{S}_n} \sigma^r$. Similar formula also appears in [22, theorem 1.4].

Now, we compute $\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q}$ on monomial of degree n. At first, note that because $\hat{H} = \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}}$, this is equal to

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = \hat{Q} \circ \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} + \boldsymbol{\sigma}_n \circ \hat{H}_{\mathrm{u}} \circ \hat{Q}$$

and using the fact that the differential commutes with the symmetric action, we get

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = \boldsymbol{\sigma}_n \circ (\hat{Q} \circ \hat{H}_{\mathrm{u}} + \hat{H}_{\mathrm{u}} \circ \hat{Q}) \,.$$

Because $\hat{H}_{\mathrm{u}}$ is a sum of operators $\hat{H}^{(i)}$, we calculate $\hat{Q} \circ \hat{H}^{(i)} + \hat{H}^{(i)} \circ \hat{Q}$. Even further, only taking $\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes(n-j-1)}$ from $\hat{Q}$, we get

$$\begin{aligned} &\left(\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes(n-j-1)}\right) \circ \left(\mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)}\right) \\ &+ \left(\mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)}\right) \circ \left(\mathbb{1}^{\otimes j} \otimes \hat{Q}^{\#} \otimes \mathbb{1}^{\otimes(n-j-1)}\right) \,. \end{aligned}$$

If $i \neq j$, in one of the terms $\hat{h}^{\#}$ commutes with $\hat{Q}^{\#}$ and in the other term it does not, so these terms subtract. If $i = j$, we get

$$\begin{aligned} &\mathbb{1}^{\otimes i} \otimes (\hat{Q}^{\#} \circ \hat{h}^{\#} + \hat{h}^{\#} \circ \hat{Q}^{\#}) \otimes (p^{\#} \circ i^{\#})^{n-i-1} \\ &= \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#} - \mathbb{1}) \otimes (p^{\#} \circ i^{\#})^{n-i-1} \\ &= \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i)} - \mathbb{1}^{\otimes(i+1)} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)} \,. \end{aligned}$$

We can therefore write

$$\hat{Q} \circ \hat{H}^{(i)} + \hat{H}^{(i)} \circ \hat{Q} = \mathbb{1}^{\otimes i} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i)} - \mathbb{1}^{\otimes(i+1)} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)} \,,$$

since there is exactly one term with $i = j$ in $\hat{Q}$. This is very good, because summing these terms from $i = 0$ to $n - 1$ results in a telescopic cancellation, leaving only

$$\hat{Q} \circ \hat{H}_{\mathrm{u}} + \hat{H}_{\mathrm{u}} \circ \hat{Q} = (p^{\#} \circ i^{\#})^{\otimes n} - \mathbb{1}^{\otimes n} \,.$$

This is symmetric, so for

$$\hat{H} = \frac{1}{n!}\boldsymbol{\sigma}_n \circ \left(\sum_{i=0}^{n-1} \mathbb{1}^{\otimes i} \otimes \hat{h}^{\#} \otimes (p^{\#} \circ i^{\#})^{\otimes(n-i-1)}\right)$$

we have

$$\hat{Q} \circ \hat{H} + \hat{H} \circ \hat{Q} = (p^{\#} \circ i^{\#})^{\otimes n} - \mathbb{1}^{\otimes n} = IP - \mathbb{1} \,,$$

where the last identity morphism is on $\mathcal{F}(V)$.

For a special deformation retract, the identities $\hat{H} \circ I = 0$ and $P \circ \hat{H} = 0$ are obviously satisfied. The identity

$$\hat{H}^2 = 0$$

follows from $h^2 = 0$, from the Koszul sign -1 when from commuting operators $\hat{h}^{\#}$, and from the annihilation conditions for $\hat{h}^{\#}$, $p^{\#}$ and $i^{\#}$.

3.1.3 A special basis

For the practical purposes of calculations, we choose bases of the three subspaces of V. We will denote the basis elements of H_Q to be a_i, the basis of $\operatorname{Im} Q$ to be b_i and the basis of C to be c_i. We also diagonalize the isomorphisms h and Q of $\operatorname{Im} d$ and C, so that

$$Q(c_i) = -b_i \quad \text{and} \quad h(b_i) = c_i \,.$$

The corresponding dual elements of $V^{\#}$ are denoted with $\alpha^i \in H_Q^{\#}$, $\beta^i \in (\operatorname{Im} Q)^{\#}$ and $\gamma^i \in C^{\#}$. The dual morphisms $Q^{\#}$ and $h^{\#}$ exchange their domain and codomain, so that

$$\hat{Q}^{\#}(\beta^i) = \gamma^i \quad \text{and} \quad \hat{h}^{\#}(\gamma^i) = \beta^i \,.$$

This is because $Q^{\#}(\beta^i) = (-1)^{|\beta^i|}(-1)^{|\beta^i|+1}\beta^i \circ Q$, which is equal to δ^i_j on c_j and similarly for $\hat{h}^{\#}$.

We formally can write these maps on $V^{\#}$ as

$$\hat{Q}^{\#} = \sum_i \gamma^i \frac{\partial}{\partial \beta^i}$$

and

$$\hat{h}^{\#} = \sum_i \beta^i \frac{\partial}{\partial \gamma^i} \,.$$

We denote the elements of $F(V)$ by the same symbols as for $V^{\#}$. With the partial derivative obeying the Leibniz rule, we can concisely write

$$Q = \sum_i \gamma^i \frac{\partial_L}{\partial \beta^i} \,,$$

because the Leibniz rule is implied in the left derivative.

For P, we have $p(a_i) = a_i$ and $i(a_i) = a_i$. Thus, the operator P is identity on all monomials composed of covectors α^i, and zero on monomials containing at least one β or γ. The I is formally an identity, but goes from $\mathcal{F}(H_Q)$ to $\mathcal{F}(V)$.

One can notice that the symmetrization of $\hat{H}$ looks like the Leibniz rule, but with $p^{\#} \circ i^{\#}$ instead of $\mathbb{1}$. However, these are projectors, so in the basis we just defined they are either zero on a element, or leave it unchanged. The symmetrization determines the number of unchanged elements, which after nontrivial manipulation gives a combinatorial factor

$$\hat{H} = \frac{1}{\hat{\eta}} \sum_i \beta^i \frac{\partial_L}{\partial \gamma^i} \,,$$

where $\hat{\eta}$ is the number of βs and γs in the monomial $\hat{H}$ is acting on. For example,

$$\begin{aligned}
\hat{H}(\alpha\beta^i\gamma^j) &= \frac{1}{2}(-1)^{|\alpha|+|\beta^i|}\alpha\beta^i\beta^j \\
\hat{Q} \circ \hat{H}(\alpha\beta^i\gamma^j) &= \frac{1}{2}(-1)^{|\beta^i|}\alpha\gamma^i\beta^j + \frac{1}{2}\alpha\beta^i\gamma^j \\
\hat{Q}(\alpha\beta^i\gamma^j) &= (-1)^{|\alpha|}\alpha\gamma^i\gamma^j \\
\hat{H} \circ \hat{Q}(\alpha\beta^i\gamma^j) &= \frac{1}{2}\alpha\beta^i\gamma^j + \frac{1}{2}(-1)^{|\gamma^i|}\alpha\gamma^i\beta^j \,.
\end{aligned}$$

and because the degree of β^i and γ^i are opposite, we get

$$(\hat{H} \circ \hat{Q} + \hat{Q} \circ \hat{H})(\alpha\beta^i\gamma^j) = \alpha\beta^i\gamma^j$$

The origin of the factor $\frac{1}{\hat{\eta}}$ can be easily seen from computing $\hat{H} \circ \hat{Q} + \hat{Q} \circ \hat{H}$

$$\begin{aligned}\hat{H} \circ \hat{Q} + \hat{Q} \circ \hat{H} &= \frac{1}{\hat{\eta}} \sum_i \beta^i \frac{\partial_L}{\partial \gamma^i} \sum_j \gamma^j \frac{\partial_L}{\partial \beta^j} + \frac{1}{\hat{\eta}} \sum_i \beta^i \frac{\partial_L}{\partial \gamma^i} \sum_j \gamma^j \frac{\partial_L}{\partial \beta^j} \\ &= \frac{1}{\hat{\eta}} \sum_i \left(\beta^i \frac{\partial_L}{\partial \beta^i} + \gamma^i \frac{\partial_L}{\partial \gamma^i} \right)\end{aligned}$$

This is a projector on variables β and γ extended like a derivative, which is why we have to divide by their total number.

3.2 Perturbing by master action

Recall from section 1.4.1 that both $\hat{Q} = \{S_0, -\}$ and $T_S = \hbar\Delta + \{S, -\}$ are differentials. We constructed a special deformation retract from the first differential using $\{S_0, -\}$ as the differential on $\mathcal{F}(V)[[\hbar]]$. It is natural to consider the full differential T_S as

$$T_S = \{S_0, -\} + \hbar\Delta + \{S - S_0, -\} \equiv \hat{Q} + \delta\,,$$

i.e. to view the full differential as a deformation of $\{S_0, -\}$.

Is this perturbation small? Yes, because $\hbar\Delta + \{S - S_0, -\}$ either adds a power of $\hbar$, or adds the polynomial degree, because we removed S_0, which is quadratic and has no $\hbar$. In the homological perturbation lemma, we need to compute $(1 - \delta\hat{H})^{-1}\delta$. Expanding

$$(1 - \delta\hat{H})^{-1}\delta = \delta + \delta\hat{H}\delta + \delta\hat{H}\delta\hat{H}\delta + \dots$$

we see that for fixed power of $\hbar$ and polynomial degree, we get contribution from a finite number of terms.

The homological perturbation lemma gives us a special deformation retract

$$\hat{H}' \circlearrowright \big(\mathcal{F}(V)[[\hbar]], \hat{Q} + \delta\big) \underset{I'}{\overset{P'}{\rightleftarrows}} (\mathcal{F}(H_Q)[[\hbar]], D')\,. \tag{41}$$

Let us look at the transferred differential D'. By lemma 3.2, we have

$$D' = P(1 - \delta\hat{H})^{-1}\delta I = P(\delta + \delta\hat{H}\delta + \dots)\,.$$

The first term of this expansion is $P(\hbar\Delta + \{S - S_0, -\})I$. Evaluating this on a function $F \in \mathcal{F}(H_Q)[[\hbar]]$, we get

$$P\hbar\Delta F + P\{S - S_0, F\}\,,$$

where we do not write the inclusion I, thinking of $\alpha \in H_Q^{\#}$ and $\alpha \in V^{\#}$ as same objects.

In first term, only Δ' survives, since there are no coordinates β and γ for it to act.

The second term, where we denote $S' \equiv S - S_0$, also has only the primed operation $\{,\}'$. Furthermore, all the terms of S' with variables β and γ survive this primed bracket $\{,\}'$ and are annihilated by P. Thus, only the terms of S' without β and γ can play a role, which we can write as

$$D'(F) = \hbar\Delta' F + \{P(S'), F\}' + \dots\,.$$

We conjecture that this is in fact a general form of this differential, i.e.

$$D'(F) = \hbar\Delta' F + \{W, F\}' \tag{42}$$

for some $W \in \mathcal{F}(H_Q)[[\hbar]]$. This then implies, because $(D')^2 = 0$, that

$$2\hbar\Delta' W + \{W, W\}' = 0\,, \tag{43}$$

i.e. *W would determine a loop homotopy algebra on H_Q*. We will show that D' indeed has a form $\hbar\Delta' + \{W, -\}$, but first we make it seem plausible using BV formalism.

3.2.1 Effective action W

There is in fact a physical construction leading to this W, given by integrating out the variables β and γ. Such integration is physically motivated by the construction of physical states, which are the homology of the differential $\hat{Q}$. The states in $\mathrm{Im}\, Q$ are then called trivial and states in C, that is not in $\mathrm{Ker}\, Q$, are called unphysical. Integrating out these degrees of freedom results in an *effective action* that is a functional of the physical states.

We therefore define the *effective action* $W \in \mathcal{F}(H_Q)[[\hbar]]$ as (see [17] for similar calculation)

$$e^{W/\hbar} \equiv \int_{L''} e^{S/\hbar}\,. \tag{44}$$

This is the BV formalism integral, which means we have to integrate over Lagrangian submanifold L'' in the space $\mathrm{Im}\, Q \oplus C$. Note that W depends on the choice L'', but we don't denote it.

This effective action satisfies the master equation in BV algebra on $\mathcal{F}(H_Q)[[\hbar]]$, which can be easily proven

$$\Delta' e^{W/\hbar} = \Delta' \int_{L''} e^{S/\hbar} = \int_{L''} \Delta' e^{S/\hbar} = \int_{L''} (\Delta - \Delta'') e^{S/\hbar}\,.$$

Here, we moved Δ' under the integral because Δ' and the integral act on different variables. This is equal to zero, because $\Delta e^{S/\hbar} = 0$ by master equation and $\int_{L''} \Delta''(\dots) = 0$ by the theorem 1.1.

We also define a normalized *effective observable*, given by integrating a functional $F \in \mathcal{F}(V)[[\hbar]]$ with weight $e^{S/\hbar}$

$$P_{L''}(F) \equiv \frac{\int_{L''} F e^{S/\hbar}}{\int_{L''} e^{S/\hbar}} = e^{-W/\hbar} \int_{L''} F e^{S/\hbar}\,,$$

and we aim to find a Lagrangian subspace L'' such that this map is the projector P' from equation 41. At first, we show that it is a chain map between the natural differentials on our spaces,

$$T_S = \hbar e^{-S/\hbar} \Delta(- \cdot e^{S/\hbar}) = \hbar\Delta + \{S, -\}\,,$$

acting on $\mathcal{F}(V)[[\hbar]]$, and

$$T_W = \hbar e^{-W/\hbar} \Delta(- \cdot e^{W/\hbar}) = \hbar\Delta' + \{W, -\}'\,,$$

acting on $\mathcal{F}(H_Q)[[\hbar]]$. To see that $P_{L''}$ is a chain map, we compute

$$\begin{aligned}
T_W \circ P_{L''}(F) &= \hbar e^{-W/\hbar}\Delta'\left(P_{L''}(F)\cdot e^{W/\hbar}\right)\\
&= \hbar e^{-W/\hbar}\Delta'\left(\int_{L''} F e^{S/\hbar}\right)\\
&= \hbar e^{-W/\hbar}\int_{L''}\Delta'\left(F e^{S/\hbar}\right)\\
&= \hbar e^{-W/\hbar}\int_{L''}(\Delta-\Delta'')\left(F e^{S/\hbar}\right)\\
&= \hbar e^{-W/\hbar}\int_{L''}\Delta\left(F e^{S/\hbar}\right)\\
&= \hbar e^{-W/\hbar}\int_{L''} e^{S/\hbar}\cdot e^{-S/\hbar}\cdot\Delta\left(F e^{S/\hbar}\right)\\
&= e^{-W/\hbar}\int_{L''} e^{S/\hbar}\cdot T_S(F)\\
&= P_{L''}\circ T_S(F)\,.
\end{aligned} \tag{45}$$

Now we show how $P_{L''}$ and P' relate, starting from an equation

$$I'\circ P' - \mathbb{1} = H'\circ T_S + T_S\circ H'\,,$$

which holds because 41 is a SDR. We evaluate this on $F\in\mathcal{F}(V)[[\hbar]]$ and integrate with weight $e^{S/\hbar}$, obtaining

$$\int_{L''} I'\circ P'(F)\cdot e^{S/\hbar} - \int_{L''} F e^{S/\hbar} = \int_{L''} H'\circ T_S(F)\cdot e^{S/\hbar} + \int_{L''} T_S\circ H'(F)\cdot e^{S/\hbar}\,.$$

We know that $\hat{H} = 1/\hat{\eta}\beta^i\frac{\partial_L}{\partial\gamma^i}$, meaning that either $\hat{H}(G)$ leaves at least one covector β in $\hat{H}(G)$, or $\hat{H}(G)=0$. Furthermore, we know $H' = \hat{H}\circ(\mathbb{1}+(1-\delta\hat{H})^{-1}\delta\hat{H})$, so also H' leaves either nothing or at least one covector β^i. If we thus choose $L'' = C\subset \operatorname{Ker}\beta^i$ for all i, the term

$$\int_{L''=C} H'\circ T_S(F)\cdot e^{S/\hbar}$$

is zero. The term

$$\int_C T_S\circ H'(F)\cdot e^{S/\hbar}$$

can be written, using similar steps as in equation 45, as

$$\begin{aligned}
\int_C T_S(H'(F))\cdot e^{S/\hbar} &= e^{W/\hbar}P_C[T_S(H'(F))]\\
&= e^{W/\hbar}T_W[P_C(H'(F))] = \hbar\Delta'\int_C H'(F) = 0\,.
\end{aligned}$$

Thus, we have

$$\int_C I'\circ P'(F)\cdot e^{S/\hbar} = \int_C F e^{S/\hbar}\,.$$

We know that $I' = I + \hat{H}\circ(1-\delta\circ\hat{H})^{-1}\circ I$ and integral of $\hat{H}\circ(1-\delta\circ\hat{H})^{-1}\circ I$ is zero, so we have

$$\int_C I\circ P'(F)\cdot e^{S/\hbar} = \int_C F e^{S/\hbar}\,.$$

The function $I \circ P'(F)$ is only a function on H_Q, which means we can take it out of the integral, which integrates the variables β and γ

$$P'(F) \int_C e^{S/\hbar} = \int_C F \cdot e^{S/\hbar} .$$

Here, we don't write I on the LHS, because we take take $\int_C$ as taking values in $\mathcal{F}(H_Q)[[\hbar]]$. Moving $\int_C e^{S/\hbar} = e^{W/\hbar}$ on the RHS, we get

$$P'(F) = P_C(F) . \tag{46}$$

The projection P' intertwines differentials T_S and D', projection P_C intertwines T_S and T_W, but because these two projections are equal, we have

$$T_W \circ P_C = P_C \circ T_S = P' \circ T_S = D' \circ P' .$$

At last, precomposing with I', for which we have $P' \circ I' = \mathbb{1}$, we get

$$T_W = D' ,$$

which holds if we choose the Lagrangian subspace of $\operatorname{Im}_Q \oplus C$. Of course, other Lagrangian subspaces give us different W which again solve a master equation. However, this change of W is only by a gauge transformation (in the space of master actions) – see [5, section 4.3, statement 6].

This computation should be viewed as a heuristic argument for considering the homological perturbation lemma for constructing new loop homotopy algebras. Mainly, we have not shown that the path integral exists even in this finite-dimensional context.

3.2.2 Perturbed differential D'

Coming back to the perturbation lemma, we want to continue studying the formula for the perturbed differential

$$D' = P \circ (1 - \delta \circ \hat{H})^{-1} \circ \delta \circ I = P \circ (\delta + \delta \circ \hat{H} \circ \delta + \dots) \circ I ,$$

where

$$\delta = \hbar\Delta + \{S', -\} .$$

We have already seen that the first term of expansion of D' gives $\hbar\Delta' + \{P(S'), -\}'$. In this section, we show that considering all terms, we get a differential in the correct form to define a loop homotopy algebra on H_Q

Theorem 3.3. *There is a function $W \in \mathcal{F}(H_Q)[[\hbar]]$ such that*

$$D' = \hbar\Delta' + \{W, -\}' .$$

Proof. A typical term in the expansion of D' looks like

$$P \circ \delta \circ \hat{H} \circ \dots \circ \hat{H} \circ \delta \circ I .$$

This term is nonzero only if, when applying it on a function $F \in \mathcal{F}(H_Q)$, we get an expression *without* any variables β or γ, because there is the projection P at the end.

Clearly, the rightmost δ operator, which acts only on a function only with variables α, reduces to

$$\delta \circ I(F) = \hbar\Delta' I(F) + \{S', I(F)\}' ,$$

because the double-primed operations are zero on F. We will abuse the notation and write F for $I(F)$, so we write

$$\delta \circ I(F) = \hbar\Delta' F + \{S', F\}' .$$

Note that this holds even without projecting out on the homology (one could say *off-shell*).

How could the statement that $D'F = \hbar\Delta' F + \{W, F\}'$ go wrong? Roughly, the operators acting on F could be derivatives of order higher than one. Of course, one such order 2 derivative, $\hbar\Delta'$, is there, but there can be no other if we want to write their action as a bracket with W. We now show that there are no such higher order derivatives.

Lemma 3.4. *There are functions $W_i \in \mathcal{F}(H_Q)[[\hbar]]$ such that we can write the perturbed differential D as*

$$D'F = \hbar\Delta' F + W_i\, \omega_\alpha^{ij} \frac{\partial_L F}{\partial \alpha^j}\,, \tag{47}$$

where ω_α^{ij} is ω^{ij} for indices i, j corresponding to α.

Proof. (of the lemma) The proof is based on tracking the number of covectors β:

- The operator $\hat{H} = -\frac{1}{\hat{\eta}}\beta^i \frac{\partial_L}{\partial\gamma^i}$ removes one γ and adds one β.
- The operator Δ'' removes one β *and* one γ.
- The bracket with S' either removes β or γ, but can add more terms. Specifically, only $\{S', -\}''$ can remove β or γ.

We see that $\hat{H}$ always adds one β (or the result is zero) and δ might or might not remove a β, which means that together they can only increase the number of covectors β. In a string of operators

$$P \circ \delta \circ \hat{H} \circ \cdots \circ \delta \circ \hat{H} \circ \delta(F)\,, \tag{48}$$

the first δ is only acting on the primed coordinates. Then the $\hat{H}$ adds one vector β. The next δ must remove this vector β, because if it didn't, the number of vectors β in this term would be at least one. Such term would be projected to zero with P.

So, denoting $\delta = \delta' + \delta''$ the decomposition of δ to primed and double-primed operators, we have that the leftmost δ is always δ' and the other operators δ are δ'' in every nonzero term of equation 48.

Thus, in the string of operators

$$P \circ \delta'' \circ \hat{H} \circ \cdots \circ \delta'' \circ \hat{H} \circ \delta'(F)\,,$$

the first operator δ' differentiates F, but the others never act on covectors α. The operator δ' contains $\hbar\Delta'$, but all terms starting with $\hbar\Delta' F$ followed by $\hat{H}$ are zero, and only the term $P \circ \hbar\Delta' \circ I(F)$ survives.

For terms starting with $\{S', F\}'$, we have $(\delta''\hat{H})^n$ acting on them. The operator $\delta'' \circ \hat{H}$ only affects the double-primed coordinates and therefore, they only act on S' in $\{S', F\}'$, determining the factors W_i:

$$P \circ (\delta'' \circ \hat{H})^n(\{S', F\}') = \left[P \circ (\delta'' \circ \hat{H})^n \left(\frac{\partial_R S}{\partial \alpha^i}\right)\right] \omega_\alpha^{ij} \frac{\partial_L F}{\partial \alpha^j}\,. \tag{49}$$

□

This result also gives some restrictions on S': we see that only terms of S' with no β dependence can appear in W_i. This corresponds to the choice of the Lagrangian subspace as $L'' = C$. Moreover, the number of variables γ in S' is also limited: in a string of operators 48, every $\delta'' \circ H$ removes two γ. The actions S' thus have to have $2n$ covectors γ in term $P \circ (\delta'' \circ \hat{H})^n \circ \delta \circ I$.

For example, looking at $P \circ \delta'' \circ \hat{H} \circ \delta(F)$, we get

$$P \circ \delta'' \circ \hat{H} \circ \delta(F) = P\left(\hbar\Delta'' H(\{S', F\}') + \{S', H(\{S', F\}')\}''\right)\,.$$

Here, the term with Δ'' can be written as

$$P\{\hbar\Delta'' H(S'), F\}' = \{\hbar P \Delta'' H(S'), F\}'\,,$$

because $\Delta'' \circ H$ has an degree 0 and does not interact with the α-derivative of $\{,\}'$. Thus, only the part of S' with two operators γ and no β contributes.

The second term $P\{S', \hat{H}(\{S', F\}')\}''$ has exactly one γ in each of the actions S', so that one is removed by $\hat{H}$ and the other by $\{\}''$.

What remains to be shown is that we can write W_i as

$$W_i = \frac{\partial_R W}{\partial \alpha^i}\,,$$

for all i, so that

$$W_i\, \omega_\alpha^{ij} \frac{\partial_L F}{\partial \alpha^j} = \{W, F\}'\,.$$

We start by considering the relation

$$D'^2(F) = 0\,.$$

Using the equation 47, we get (using a shorthand $W_i\, \omega_\alpha^{ij} \equiv W^j$)

$$\begin{aligned} D'^2(F) &= \hbar\Delta'\hbar\Delta'(F) + \hbar\Delta'\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) \\ &\quad + W^j \frac{\partial_L}{\partial \alpha^j}\left(\hbar\Delta' F\right) + W^i \frac{\partial_L}{\partial \alpha^i}\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right)\,. \end{aligned} \tag{50}$$

The first term of RHS is 0, because $\Delta'^2 = 0$. We can write the second term as

$$\hbar\Delta'\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) = \hbar\Delta'\left(W^j\right) \frac{\partial_L F}{\partial \alpha^j} + (-1)^{|W^j|} W^j \hbar\Delta'\left(\frac{\partial_L F}{\partial \alpha^j}\right) + (-1)^{|W^j|}\{W^j, \frac{\partial_L F}{\partial \alpha^j}\}'\,.$$

Because D' has degree 1, we have $|W^i| - |\alpha^i| = 1$. If we commute Δ' with $\frac{\partial_L}{\partial \alpha^j}$ in the middle term, we get

$$(-1)^{|W^j|} W^j \hbar\Delta'\left(\frac{\partial_L F}{\partial \alpha^j}\right) = -W^j \frac{\partial_L}{\partial \alpha^j} \hbar\Delta' F\,,$$

which cancels with the third term in RHS of equation 50. In the last term of 50, we have

$$W^i \frac{\partial_L}{\partial \alpha^i}\left(W^j \frac{\partial_L F}{\partial \alpha^j}\right) = (-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial \alpha^i}\frac{\partial_L}{\partial \alpha^j} + W^i \frac{\partial_L W^j}{\partial \alpha^i}\frac{\partial_L F}{\partial \alpha^j}\,.$$

Using the standard procedure, we show that the first term is zero

$$\begin{aligned} (-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial \alpha^i}\frac{\partial_L}{\partial \alpha^j} &= (-1)^{|W^i||\alpha^j| + |W^i||W^j| + |\alpha^i||\alpha^j|} W^j W^i \frac{\partial_L}{\partial \alpha^j}\frac{\partial_L}{\partial \alpha^i} \\ &= (-1)^{|W^j||\alpha^i| + |W^j||W^i| + |\alpha^j||\alpha^i|} W^i W^j \frac{\partial_L}{\partial \alpha^i}\frac{\partial_L}{\partial \alpha^j} \\ &= -(-1)^{|W^i||\alpha^j|} W^i W^j \frac{\partial_L}{\partial \alpha^i}\frac{\partial_L}{\partial \alpha^j} \end{aligned}$$

after using $|W^i| - |\alpha^i| = 1$.

Expanding the bracket $\{\}'$, we finally get

$$\begin{aligned} D'^2(F) &= \hbar\Delta'\left(W^j\right) \frac{\partial_L F}{\partial \alpha^j} + (-1)^{|W^j|}\{W^j, \frac{\partial_L F}{\partial \alpha^j}\}' + W^i \frac{\partial_L W^j}{\partial \alpha^i}\frac{\partial_L F}{\partial \alpha^j} \\ &= (-1)^{|W^j|} \frac{\partial_R W^j}{\partial \alpha^k} \omega_\alpha^{kl} \frac{\partial_L}{\partial \alpha^l}\frac{\partial_L F}{\partial \alpha^j} + \hbar\Delta'\left(W^j\right)\frac{\partial_L F}{\partial \alpha^j} + W^i \frac{\partial_L W^j}{\partial \alpha^i}\frac{\partial_L F}{\partial \alpha^j}\,. \end{aligned}$$

Because this identity holds for all F, we have two identities, for the order 1 and order 2 differential operators. We will see that these two identities encode the two identities from the definition of loop homotopy algebra.

The second order differential operator has to be zero

$$(-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}\frac{\partial_L}{\partial\alpha^l}\frac{\partial_L F}{\partial\alpha^j}=0\,,$$

which means that a (graded) symmetric part of $(-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}$ with respect to indices l, j is zero:

$$0=(-1)^{|W^j|}\frac{\partial_R W^j}{\partial\alpha^k}\omega_\alpha^{kl}+(-1)^{|\alpha^l||\alpha^j|}(-1)^{|W^l|}\frac{\partial_R W^l}{\partial\alpha^k}\omega_\alpha^{kj}\,.$$

Reverting the substitution $W_i\,\omega_\alpha^{ij}\equiv W^j$, we can write

$$0=(-1)^{|W^j|}\frac{\partial_R W_i}{\partial\alpha^k}\omega_\alpha^{ij}\omega_\alpha^{kl}+(-1)^{|\alpha^l||\alpha^j|}(-1)^{|W^l|}\frac{\partial_R W_m}{\partial\alpha^k}\omega_\alpha^{ml}\omega_\alpha^{kj}\,.$$

Now we contract this with $\omega_{\alpha,jp}\,\omega_{\alpha,lq}$, the inverse matrices to ω_α^{ij}. Using the degree of ω and $|W^i|-|\alpha^i|=1$, the sign factors give

$$\frac{\partial_R W_p}{\partial\alpha^q}=(-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R W_q}{\partial\alpha^p}\,. \tag{51}$$

This is the necessary condition for W to exist, because if it does, we have

$$\frac{\partial_R W_p}{\partial\alpha^q}=\frac{\partial_R}{\partial\alpha^q}\frac{\partial_R W}{\partial\alpha^p}=(-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R}{\partial\alpha^p}\frac{\partial_R W}{\partial\alpha^q}=(-1)^{|\alpha^p||\alpha^q|}\frac{\partial_R W_q}{\partial\alpha^p}\,.$$

Special case of relation 51 is following for $p=q$

$$\frac{\partial_R W_p}{\partial\alpha^p}=(-1)^{|\alpha^p|}\frac{\partial_R W_p}{\partial\alpha^p}\,, \tag{52}$$

which for fermionic α^p implies that W_p *contains no* α^p. This is important for us, because now we can integrate W^p.

Lemma 3.5. *A function W defined as*

$$W=\sum_i\left(\int_0^1 \mathrm{d}t\,W_i|_{\alpha\to t\alpha}\right)\alpha^i \tag{53}$$

satisfies

$$\frac{\partial_R W}{\partial\alpha^k}=W_k\,.$$

This is just application of the homotopy operator for differential forms to the graded context – see e.g. [23].

Proof. (of the lemma) Directly calculating, we get

$$\begin{aligned}
\frac{\partial_R W}{\partial\alpha^k}&=\int_0^1 \mathrm{d}t\,W_k|_{\alpha\to t\alpha}+\sum_i\left(\int_0^1\mathrm{d}t\,(-1)^{|\alpha^i||\alpha^k|}\frac{\partial_R W_i}{\partial\alpha^k}|_{\alpha\to t\alpha}\cdot t\right)\alpha^i\\
&=\int_0^1 \mathrm{d}t\,W_k|_{\alpha\to t\alpha}+\sum_i\left(\int_0^1\mathrm{d}t\,\frac{\partial_R W_k}{\partial\alpha^i}|_{\alpha\to t\alpha}\alpha^i\cdot t\right)\\
&=\int_0^1 \mathrm{d}t\,W_k|_{\alpha\to t\alpha}+\int_0^1\mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(W_k|_{\alpha\to t\alpha}\right)\cdot t\\
&=\int_0^1 \mathrm{d}t\,W_k|_{\alpha\to t\alpha}+\left[W_k|_{\alpha\to t\alpha}\cdot t\right]_{t=0}^{t=1}-\int_0^1\mathrm{d}t\,W_k|_{\alpha\to t\alpha}\\
&=W_k\,.
\end{aligned}$$

□

Because we proved that $D' = \hbar\Delta' + \{W, -\}'$ and $D'^2 = 0$, we get (see section 1.4.1)

$$2\hbar\Delta' W + \{W, W\}' = 0,$$

which means that W determines a loop homotopy Lie algebra.

From W_i, we constructed the function W directly. An alternative proof would be defining loop homotopy algebra operations using W_k. If we decompose $W_k \frac{\partial_L}{\partial \alpha^k}$ into terms of homogeneous weight and $\hbar$ power, by dualizing them we get the loop homotopy algebra operations $\tilde{\lambda}_n^g : H_Q^{\otimes n} \to H_Q$. If we then construct

$$\tilde{s}_{n+1}^g \equiv \omega \circ (\mathbb{1} \otimes \tilde{\lambda}_n^g)\,,$$

the graded symmetry of these $\tilde{s}$ follows from equation 51. Then, we can use them to construct the action W as in section 2.3, which is a solution of a quantum master equation and thus determines a loop homotopy algebra on H_Q. □

It can be easily seen that this transferred action W determines trivial differential, i.e. it has no quadratic $\hbar^0$ part. The $\hbar^0$ components of the action can come only from the $\{S', -\}''$ part of δ'', and each of these terms adds at least two covectors. So the lowest number of covectors in W can come from the term $P \circ \delta \circ I$ of the expansion, i.e. $P(S')$, which is at least cubic.

This effective action thus (up to quantum corrections) determines the scattering completely, since there is no propagator to form more complicated Feynman diagrams other than just vertices. This suggests a name *effective S-matrix* for W.

Another characterization of W can come from the theory of the homotopy algebras. For (classical) homotopy algebras, one usually finds a *minimal model* of the algebra, which is a quasi-isomorphic homotopy algebra defined on the homology of the original homotopy algebra (see [4] for details). This is exactly what we did. A definition of a loop homotopy algebra morphism hasn't explicitly appeared in literature yet, but we can postulate that the transferred action W together with the perturbed chain maps P' and I' are a minimal model. Note that dualizing the map $P' = P + P(1 - \delta\hat{H})^{-1}\delta\hat{H}$, one gets $i + h(\dots)$, which is a symplectomorphism, as one would expect.

Definition 3.6. A minimal model of an loop homotopy Lie algebra is given by the transferred action W and the perturbed morphisms P', I' of homological perturbation lemma. △

In the general case of algebras over modular operads, the minimal model was found by a similar diagrammatic expansion by Chuang and Lazarev [9].

The last point we make is connecting this calculation to the path integral. We have shown that the effective action W can be computed using the path integral. Writing

$$W = \hbar \ln \int_C e^{S_0/\hbar} e^{S'/\hbar}\,,$$

the perturbative series obtained this way would consist of vertices coming from $S'/\hbar$ paired by edges coming from the inverse of S_0, the propagator. The kinetic term is $S_0 = \frac{1}{2}\omega(\mathbb{1} \otimes Q)$, thus its inverse pairs two variables γ using h (inverse of Q). We see that this corresponds to the operator $(1 - \delta \circ \hat{H})^{-1} \circ \delta$ from the perturbation lemma, with the terms of expansion of the inverse operator corresponding to collecting all the Feynman graphs with definite number of edges.

Appendix A On signs, degrees and factors

In this appendix, we summarize some conventions we use and also collect useful results connected with these conventions.

A.1 Graded vector spaces

Throughout this paper, we use graded vector spaces V_n, see e.g. [24, section 1.5]. Specifically, we use Koszul sign convention, e.g

$$(\phi \otimes \psi)(v \otimes w) \equiv (-1)^{|\psi||v|}\phi(v) \otimes \psi(w)\,.$$

Sometimes, we use $(-1)^{|-|}$ for the operator

$$v \mapsto (-1)^{|v|}v\,.$$

We define a dual of a graded vector space by $(V^{\#})_n = (V_{-n})^{\#}$. The transpose (or dual) of a map $\phi : V \to W$ is defined (for $w \in W^{\#}$) as

$$\phi^{\#}(w) = (-1)^{|w||\phi|+1} w \circ \phi\,,$$

Note that the transpose map has the same degree as the original one.

The grading of will be also referred to as a *ghost number*.

On the (symmetric) tensor powers of V, there is another grading, by the tensor power. We will call this grading *weight*.

A.1.1 Symmetric action

On n-fold tensor products of graded vector spaces we usually use the right $\mathbb{S}_n$ action, defined as

$$\sigma^r(v_1 \otimes \cdots \otimes v_n) = \epsilon(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}\,. \tag{54}$$

The sign $\epsilon(\sigma)$ is defined as the sign coming from commuting graded elements. If all v_i have even degree, it is 1. We write $\chi(\sigma)$ for $\epsilon(\sigma)$ multiplied by the sign of the permutation. We choose the right action mostly because it is shorter to write $v_{\sigma(i)}$ than $v_{\sigma^{-1}(i)}$. We also define a projector

$$\boldsymbol{\sigma}_n \equiv \sum_{\sigma \in \mathbb{S}_n} \frac{1}{n!}\sigma^r\,.$$

Unshuffles $\sigma \in \mathrm{unsh}(l, n-l)$ are permutations in $\mathbb{S}_n$ for which $\sigma(1) < \cdots < \sigma(l)$ and $\sigma(l+1) < \cdots < \sigma(n)$. Unshuffles from $\mathrm{unsh}(i_1, i_2, \ldots, i_k)$ with $\sum_m i_m = n$ are defined analogously.

A.1.2 Suspensions etc

By a *suspension* of a graded vector space V, we mean a graded vector space $(\uparrow V)_n = V_{n-1}$, which takes $v \in V_n$ into $\uparrow v \in (\uparrow V)_{n+1}$, i.e. it increases the degree by 1. Similarly, desuspension of V is $(\downarrow V)_n = V_{n+1}$. A reflection of V is defined as $(\mathbf{r}V)_n = V_{-n}$. We have $\mathbf{r} \circ \uparrow = \downarrow \circ \mathbf{r}$ and $\mathbf{r} \circ \downarrow = \uparrow \circ \mathbf{r}$.

For map $\phi : V^{\otimes n} \to W$, we define maps

$$\begin{aligned}
\phi^{\uparrow} &: (\uparrow V)^{\otimes n} \to W \text{ as } & \phi^{\uparrow} &\equiv \phi \circ \downarrow^{\otimes n}\,,\\
\phi^{\downarrow} &: (\downarrow V)^{\otimes n} \to W \text{ as } & \phi^{\downarrow} &\equiv \phi \circ \uparrow^{\otimes n}\,,\\
\phi_{\uparrow} &: V^{\otimes n} \to (\uparrow W) \text{ as } & \phi_{\uparrow} &\equiv \uparrow \circ \phi\,,\\
\phi_{\downarrow} &: V^{\otimes n} \to (\downarrow W) \text{ as } & \phi_{\downarrow} &\equiv \downarrow \circ \phi\,.
\end{aligned}$$

Note that for the first two cases, we imply a Koszul sign when commuting vectors with $\uparrow$. What happens with degrees of these maps?

- For $\phi^{\uparrow}$, the degree *decreases* by n, because now the map takes vectors of higher degree to produce the same degree in results. Similarly, desuspending the vector space V *increases* the degree of ϕ by n.
- For suspending and desuspending W, we simply have $|\phi_{\uparrow}| = |\phi| + 1$ and $|\phi_{\downarrow}| = |\phi| - 1$.
- A special case that will be useful is when $\phi : V^{\otimes n} \to V$ and we suspend or desuspend the arguments and results at the same time. In the case of suspension, the degree of ϕ decreases by $n-1$, for desuspension it increases by $n-1$.
- Another special case is a map $\omega : V^{\otimes n} \to \mathbb{k}$. Since $\mathbb{k}$ as a graded vector space is concentrated in degree 0, ω is nonzero only on vectors of total degree $-|\omega|$. Again, suspending V decreases the degree of ω by n and vice versa.

For general map $\phi : V^{\otimes n} \to W$, the reflection of only arguments or results would not give a map of definite degree. For $\phi : V^{\otimes n} \to V$, we can define

$$\phi_{\mathbf{r}} \equiv \mathbf{r} \circ \phi \circ \mathbf{r}^{\otimes n} .$$

which has degree equal to $-|\phi|$. Similarly, for $\omega : V^{\otimes n} \to \mathbb{k}$, we can define

$$\omega_{\mathbf{r}} \equiv \omega \circ \mathbf{r}^{\otimes n} ,$$

which again has a degree opposite to that of ω. Of course, there is no Koszul sign for commuting with $\mathbf{r}$, since it has no well-defined degree.

A.2 Symmetric powers

We choose to represent the symmetric tensor power $\mathrm{Sym}^n(V)$ as the subspace of $V^{\otimes n}$ of tensors which are graded symmetric, i.e. for which $T = \sigma^r T$ for all $\sigma \in \mathbb{S}_n$. We define the an associative symmetric product $\odot$ as (see [25, section 4.5])

$$v_1 \odot \cdots \odot v_n = \boldsymbol{\sigma}[v_1 \otimes \cdots \otimes v_n] .$$

With this convention

$$V \odot W = \boldsymbol{\sigma}(V \otimes W) .$$

We also sometimes omit $\odot$ and write

$$v_1 v_2 \equiv v_1 \odot v_2 .$$

A.2.1 Pairing with dual symmetric powers

For $v_1 \odot \cdots \odot v_n \in \mathrm{Sym}^n V$ and $\phi^1 \odot \cdots \odot \phi^n \in \mathrm{Sym}^n V^{\#}$, we define their pairing as

$$\phi^1 \odot \cdots \odot \phi^n (v_1 \odot \cdots \odot v_n) \equiv \sum_{\sigma \in \mathbb{S}_n} \epsilon(\sigma) \phi^1(v_{\sigma(1)}) \dots \phi^n(v_{\sigma(n)}) ,$$

i.e. without signs from commuting ϕ and v.

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