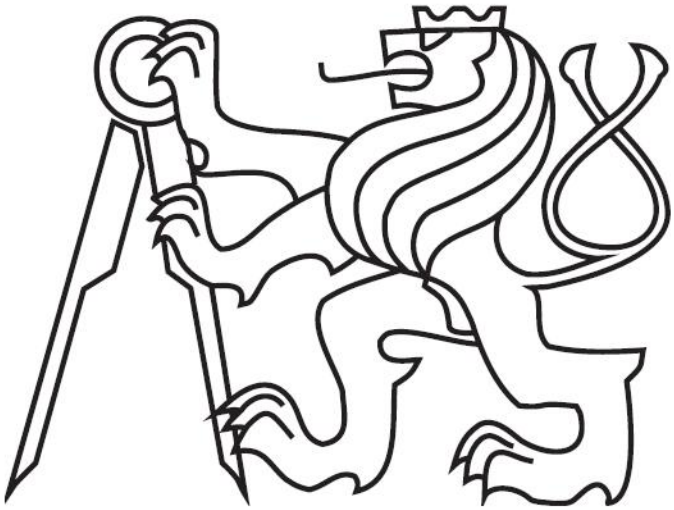
Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Department of Physics
Field of Study: Experimental Nuclear and Particle Physics



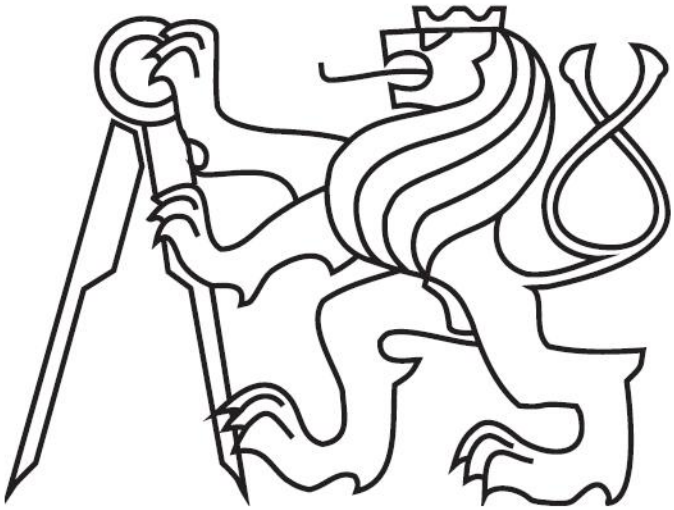
Multiplicity Fluctuations

RESEARCH PROJECT

Author: Josef Uchytil
Supervisor: doc. Mgr. Boris Tomášik, Ph.D.
Academic Year: 2016/2017

České vysoké učení technické v Praze
Fakulta jaderná a fyzikálně inženýrská

Katedra fyziky
Obor: Experimentální jaderná a částicová fyzika



Fluktuace multiplicity

VÝZKUMNÝ ÚKOL

Vypracoval: Josef Uchytil
Vedoucí práce: doc. Mgr. Boris Tomášik, Ph.D.
Akademický rok: 2016/2017

Prohlášení

Prohlašuji, že jsem svůj výzkumný úkol vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty) uvedené v přiloženém seznamu.

Nemám závažný důvod proti použití tohoto školního díla ve smyslu § 60 Zákona č. 121/200 Sb., o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze dne 22. 9. 2017

.....
Josef Uchytil

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Katedra: fyziky **Akademický rok:** 2016/2017

VÝZKUMNÝ ÚKOL

Student: Bc. Josef Uchytil

Obor: Experimentální jaderná a částicová fyzika

Vedoucí úkolu: doc. Mgr. Boris Tomášik, Ph.D., FJFI ČVUT

Název úkolu (česky/anglicky):

Fluktuace multiplicity / Multiplicity fluctuations

Pokyny pro vypracování:

Motivace: Počty částic produkované v ultrarelativistických jaderných srážkách jsou dobře popsané statistickým modelem. V tomto modelu závisí výtěžky na teplotě a chemickém potenciálu. Statistická fyzika však dokáže předpovědět i fluktuace multiplicity. Ty pak mohou být porovnány s daty z experimentu.

Postup:

- Nastudovat a v textu práce objasnit způsob výpočtu fluktuací multiplicity ve statistickém modelu.
- Určit velikost fluktuací multiplicity pro jednoduchý model, ve kterém zachovávající se kvantové číslo nese jenom jeden druh částic.
- Vyšetřit fluktuace multiplicity v modelu rezonančního plynu, kde se uvažuje produkce částic z rozpadů rezonancí a předpokládá se chemická rovnováha.
- Vyšetřit fluktuace multiplicity v modelu rezonančního plynu, kde bude vzata do úvahy i možnost chemické nerovnováhy.
- Zaměřit se na fluktuace zachovávajících se nábojů a odvodit, jak se na fluktuacích počtu částic projeví, když je částic málo a je potřeba respektovat zákony zachování náboje (elektrický náboj, baryonové číslo, podivnost…).

Součástí zadání výzkumného úkolu je jeho uložení na webové stránky katedry.

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Datum odevzdání: 30.06.2017 ***vedoucí katedry***

Poděkování

Děkuji doc. Mgr. Borisi Tomášikovi, Ph.D. za vedení mého výzkumného úkolu a podnětné návrhy, které jej obohatily.

Josef Uchytil

Title:
Multiplicity Fluctuations

Author: Josef Uchytil

Field of Study: Experimental Nuclear and Particle Physics
Kind of paper: Research Project

Supervisor: doc. Mgr. Boris Tomášik, Ph.D.
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Abstract:
The number of particles produced in ultra-relativistic nucleus-nucleus collisions is well described by the statistical model. In this model, the particle yields depend on temperature and chemical potential. However, statistical physics can also predict multiplicity fluctuations, which can subsequently be compared to experimental data. The primary aim of this research project is to provide information on how to compute multiplicity fluctuations within the statistical model. Also, ways of determining multiplicity fluctuations in a simple model, where only one quantum number is conserved, will be discussed and the method of the saddle-point expansion will be introduced. Furthermore, the multiplicity fluctuations in a hadron resonance gas model will be investigated for both the chemical equilibrium and the chemical non-equilibrium. The implementation of conservation laws concerning the charges will also be introduced.

Key words: Statistical model, Heavy-ion collisions, Multiplicity fluctuations, Charge conservation, Saddle-point expansion

Název práce:
Fluktuace multiplicity

Autor: Josef Uchytil

Obor: Experimentální jaderná a částicová fyzika
Druh práce: Výzkumný úkol

Vedoucí práce: doc. Mgr. Boris Tomášik, Ph.D.
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Abstrakt:

Počty částic produkované v ultrarelativistických jaderných srážkách jsou dobře popsány statistickým modelem. V tomto modelu závisí výtěžky na teplotě a chemickém potenciálu. Statistická fyzika však dokáže předpovědět i fluktuace multiplicity. Ty pak mohou být porovnány s daty z experimentu. Budou představeny metody výpočtů fluktuací multiplicity ve statistickém modelu, kterých následně bude využito k určení fluktuací multiplicity pro jednoduchý model - v našem případě pro pionový plyn. Dále budou vyšetřeny fluktuace multiplicity v modelu hadronového plynu, a to jak pro chemickou rovnováhu, tak pro chemickou nerovnováhu. Budou rovněž představeny metody výpočtů fluktuací zachovávajících se nábojů s uvážením zákonů zachování.

Klíčová slova: Statistický model, těžko-iontové srážky, fluktuace multiplicity, zachování náboje, rozvoj okolo lokálního extrému

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Introduction

The number of particles produced in ultra-relativistic nucleus-nucleus collisions is well described by the statistical model. In this model, the particle yields depend on temperature and chemical potential. In this research project, we will try to describe higher moments of the multiplicity distribution.

In **Chapter 1**, the statistical moments will be formally introduced, defined and elaborated. We will focus on the first four central moments especially, as those are the most important ones when considering the heavy-ion collisions. They are called **mean, variance, skewness and kurtosis** and they contribute significantly to a better understanding of the heavy-ion collisions and the subsequent particle production, which will also be emphasized in the text. Also, the canonical and grandcanonical formalism will be introduced as well as the so-called "scaled variance" which is widely used when describing fluctuations.

Chapter 2 concerns the application of the mathematical apparatus obtained in Chapter 1 in exploring multiplicity fluctuations within a simple model, where only one particle species bears the conserved quantum number. In this research project, the simple pion gas was chosen. After that, the formalism for a full hadron gas will be provided and the method called "saddle-point expansion" will be introduced.

In **Chapter 3**, the topic of chemical equilibrium and the fluctuations there will be elaborated. At the very first, the exact charge conservations in statistical systems will be introduced and the quantity called the *microscopic correlator* within the grandcanonical ensemble (GCE) will be defined, which is also very useful for the description of fluctuations. Subsequently, the conservation laws will be imposed into the equations using delta functions. Then the CE form of the microscopic correlator will be introduced based on those conservation laws.

Afterwards, the fluctuations in a hadron resonance gas model will be introduced and the corresponding thermodynamic susceptibilites - similar to Chapter 1 - will be defined and the first four cumulants in the ideal hadron gas will be written down. At the end of Chapter 3, the loss of chemical equilibrium and the chemical freeze-out parametrization will be dealt with and the effect of resonance decays will be taken into account, which includes the generalization of the first four cumulants in the ideal hadron gas for the case that the effect of resonance decays is assumed. Finally, the particle correlation after resonance decays parametrized by the so-called "generating function" will be introduced.

In **Chapter 4**, the formalism laid down in Chapter 3 will be further generalized in order to account for the case of chemical non-equilibrium. We will reintroduce the formula for

the particle pressure and adjust it to the state of chemical non-equilibrium. Furthermore, we will write down the formulae for particle fluctuations as well as other cumulants from Chapter 3 and implement the state of chemical non-equilibrium.

At the end, a summary of the information on multiplicity fluctuations we have collected so far will be provided, along with a brief outlook.

Chapter 1

Calculation of the statistical moments within the statistical model

Statistical moments are an important mathematical tool used to describe and calculate multiplicity fluctuations in the statistical model. The m-th central moment $\varphi_m(X)$, where $m \in \mathbb{N}$ is defined as follows:

$$\varphi_m(X) = E(X - EX)^m$$

where EX is the mean value of the statistical quantity X. We will further concentrate on the first four moments only, as those are of great significance. They are defined and called as follows:

mean: $M = \varphi_1$
variance: $\sigma^2 = \varphi_2$
skewness: $S = \varphi_3/\varphi_2^{3/2}$
kurtosis: $\kappa = \varphi_4/\varphi_2^2$

Sometimes, a constant -3 is added to kurtosis, because it may or may not be included, which depends on whether we want the kurtosis of the Gauss distribution to be equal to zero. In further calculations, this factor is not accounted for. The importance of skewness and kurtosis becomes obvious from Figure 1.1, where also the meaning of those two moments is depicted: skewness measures the assymetry of the probability distribution, kurtosis its "tailedness".

1.1 Grandcanonical and canonical formalism

We usually assume that we work with grandcanonical or canonical ensemble, whose event-by-event distributions of conserved quantities are characterized by the moments (M, σ, S, κ) defined above. In order to be able to directly compare theoretical predictions and experimental measurements, we also introduce the following:

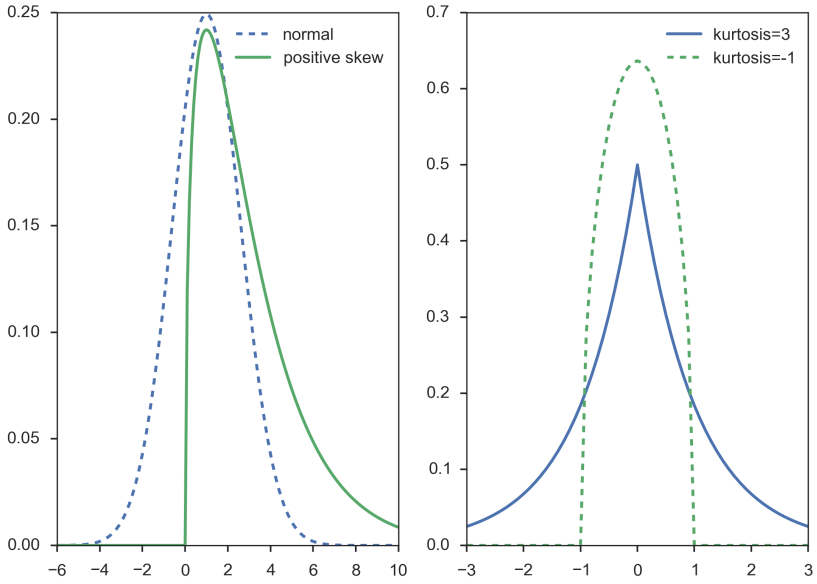

Figure 1.1: Explanation of skewness and kurtosis [1].

$$S\sigma = \varphi_3/\varphi_2 \tag{1.1}$$

$$\kappa\sigma^2 = \varphi_4/\varphi_2 \tag{1.2}$$

$$M/\sigma^2 = \varphi_1/\varphi_2 \tag{1.3}$$

$$S\sigma^3/M = \varphi_3/\varphi_1 \tag{1.4}$$

The grandcanonical partition (GC) function is given by

$$Z_{GC}(\lambda_j) = \prod_j \exp\left[\sum_{n_j=1}^{+\infty} \frac{z_j(n_j)\lambda_j^{n_j}}{n_j}\right] \tag{1.5}$$

and the single particle partition function by

$$z_j(n_j) = (\mp 1)^{n_j+1} \frac{g_j V}{2\pi^2 n_j} T m_j^2 K_2\left(\frac{n_j m_j}{T}\right). \tag{1.6}$$

The products runs over all types of hadrons and the sum is necessary due to the quantum statistical distribution.

Furthermore, K_2 is the modified Bessel function (see Appendix), V is the volume of the hadron gas,

$$\lambda_j = \exp(\frac{\mu_j}{T})$$

is the fugacity for each particle species j, m_j is the hadron mass, μ_j is the chemical potential of a particle species j,

$$g_j = 2J_j + 1$$

is the spin degeneracy and the upper sign holds for fermions, lower sign for bosons.

Canonical formalism is a little more complicated, as it cannot be factorized into one-species expressions, as is the case for the GC formalism. However, we may introduce the Wick-rotated fugacities:

$$\lambda_j = \exp[i \sum_i q_{i,j} \phi_i]$$

and the canonical partition function will now be expressed as:

$$Z_{\vec{Q}} = \left[\prod_{i=1}^{3} \frac{1}{2\pi} \int_0^{2\pi} d\phi_i e^{-iQ_i\phi_i} \right] Z_{GC}(\lambda_j) \tag{1.7}$$

where Z_{GC} is the GC partition function given by Eq. (1.5).

We will now introduce the vector of total charges

$$\vec{Q} = (Q_1, Q_2, Q_3) = (B, S, Q)$$

and the vector of charges of the hadron species j

$$\vec{q_j} = (q_{1,j}, q_{2,j}, q_{3,j}) = (b_j, s_j, q_j)$$

where Q, B, S denote the charge, the baryon number and the strangeness, respectively.

Let h be a set of hadron species with the corresponding fugacity factor λ_h. We may then write

$$\lambda_j \to \lambda_h \lambda_j$$

and have now everything we need to write down the explicit form of the first four statistical moments:

$$\langle N_h \rangle = \frac{1}{Z_{\vec{Q}}} \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} |_{\lambda_h = 1} = \sum_{j \in h} \sum_{n_j=1}^{\infty} z_j(n_j) \frac{Z_{\vec{Q} - n_j \vec{q_j}}}{Z_{\vec{Q}}} \tag{1.8}$$

$$\langle N_h^2 \rangle = \frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right] |_{\lambda_h = 1} = \sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \frac{Z_{\vec{Q} - n_j \vec{q_j}}}{Z_{\vec{Q}}} + \sum_{j \in h} \sum_{n_j=1}^{+\infty} z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q} - n_j \vec{q_j} - n_k \vec{q_k}}}{Z_{\vec{Q}}} \tag{1.9}$$

$$\langle N_h^3 \rangle = \frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right) \right] |_{\lambda_h = 1} =$$

$$\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j^2 z_j(n_j) \frac{Z_{\vec{Q}-n_j\vec{q_j}}}{Z_{\vec{Q}}} +$$

$$3 \left[\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}}}{Z_{\vec{Q}}} \right] +$$

$$\sum_{j \in h} \sum_{n_j=1}^{+\infty} z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k)$$

$$\sum_{l \in h} \sum_{n_l=1}^{+\infty} z_l(n_l) \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}-n_l\vec{q_l}}}{Z_{\vec{Q}}} \quad (1.10)$$

$$\langle N_h^4 \rangle =$$

$$\frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right) \right) \right] |_{\lambda_h = 1} =$$

$$\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j^3 z_j(n_j) \frac{Z_{\vec{Q}-n_j\vec{q_j}}}{Z_{\vec{Q}}} +$$

$$4 \left[\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j^2 z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}}}{Z_{\vec{Q}}} \right]$$

$$+ 3 \left[\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} n_k z_k(n_k) \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}}}{Z_{\vec{Q}}} \right]$$

$$+ 6 \left[\sum_{j \in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \sum_{l \in h} \sum_{n_l=1}^{+\infty} z_l(n_l) \right.$$

$$\left. \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}-n_l\vec{q_l}}}{Z_{\vec{Q}}} \right]$$

$$+ \left[\sum_{j \in h} \sum_{n_j=1}^{+\infty} z_j(n_j) \sum_{k \in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \sum_{l \in h} \sum_{n_l=1}^{+\infty} z_l(n_l) \right.$$

$$\left. \sum_{m \in h} \sum_{n_m=1}^{+\infty} z_m(n_m) \frac{Z_{\vec{Q}-n_j\vec{q_j}-n_k\vec{q_k}-n_l\vec{q_l}-n_m\vec{q_m}}}{Z_{\vec{Q}}} \right] \quad (1.11)$$

1.2 Asymptotic fluctuations in the canonical ensemble

The canonical partition function is given by Eq. (1.7). We will now introduce a way to compute this integral using the so-called "saddle-point expansion".
The integration is performed on the complex **w** unit circle parametrized as:

$$w_i = \exp[i\phi_i].$$

The canonical partition function may then be written as

$$Z_{\vec{Q}} = \frac{1}{(2\pi i)^3} \oint dw_B \oint dw_S \oint dw_Q w_B^{-B-1} w_S^{-S-1} w_Q^{-Q-1} \exp \sum_j z_{j(1)} w_B^{b_i} w_S^{s_i} w_Q^{q_i} \quad (1.12)$$

where $z_j(1)$ is the one-particle partition function given by

$$z_j(1) = (2J_j + 1)\frac{V}{(2\pi)^3} \int d^3p \exp\left[-\sqrt{p^2 + m_j^2}\right]. \quad (1.13)$$

Obviously:

$$w_B^{-B} = \exp[-B \ln w_B], \quad (1.14)$$

$$w_Q^{-Q} = \exp[-Q \ln w_Q], \quad (1.15)$$

and

$$w_S^{-S} = \exp[-S \ln w_S] \quad (1.16)$$

where Q, B, S denote the charge, the baryon number and the strangeness, respectively. Let

$$g(\vec{w}) = w_B^{b_j - 1} w_S^{s_j - 1} w_Q^{q_j - 1}, \quad (1.17)$$

$$\rho_B = \frac{B}{V}, \quad (1.18)$$

$$\rho_S = \frac{S}{V}, \quad (1.19)$$

$$\rho_Q = \frac{Q}{V} \quad (1.20)$$

and

$$f(\vec{w}) = -\rho_B \ln w_B - \rho_S \ln w_S - \rho_Q \ln w_Q + \sum_k \frac{z_{k(1)}}{V} w_B^{b_k} w_S^{s_k} w_Q^{q_k}. \quad (1.21)$$

We may now write

$$Z_{\vec{Q}-\vec{q_j}} = \frac{1}{(2\pi i)^3} \oint dw_B \oint dw_S \oint dw_Q g(\vec{w}) \exp[V f(\vec{w})] \quad (1.22)$$

1.2.1 Scaled variance

Fluctuations can be described by using the so-called "scaled variance" ω [7] of a multiplicity distribution:

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} \tag{1.23}$$

where N is the multiplicity of any hadron species, which includes primary or final (i.e. after resonance decays [7] hadrons or the sum of an arbitrary number of hadron species. Once the quantum statistics is neglected and in the absence of any other dynamical effects, the multiplicity distribution of any primary hadron is a Poisson, which means $\omega = 1$. Having performed the calculations in Eq. (1.8) and Eq. (1.9), we can rewrite the scaled variance as the sum of a Poissonian term, which means 1, and a canonical correction term:

$$\omega = 1 + \frac{\sum_{j\in h} \langle N_j \rangle \sum_{k\in h} z_{k(1)} \left(\frac{Z_{\vec{Q}-\vec{q_k}-\vec{q_j}}}{Z_{\vec{Q}-\vec{q_j}}} - \frac{Z_{\vec{Q}-\vec{q_k}}}{Z_{\vec{Q}}} \right)}{\sum_{j\in h} \langle N_j \rangle} \tag{1.24}$$

1.2.2 Saddle-point expansion

We may now introduce the method of the saddle-point expansion. Let $\vec{w_0} = (\lambda_B, \lambda_S, \lambda_Q)$ be the saddle point. Then obviously

$$\frac{\partial f(\vec{w})}{\partial w_k}|_{\vec{w_0}} = 0.$$

We will now try to find the explicit solution of a complex d-dimensional integral

$$I(\nu) = \left[\prod_{k=1}^{d} \int_{\Gamma_k} dw_k \right] g(\vec{w}) e^{\nu f(\vec{w})}$$

where Γ_k is the k-th path of integration.

If ν is large, then only a small segment around the saddle point $\vec{w_0}$ contributes to the total integral value. We may then write

$$I(\nu) \simeq e^{\nu f(\vec{w_0})} \frac{1}{(2\pi)^d} \left[\prod_{k=1}^{d} \int_{-\infty}^{+\infty} dt_k \right] g(\vec{w(t)}) e^{-\frac{1}{2}\nu \vec{t}^T \mathbf{H} \vec{t}}$$

where $\mathbf{H}$ is the Hessian matrix of $f(\vec{w})$.

We may now summarize the procedure as follows: at first we choose a **real** integration variable t_k:

$$w_k - w_{0k} = e^{i\phi_k} t_k$$

where ϕ_k denotes the phase.

Consequently, the original path is "deformed" into a **line** in the complex plane.

After that, we expand $g(\vec{w})$ into a Taylor series around $\vec{w} = \vec{w_0}$.

Finally, we assume that $\mathbf{H}$ is diagonalizable, so we can find a matrix $\mathbf{A}$ such that $\mathbf{H'} = \mathbf{AHA^T}$.
The integral will now have the following expression:

$$I(\nu) \simeq \exp(\nu f(\vec{w_0}))\sqrt{\frac{1}{(2\pi\nu)^d det\mathbf{H}}}\big[g(\vec{w_0})+ \\ \frac{1}{\nu}\left[-\frac{1}{2}\sum_{k,m=1}^{d}\frac{\partial^2 g(\vec{w})}{\partial w_k \partial w_m}|_{\vec{w_0}}\left(\sum_{i=1}^{d}\frac{A_{im}A_{ik}}{h_i}\right)|_{\vec{w_0}}+ \\ \sum_{k=1}^{d}\alpha_i\frac{\partial g(\vec{w})}{\partial w_i}|_{\vec{w_0}} + \gamma g(\vec{w_0})\right]\Big]. \quad (1.25)$$

where γ and α_i are constants dependent only on function f and its derivatives.

Chapter 2

Multiplicity fluctuations for a simple model

2.1 Multiplicity fluctuations for a classical pion gas

Since pions are composed of u and d quarks only and they are mesons, it is obvious that

$$\vec{Q} = (0, 0, Q)$$

and the saddle point

$$w_0 = \lambda_Q.$$

Furthermore, the only charged pions are π^+ and π^-, which is why they are the only ones considered. Let $\nu = V$, $g(w) = 1/w$, $f(w) = -\rho_Q \ln w + \frac{z_\pi}{V}(w + \frac{1}{w})$. Then using Eq. (1.25) we obtain

$$Z_Q = \frac{1}{2\pi i} \oint dw_q w_q^{-Q-1} \exp\left[\sum_{j=\pm 1} z_{\pi(1)} w_Q^{q_j}\right] \tag{2.1}$$

For the spin s, pion mass m and the single-particle partition function, we assume the following:

$J_{\pi^+} = J_{\pi^-} = 0$; $m_{\pi^+} = m_{\pi^-} = 139.57$ MeV

$$z_{j(1)} = (2J_j + 1)\frac{V}{(2\pi)^3} \int d^3p \exp(-\sqrt{p^2 + m_j^2}) = \frac{V}{(2\pi)^3} \int d^3p \exp(-\sqrt{p^2 + m_j^2}) \tag{2.2}$$

The partition function Z_Q^π will have the following form:

$$\begin{aligned} Z_Q^\pi &= \frac{Z_{GC}}{\lambda_Q^Q} \sqrt{\frac{1}{2\pi f''(\lambda_Q)}} \left[\frac{1}{\lambda_Q} + \frac{1}{V}\left(\frac{\gamma(\lambda_Q)}{\lambda_Q} - \frac{\alpha(\lambda_Q)}{\lambda_Q^2} - \frac{1}{\lambda_Q^3 f''(\lambda_Q)}\right)\right. \\ &+ \left. O(V^{-2})\right] \end{aligned} \tag{2.3}$$

and

$$Z^{\pi}_{Q-Q_i} = \frac{Z_{GC}}{\lambda_Q^{Q+1}}\sqrt{\frac{1}{2\pi f''(\lambda_Q)}}[1+\frac{1}{V}[\gamma(\lambda_Q)+(q_j-1)\frac{\alpha(\lambda_Q)}{\lambda_Q} - \frac{1}{2}(q_j-1)(q_j-2)\frac{1}{\lambda_Q^2 f''(\lambda_Q)}]+O(V^{-2})]. \quad (2.4)$$

If we assume the thermodynamical limit $V \to \infty$, we obtain the mean number of pions:

$$\langle \pi^{\pm} \rangle = z_\pi \frac{Z^{\pi}_{Q\mp1}}{Z^{\pi}_{Q}} = z_\pi \lambda_Q^{\pm1} + O(V^{-1}) = \langle \pi^{\pm} \rangle_{GC} + O(V^{-1}) \quad (2.5)$$

which suggests that in the thermodynamical limit, the mean number of pions corresponds with that in grandcanonical formalism.

We may now write down the final expressions of scaled variances (as defined above in Eq. (1.23) and Eq. (1.24)) for both net charge and total particle number distributions:

$$\lim_{V\to\infty} \omega_{\pm} = 1 - \frac{\langle \pi^{\pm} \rangle_{GC}}{\langle \pi^{+} \rangle_{GC} + \langle \pi^{-} \rangle_{GC}} \quad (2.6)$$

$$\lim_{V\to\infty} \omega_{ch} = 1 - \frac{\langle \pi^{+} \rangle_{GC} - \langle \pi^{-} \rangle_{GC}}{\langle \pi^{+} \rangle_{GC} + \langle \pi^{-} \rangle_{GC}} \quad (2.7)$$

where $\omega_{\pm}$ denotes positive and negative pions, respectively, ω_{ch} denotes all charged pions and $\langle \pi^{\pm} \rangle_{GC}$ denote the mean multiplicities of the respective pions in the GC ensemble.

2.2 Multiplicity fluctuations for a full hadron gas

We may now consider the general case of multi-hadron gas carrying total charges $\{Q_l\} = (B, S, Q)$ [7]. Assuming the asymptotic expansion of the canonical partition function, the Hessian matrix of the function f, which is defined in Eqs (1.21) and (1.22), is diagonalizable and takes the following form:

$$H_{ln}(\vec{w}_0) = \frac{\partial^2 f(\vec{w})}{\partial w_{Q_l} \partial w_{Q_n}}|_{\vec{w}_0} = \frac{1}{V\lambda_{Q_l} Q_n}\left[Q_l\delta_{ln} + \sum_j q_{l,j}(q_{n,j}-\delta_{ln})\langle N_j \rangle_{GC}\right] \quad (2.8)$$

where $q_{l,j}$ is the l^{th} charge of the j^{th} hadron species. Due to the symmetry and the matrix being real, we may perform the diagonalization using an orthogonal matrix $\mathbf{A}$:

$$\mathbf{H'} = diag(h_1, h_2, h_3) = \mathbf{AHA^T} \quad (2.9)$$

where h_l are the Hessian eigenvalues.

The Eq. (1.22) can now - using the approximation (1.25) and the functions f and g defined as (1.21) and (1.17), respectively - be rewritten as

$$Z_{\vec{Q}-\vec{q_j}} = \frac{Z_{GC}}{\lambda_B^{B+1}\lambda_S^{S+1}\lambda_Q^{Q+1}}\sqrt{\frac{1}{(2\pi)^3 V^3 detH}}\lambda_B^{b_j}\lambda_S^{s_j}\lambda_Q^{q_j}\left\{1+\frac{1}{V}\left[-\frac{1}{2\lambda_B^2}(b_j-1)(b_j-2)\left(\sum_{i=1}^{3}\frac{A_{i1}A_{i1}}{h_i}\right)\right.\right. \tag{2.10}$$
$$-\frac{1}{2\lambda_S^2}(s_j-1)(s_j-2)\left(\sum_{i=1}^{3}\frac{A_{i2}A_{i2}}{h_i}\right)-\frac{1}{2\lambda_Q^2}(q_j-1)(q_j-2)\left(\sum_{i=1}^{3}\frac{A_{i3}A_{i3}}{h_i}\right)$$
$$-\frac{1}{2\lambda_B\lambda_S}(b_j-1)(s_j-2)\left(\sum_{i=1}^{3}\frac{A_{i1}A_{i2}}{h_i}\right)-\frac{1}{2\lambda_B\lambda_Q}(b_j-1)(q_j-2)\left(\sum_{i=1}^{3}\frac{A_{i1}A_{i3}}{h_i}\right)$$
$$\left.\left.-\frac{1}{2\lambda_Q\lambda_S}(q_j-1)(s_j-2)\left(\sum_{i=1}^{3}\frac{A_{i2}A_{i3}}{h_i}\right)+\gamma+\frac{\alpha_B(b_j-1)}{\lambda_B}+\frac{\alpha_S(s_j-1)}{\lambda_S}+\frac{\alpha_Q(q_j-1)}{\lambda_Q}\right]\right\}$$
$$+O(V^{-2}).$$

where γ and α_i are constants dependent only on function f and its derivatives (as is the case in Eq. (1.25)).

It should be noted here that the hessian matrix $\mathbf{H}$ and its eigenvalues, as well as the diagonalizing matrix $\mathbf{A}$, is independent of the volume V and so is function f given by Eq. (1.21).

The expansion above has been performed for the partition function $Z_{\vec{Q}-\vec{q_j}}$ given by Eq. (1.22). If we do the same with the partition function $Z_{\vec{Q}}$ given by Eq. (1.12), we immediately obtain

$$\lim_{V\to\infty}\frac{Z_{\vec{Q}-\vec{q_j}}}{Z_{\vec{Q}}} = \lambda_B^{b_j}\lambda_S^{s_j}\lambda_Q^{q_j} \equiv \lambda_j. \tag{2.11}$$

That means that if the large volume limit is taken into account, the suitability of the usual grand-canonical formalism is restored, which can be expressed as

$$\lim_{V\to\infty} z_{j(1)}\frac{Z_{\vec{Q}-\vec{q_j}}}{Z_{\vec{Q}}} = z_{j(1)}\lambda_B^{b_j}\lambda_S^{s_j}\lambda_Q^{q_j} = \langle N\rangle_{GC}\,. \tag{2.12}$$

Let us now denote:

$$M_{ln} = \sum_{i=1}^{3}\frac{A_{il}A_{in}}{h_i}. \tag{2.13}$$

We may now - using Eq. (2.10) - immediately write

$$\frac{Z_{\vec{Q}-\vec{q_j}-\vec{q_k}}}{Z_{\vec{Q}-\vec{q_j}}} = -\frac{\lambda_B^{b_k}\lambda_S^{s_k}\lambda_Q^{q_k}}{V}\left[\frac{b_k b_j}{\lambda_B^2}M_{11}+\frac{s_k s_j}{\lambda_S^2}M_{22}+\frac{q_k q_j}{\lambda_Q^2}M_{33}\right. \tag{2.14}$$
$$\left.+\frac{b_k s_j+s_k b_j}{\lambda_B\lambda_S}M_{12}+\frac{b_k q_j+b_j q_k}{\lambda_B\lambda_Q}M_{13}+\frac{s_k q_j+q_k s_j}{\lambda_Q\lambda_S}\right]+O(V^{-2})$$
$$\equiv \frac{\lambda_k}{V}C_{jk}+O(V^{-2}).$$

The scaled variance as defined by Eq. (1.24) will now have the following form:

$$\omega_h = 1 + \frac{\sum_{j\in h} \langle N_j \rangle_{GC} \sum_{k\in h} \langle N_k \rangle_{GC} C_{jk}}{V \sum_{j\in h} \langle N_j \rangle_{GC}} + O(V^{-1}) \tag{2.15}$$

We will now for a moment consider the scaled variance ω_j of a single hadron species. We will also neglect baryon number and strangeness, thus considering electric charge only. The hessian matrix is obviously diagonal with only one non-zero element in the matrix $\mathbb{M}$:

$$M_{33} = \left(\frac{\partial^2 f}{\partial w_Q^2} |_{\vec{w}_0} \right)^{-1} \tag{2.16}$$

Furthermore, if we neglect hadrons with two or more units of electric charge, we obtain the following expression for C_{jk}[7]:

$$\begin{aligned} C_{jk} &= \frac{q_k q_j}{\lambda_Q^2} M_{33} = \frac{q_k q_j}{\lambda_Q^2} H_{33}^{-1} = \frac{q_k q_j}{\lambda_Q^2} \frac{V \lambda_Q^2}{Q + \sum_j \langle N_j \rangle_{GC} q_j (q_j - 1)} \\ &= \frac{V q_k q_j}{Q + 2 \langle h^- \rangle_{GC}} = \frac{V q_k q_j}{\langle h^+ \rangle_{GC} + \langle h^- \rangle_{GC}} \end{aligned} \tag{2.17}$$

where $\langle h^{\pm} \rangle$ denotes the mean multiplicities of positive and negative hadrons, respectively, and $Q = \langle h^+ \rangle_{GC} - \langle h^- \rangle_{GC}$.

If we plug this expression into Eq. (2.15) and assume that for a single hadron species $j = k$ and $q_j^2 = 1$, we immediately obtain:

$$\lim_{V\to\infty} \omega_j = 1 + \frac{\langle N_j \rangle_{GC}}{\langle h^+ \rangle_{GC} + \langle h^- \rangle_{GC}}. \tag{2.18}$$

Similarly, the scaled variances of all negative, positive and charged hadrons now read as follows:

$$\lim_{V\to\infty} \omega_{\pm} \sim 1 - \frac{\langle h^{\pm} \rangle_{GC}}{\langle h^+ \rangle_{GC} + \langle h^- \rangle_{GC}} \tag{2.19}$$

$$\lim_{V\to\infty} \omega_{ch} \sim 1 - \frac{\langle h^+ \rangle_{GC} - \langle h^- \rangle_{GC}}{\langle h^+ \rangle_{GC} + \langle h^- \rangle_{GC}} \tag{2.20}$$

where $\omega_{\pm}$ denotes positive and negative hadrons, respectively, ω_{ch} denotes all charged hadrons and $\langle h^{\pm} \rangle_{GC}$ denote the mean multiplicities of the respective hadrons in the GC ensemble.

This is obviously similar to the classical pion gas case, both of which was also proved in [7].

Chapter 3

Multiplicity fluctuations for a resonance gas model with chemical equilibrium

Since we have at this point presented all the necessary formalism concerning the calculation of multiplicity fluctuations, it seems only fitting that we now proceed towards systems where chemical equilibrium is a priori assumed. At first, we will present a certain generalization of what we laid down above.

As statistical models provide a valid description of hadron multiplicities in relativistic nucleus-nucleus collisions [2], we may further concentrate on multiplicity fluctuations in **high energy** nuclear collisions. Last but not least, the *microscopic correlator* providing a possibility to calculate the fluctuations of different observables [2], as well as the generating function comprising the inclusion of resonance decays will be introduced.

3.1 Exact charge conservations in statistical systems

Let us now assume the canonical ensemble (CE). Our primary aim is to include different types of hadrons while keeping (B, S, Q) (introduced in Chapter 1) exactly fixed. For simplicity, we will assume a system of non-interacting Bose or Fermi particles, characterized by their occupation numbers $n_{p,i}$, with i denoting the particle species and p the particle momentum. For fermions, the occupation numbers are $n_{p,i} = 0, 1$, whereas for Bosons, they are $n_{p,i} = 0, 1, \ldots$. The grand canonical ensemble (GCE) average values and the fluctuations of $n_{p,i}$ are the following:

$$\langle n_{p,i} \rangle = \frac{1}{\exp[(\sqrt{p^2 + m_i^2} - \mu_i)/T] - \gamma_i} \tag{3.1}$$

$$v_{p,i}^2 \equiv \left\langle \Delta n_{p,i}^2 \right\rangle = \left\langle (n_{p,i} - \langle n_{p,i} \rangle)^2 \right\rangle = \langle n_{p,i} \rangle \left(1 + \gamma_i \langle n_{p,i} \rangle\right) \tag{3.2}$$

where $\gamma_i \in \{-1, 0, 1\}$ denotes the Fermi distribution, the Boltzmann approximation and the Bose distribution, respectively.

The chemical potential μ_i is given by the following equation:

$$\mu_i = q_i\mu_Q + b_i\mu_B + s_i\mu_S \tag{3.3}$$

where q_i, b_i, s_i denote the electric charge, the baryon number and the strangeness of a particle of species i, respectively; μ_Q, μ_B, μ_S are the corresponding chemical potentials regulating the average values of these global conserved charges in the GCE [2].

We will now lay down the formalism concerning the average number of particles of species i denoted as $\langle N_i \rangle$, the number of positive, negative and all charged particles denoted as N_+, N_- and N_{ch}, respectively [2]:

$$\langle N_i \rangle \equiv \sum_p \langle n_{p,i} \rangle = \frac{g_i V}{2\pi^2} \int_0^\infty p^2 dp \, \langle n_{p,i} \rangle \,, \tag{3.4}$$

$$\langle N_+ \rangle = \sum_{i,q_i>0} \langle N_i \rangle \,, \quad \langle N_- \rangle = \sum_{i,q_i<0} \langle N_i \rangle \,, \quad \langle N_{ch} \rangle = \sum_{i,q_i\neq 0} \langle N_i \rangle \tag{3.5}$$

where V is the system volume and g_i is the degeneracy factor of particle of species i (a number of spin states). The momentum integral (3.4) holds in the thermodynamic limit $V \to \infty$.

We may now define the *microscopic correlator* [2]:

$$\langle \Delta n_{p,i} \Delta n_{k,j} \rangle = v_{p,i}^2 \delta_{ij}\delta_{pk} = \langle n_{p,i} \rangle \left(1 + \gamma_i \langle n_{p,i} \rangle\right) \delta_{ij}\delta_{pk} \tag{3.6}$$

where the average values of $\langle n_{p,i} \rangle$ are given by Eq. (3.1).

The aforementioned enables us to calculate the fluctuations of different observables in the GCE. Due to the presence Kronecker factors in Eq. (3.6), it becomes obvious that only the Bose and Fermi effects for the fluctuations of identical particles on the same level are relevant in the GCE, as this has to correspond with Eq. (3.2).

In order to account for the effect of exact conservation laws, we will introduce the equilibrium probability distribution denoted $W(\Delta n_{p,i})$ describing the deviations of different sets $\{n_{p,i}\}$ of the occupation numbers from their average value given by Eq. (3.1).

If the GCE is assumed, each $\Delta n_{p,i}$ fluctuates independently according approximately to the Gauss distribution law for $\Delta n_{p,i}$ [2], with mean square deviation being $\Delta n_{p,i}^2 = v_{p,i}^2$:

$$W_{g.c.e.}(\Delta n_{p,i}) \propto \prod_{p,i} \exp\left[-\frac{(\Delta n_{p,i})^2}{2v_{p,i}^2}\right] \tag{3.7}$$

This may make an elaboration necessary. We may consider the sum of $n_{p,i}$ in a small momentum volume $(\Delta p)^3$ with the center at p. Once $(\Delta p)^3$ is fixed and $V \to \infty$, the average number of particles inside $(\Delta p)^3$ will be large. Each particle configuration inside $(\Delta p)^3$ consists of $(\Delta p)^3 \cdot gV/(2\pi)^3 \gg 1$ statistically independent terms, each with average value $\langle n_{p,i} \rangle$ given by Eq. (3.1) and variance $v_{p,i}^2$ given by Eq. (3.2) [2]. Using the central limit theorem, we can immediately prove that the fluctuations inside $(\Delta p)^3$ should be Gaussian. Since $n_{p,i}$ is always convolved with a smooth function of p, we can write the Gaussian distribution sum directly for $n_{p,i}$.

3.1.1 Imposition of exact conservation laws

The primary aim is to provide an explicit form of the microscopic correlator defined by Eq. (3.6) with three conserved charges Q, B, S (denoting the electric charge, the baryon number and the strangeness, respectively - see previous Chapters) in the CE, which means that global charge conservation laws are imposed on each **microscopic** state of the system. The respective charges - for example the electric charge Q - can be written in the form

$$Q \equiv \sum_{p,i} q_i \Delta n_{p,i}.$$

We introduce an **exact conservation law** as the restriction on the sets of the occupation numbers $\{n_{p,i}\}$, which means only those sets satisfying

$$\Delta Q = \sum_{p,i} q_i \Delta n_{p,i} = 0$$

can be realized. We may now implement the three conservation laws into the distribution given by Eq. (3.7):

$$\begin{aligned} W_{c.e.}(\Delta n_{p,i}) &\propto \prod_{p,i} \exp\left[-\frac{(\Delta n_{p,i})^2}{2v_{p,i}^2}\right] \cdot \delta\left(\sum_{p,i} q_i \Delta n_{p,i}\right) \cdot \delta\left(\sum_{p,i} b_i \Delta n_{p,i}\right) \cdot \delta\left(\sum_{p,i} s_i \Delta n_{p,i}\right) \\ &\propto \int_{-\infty}^{+\infty} d\lambda_q d\lambda_b d\lambda_s \prod_{p,i} \exp\left[-\frac{(\Delta n_{p,i})^2}{2v_{p,i}^2} + i\lambda_q q_i \Delta n_{p,i} + i\lambda_b b_i \Delta n_{p,i} + i\lambda_s s_i \Delta n_{p,i}\right]. \end{aligned} \tag{3.8}$$

In further calculations, we will use the generalized form of distribution (3.8), which uses the integration along imaginary axis in λ-space. By completing squares, we can easily obtain:

$$\begin{aligned} W_{c.e.}(\Delta n_{p,i}; \lambda_q, \lambda_b, \lambda_s) &\propto \prod_{p,i} \exp\left[-\frac{(\Delta n_{p,i} - \lambda_q v_{p,i}^2 q_i - \lambda_b v_{p,i}^2 b_i - \lambda_s v_{p,i}^2 s_i)^2}{2v_{p,i}^2}\right. \\ &\left. + \frac{\lambda_q^2}{2} v_{p,i}^2 q_i^2 + \frac{\lambda_b^2}{2} v_{p,i}^2 b_i^2 + \frac{\lambda_s^2}{2} v_{p,i}^2 s_i^2 - \lambda_q \lambda_s v_{p,i}^2 q_i s_i - \lambda_q \lambda_b v_{p,i}^2 q_i b_i - \lambda_b \lambda_s v_{p,i}^2 b_i s_i\right]. \end{aligned} \tag{3.9}$$

We may now perform the CE averaging as follows

$$\langle \cdots \rangle_{c.e.} = \frac{\int_{-i\infty}^{+i\infty} d\lambda_q d\lambda_b d\lambda_s \int_{-\infty}^{+\infty} \prod_{p,i} dn_{p,i} \cdots W_{c.e.}(\Delta n_{p,i}; \lambda_q, \lambda_b, \lambda_s)}{\int_{-i\infty}^{+i\infty} d\lambda_q d\lambda_b d\lambda_s \int_{-\infty}^{+\infty} \prod_{p,i} dn_{p,i} W_{c.e.}(\Delta n_{p,i}; \lambda_q, \lambda_b, \lambda_s)} \tag{3.10}$$

and the CE microscopic correlator reads

$$\begin{aligned}\langle \Delta n_{p,i} \Delta n_{k,j} \rangle_{c.e.} &= v^2_{p,i}\delta_{ij}\delta_{pk} \\ &- \frac{v^2_{p,i}v^2_{k,j}}{|A|}[q_i q_j M_{qq} + b_i b_j M_{bb} + s_i s_j M_{ss} + (q_i s_j + q_j s_i)M_{qs} \\ &- (q_i b_j + q_j b_i)M_{qb} - (b_i s_j + b_j s_i)M_{bs}]\end{aligned} \tag{3.11}$$

where the first term on the right hand side corresponds to the microscopic correlator in the GCE as given by (3.6), while the other terms appear due to the global CE charge conservations and the resulting (anti)correlations among different particles (i≠j) and $|A|$ is the determinant of matrix

$$A = \begin{pmatrix} \Delta(q^2) & \Delta(bq) & \Delta(sq) \\ \Delta(qb) & \Delta(b^2) & \Delta(sb) \\ \Delta(qs) & \Delta(bs) & \Delta(s^2) \end{pmatrix}$$

with matrix elements being equal to

$$\Delta(q^2) \equiv \sum_{p,i} q_i^2 v^2_{p,i},$$

$$\Delta(qb) \equiv \sum_{p,i} q_i b_i v^2_{p,i}$$

etc. The sum $\sum_{p,i}$ means that we integrate over momentum p and sum over hadron-resonance species i. M_{ij} stand for the corresponding minors of the matrix A, e.g.:

$$M_{qs} = det \begin{pmatrix} \Delta(qb) & \Delta(b^2) \\ \Delta(qs) & \Delta(bs) \end{pmatrix}$$

The microscopic correlator can thus be used to calculate correlations and fluctuations of different physical quantities in the canonical ensemble.

We may now present the particle number fluctuations. The correlations in the GCE and CE, respectively, read as follows:

$$\langle \Delta N_i \Delta N_j \rangle_{g.c.e.} = \sum_{p,k} \langle \Delta n_{p,i} \Delta n_{k,j} \rangle = \sum_p v^2_{p,i} \tag{3.12}$$

and

$$\langle \Delta N_i \Delta N_j \rangle_{.c.e.} = \sum_{p,k} \langle \Delta n_{p,i} \Delta n_{k,j} \rangle_{c.e.} \,. \tag{3.13}$$

The CE scaled variance now has the following form:

$$\omega^i_{c.e.} \equiv \frac{\langle (\Delta N_i)^2 \rangle_{c.e.}}{\langle \Delta N_i \rangle_{c.e.}} = \omega^i_{g.c.e.}[1 \tag{3.14}$$

$$-\frac{\sum_k v^2_{k,i}}{|A|}(q_i^2 M_{qq} + b_i^2 M_{bb} + s_i^2 M_{ss} + 2q_i s_i M_{qs} - 2q_i b_i M_{qb} - 2b_i s_i M_{bs})]$$

where

$$\omega^i_{g.c.e.} \equiv \frac{\langle(\Delta N_i)^2\rangle_{c.e.}}{\langle\Delta N_i\rangle_{c.e.}} = \frac{\sum_p v^2_{p,i}}{\sum_p \langle n_{p,i}\rangle} \tag{3.15}$$

We have used the fact that $\langle N_i\rangle_{c.e.} = \langle N_i\rangle$ at $V \to \infty$, where $\langle N_i\rangle$ is defined by Eq. (3.4). It should be pointed out that the particle number fluctuations and correlations in the CE - albeit different from those in the GCE - can obviously be obtained in terms of quantities calculated within the GCE. However, the method cannot be used to obtain the finite volume corrections, as Eq. (3.14) is obtained in the thermodynamic limit and thus V-independent.

3.2 Fluctuations in a hadron resonance gas model

We may describe fluctuations in the number of particles of species i in a thermally and chemically equilibrated Hadron Resonance Gas (HRG) using the corresponding susceptibilities defined as

$$\chi^{(i)}_l = \frac{\partial^l (P/T)^4}{\partial(\mu_i/T)^l}\,|_T \tag{3.16}$$

where $l \in \mathbb{N}$.

The susceptibilities can be related to the cumulants of the distribution of particle i via

$$\chi^{(i)}_1 = \frac{1}{VT^3}\langle N_i\rangle_c = \frac{1}{VT^3}\langle N_i\rangle \tag{3.17}$$

$$\chi^{(i)}_2 = \frac{1}{VT^3}\left\langle(\Delta N_i)^2\right\rangle_c = \frac{1}{VT^3}\left\langle(\Delta N_i)^2\right\rangle \tag{3.18}$$

$$\chi^{(i)}_3 = \frac{1}{VT^3}\left\langle(\Delta N_i)^3\right\rangle_c = \frac{1}{VT^3}\left\langle(\Delta N_i)^3\right\rangle \tag{3.19}$$

$$\chi^{(i)}_4 = \frac{1}{VT^3}\left\langle(\Delta N_i)^4\right\rangle_c = \frac{1}{VT^3}\left(\left\langle(\Delta N_i)^4\right\rangle - 3\left\langle(\Delta N_i)^2\right\rangle^2\right) \tag{3.20}$$

where $\Delta N_i = N_i - \langle N_i\rangle$ and the subscript c denotes the corresponding cumulant value.

It is obvious that the first three cumulants are equal to the corresponding central moments, but the fourth cumulant is given by a combination of fourth and second central moments (for the corresponding formalism see Chapter 1). The cumulants will be discussed later on in this Chapter. If we assume an equilibrium HRG model in the GCE formulation, thermally produced and non-interacting particles and anti-particles are uncorrelated [8]. The susceptibilities of the net-distributions can thus be written as:

$$\chi^{net,i}_l = \chi_l + (-1)^l \chi^{\bar{i}}_l \tag{3.21}$$

where $\bar{i}$ denotes the species of the antiparticle and i the species of the particle.

As we have already mentioned in Chapter 1, some ratios of the susceptibilities can be expressed in terms of the first two central moments, those being the mean M, the variance σ, and in terms of the skewness S and the kurtosis κ, as we can see in Eq. (1.1), (1.2), (1.3), (1.4).

The dependence of susceptibility ratios (1.1), (1.2) and (1.3) on the collision energy $\sqrt{s}$ is depicted in Fig. 3.1. The **full squares** depict experimental data on net proton fluctuations as measured by the STAR collaboration for the two most central collision classes (0-10%). **Empty circles** stand for the susceptibility ratios for the net baryon number fluctuations in the full HRG model, the **empty triangles** show the corresponding ratios for the net proton fluctuations with respect to primordial protons and anti/protons. The **solid curves** show the corresponding Skellam limits for a Boltzmann gas of baryons and anti-baryons.

We may now write down the specific equilibrium pressure P, which is given by the sum of the partial pressures of all particle species i included in the model [8]:

$$P/T^4 = \frac{1}{VT^3} \sum_i \ln Z_{m_i}^{M/B}(V, T, \mu_B, \mu_Q, \mu_S), \tag{3.22}$$

where

$$\ln Z_{m_i}^{M/B} = \mp \frac{V g_i}{(2\pi)^3} \int d^3k \ln(1 \mp z_i \exp(-\epsilon_i/T)). \tag{3.23}$$

The single-particle energy (see Chapter 1) is equal to

$$\epsilon_i = \sqrt{k^2 + m_i^2}$$

with m_i being the particle mass, g_i the degeneracy factor, V the volume and z_i being the fugacity given by

$$z_i = \exp((B_i\mu_B + Q_i\mu_Q + S_i\mu_S)/T) \equiv \exp(\mu_i/T). \tag{3.24}$$

We may also perform the partial derivative of the pressure with respect to the particle chemical potential μ_i, which gives us the density of particles i:

$$n_i(T, \mu_i) = \frac{g_i}{(2\pi)^3} \int d^3k f_{FD/BE}(T, \mu_i) \tag{3.25}$$

where $f_{FD/BE}$ is the Fermi-Dirac/Bose-Einstein distribution function for (anti-)baryons or mesons.

3.2.1 The first four cumulants in the ideal hadron gas

In this subsection, the aforementioned cumulants of primary particles i will be discussed [9]. We may plug Eq. (3.24) into Eq. (3.23), thus immediately obtaining

$$\ln Z_i(T, V, \mu_i) = \frac{V g_i}{2\pi^2} \int_0^{+\infty} \pm p^2 dp \ln[1 \pm \exp(-(E_i - \mu_i)/T)], \tag{3.26}$$

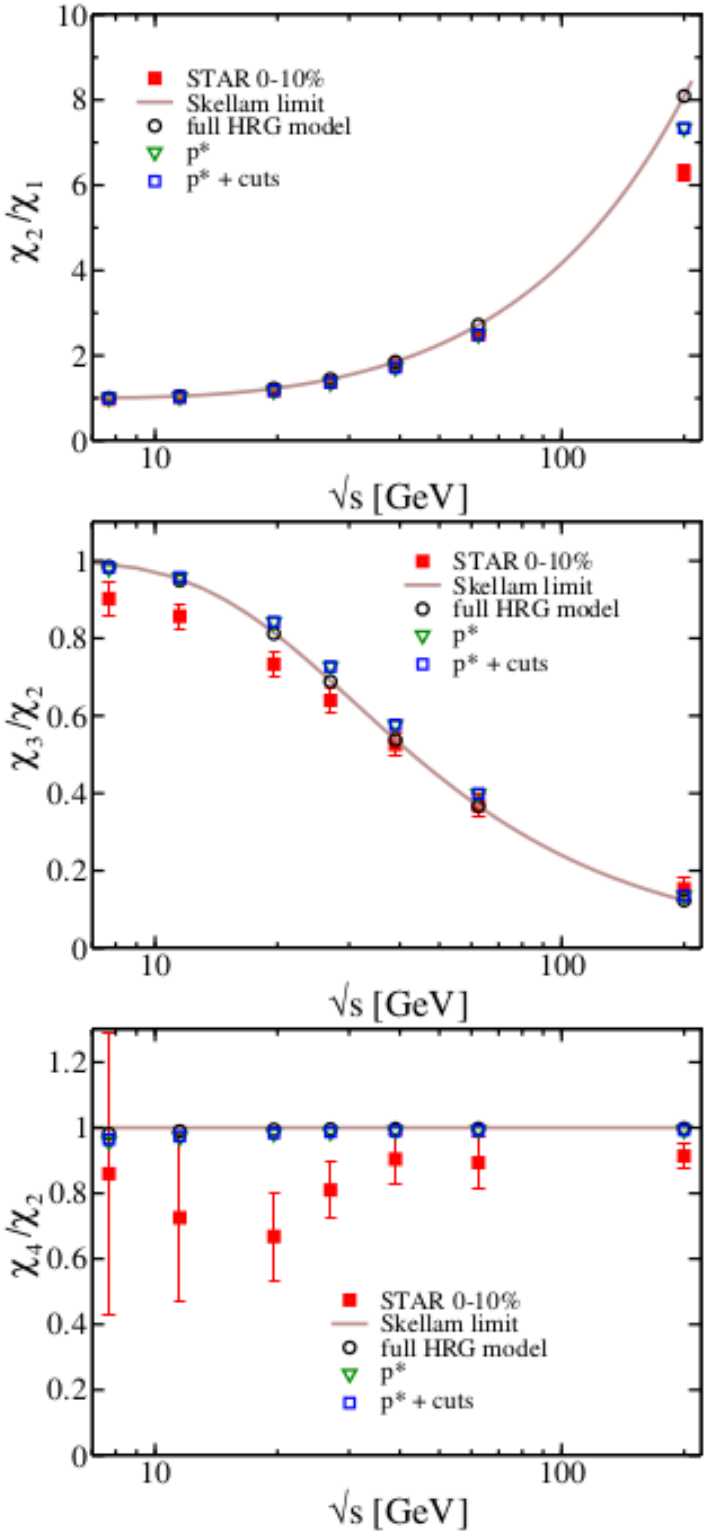

Figure 3.1: Ratios of susceptibilities as function of the collision energy. Taken from [8].

where $E_i = \sqrt{p^2 + m_i^2}$ is the single particle energy.

Using Eq.(3.26), we may now calculate the first four cumulants. The mean number of primary particles i is calculated (see formalism in Chapter 1) as follows:

$$C_1 = M = \langle N_i \rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right) \ln Z_i \right]_{T,V} = \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i \tag{3.27}$$

where

$$n_i = \frac{1}{\exp[(E_i - \mu_i)/T] \pm 1}.$$

The variance and higher order cumulants have the following form:

$$C_2 = \sigma^2 = \left\langle (\Delta N_i)^2 \right\rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^2 \ln Z_i \right]_{T,V} \tag{3.28}$$

$$= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp n_i), \tag{3.29}$$

$$C_3 = \left\langle (\Delta N_i)^3 \right\rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^3 \ln Z_i \right]_{T,V} \tag{3.30}$$

$$= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp 3n_i + 2n_i^2), \tag{3.31}$$

$$C_4 = \left\langle (\Delta N_i)^4 \right\rangle - 3 \left\langle (\Delta N_i)^2 \right\rangle^2 = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^4 \ln Z_i \right]_{T,V} \tag{3.32}$$

$$= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp 7n_i + 12n_i^2 \mp 6n_i^3). \tag{3.33}$$

3.3 Loss of chemical equilibrium and chemical freeze-out parametrization

We may conclude that the chemical composition of a HRG in local thermal and chemical equilibrium is determined by the **independent chemical potentials** μ_i of each individual species, their masses and the temperature [8]. However, the created matter expands rapidly, causing the density to decrease and leading to an enhancement of the particle mean free path. Consequently, there must be a specific set of parameters $(T^{fo}, \mu_B^{fo}, \mu_S^{fo}, \mu_Q^{fo})$, where reactions like baryon-antibaryon annihilation ($p\bar{p} \rightarrow \pi\pi\pi\pi\pi$) become too rare to maintain chemical equilibrium among different particle species [8]. This particular set of parameters describes the chemical freeze-out. The chemical freeze-out is an instant at which chemical equilibrium is lost, the chemical composition of the gas is frozen-out and after which only elastic scatterings occur frequently enough to maintain local thermal equilibrium until even these become too rare and the particles

start to stream freely after the kinetic freeze-out [8].

We may assume that chemical equilibrium is not completely lost just after the chemical freeze-out. If the temperature T is high enough, specific reactions in form of resonance regenerations and decays (e.g. $\pi\pi \to \rho \to \pi\pi$) continue to occur, which means that resonances are still in chemical equilibrium with their decay products.

We may assume the hadronic matter to be in a state of partial chemical equillibrium, which means that the chemical potentials of all stable hadrons μ_h become $T-$dependent, while the chemical potentials of the resonances (whose effects will be discussed in the next Section) μ_R become functions of the μ_h:

$$\mu_R = \sum_h \mu_h \langle n_h \rangle_R .$$

The sum runs over all stable hadrons and

$$\langle n_h \rangle_R \equiv \sum_r b_r^R n_{h,r}^R$$

is the decay-channel averaged number of hadrons h produced in the decay of resonance R, where b_r^R is the branching ratio of the decay-channel and $n_{h,r}^R = 0, 1, \ldots$ is the number of hadrons h formed in that specific decay-channel.

The temperature is parametrized by a polynomial function of μ_B:

$$T^{fo}(\mu_B^{fo}) = a - b(\mu_B^{fo})^2 - c(\mu_B^{fo})^4 \tag{3.34}$$

where $a = (0.166 \pm 0.002)$GeV, $b = 0.139 \pm 0.016$GeV^{-1}, $c = (0.053 \pm 0.021)$GeV^{-3}.

The baryon-chemical potential can be given as a function of $\sqrt{s}$:

$$\mu_B^{fo}(\sqrt{s}) = \frac{d_B}{1 + e_B\sqrt{s}} \tag{3.35}$$

where $d_B = (1.308 \pm 0.028)$GeV, $e_B = (0.273 \pm 0.008)$GeV^{-1}.

All the parameter values are taken from [8]. If we want to investigate the $\sqrt{s}-$dependence of the electric charge and strangeness chemical potentials -μ_B and μ_S- we have to require the following [8]:

$$n_S^{(net)}(T, \mu_B, \mu_S, \mu_Q) = 0, \tag{3.36}$$

$$n_Q^{(net)}(T, \mu_B, \mu_S, \mu_Q) = x n_B^{(net)}(T, \mu_B, \mu_S, \mu_Q). \tag{3.37}$$

where $x \in \langle 0, 1 \rangle$., e. g. $x \simeq 0.4$ for Au + Au and Pb + Pb collisions present in the initial state [8].

Just as in case of Eq. (3.35), μ_Q^{fo} and μ_S^{fo} can be parametrized as functions of $\sqrt{s}$. Here, the parameters are $d_Q = -0.0202$GeV, $e_Q = 0.125$GeV^{-1} and $d_s = 0.224$GeV, $e_S = 0.184$GeV^{-1}.

3.4 Effect of resonance decays

We will now finally take the resonance decays into account. As we have already mentioned, the chemical potential of the resonances μ_R depends on the chemical potential of stable hadron species μ_h. As such, the resonances significantly affect the evolution of the created strongly interacting hadronic matter and their decays exercise a major influence on the final numbers of the stable hadrons and the fluctuations therein [8]. We may now consider the derivative of P/T^4 with respect to μ_h/T as defined in Eq. (3.16). Considering that only the chemical potentials μ_h are independent of each other (while the μ_R depend on μ_h), we obtain

$$VT^3 \frac{\partial(P/T^4)}{\partial(\mu_h/T)}|_T = \langle N_h \rangle + \sum_R \langle N_R \rangle \langle n_h \rangle_R \tag{3.38}$$

where $\langle N_h \rangle$ and $\langle N_R \rangle$ are the means of the primordial numbers of hadrons and resonances, respectively. The sum runs over all the resonances in the model.

In agreement with the QCD equations of state [8], there are 26 particle species we consider stable, those being: $\pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \bar{K}_0, \eta$ and $p, d, \lambda^0, \sigma^+, \sigma^0, \sigma^-, \Xi^0, \Xi^-, \Omega^-$ and their respective anti-baryons.

We will now demonstrate this using the example of fluctuations in the final numbers of **protons and neutrons**. Since μ_R is μ_p-dependent and under assumption of fixed, average numbers of produced protons as determined by the branching ratios of the resonance decays, we may write:

$$\left\langle \hat{N}_p \right\rangle = \langle N_p \rangle + \sum_R \langle N_R \rangle \langle n_p \rangle_R \tag{3.39}$$

$$\left\langle (\Delta \hat{N}_p)^2 \right\rangle = \left\langle (\Delta N_p)^2 \right\rangle + \sum_R \left\langle (\Delta N_R)^2 \right\rangle \langle n_p \rangle_R^2 \tag{3.40}$$

$$\left\langle (\Delta \hat{N}_p)^3 \right\rangle = \left\langle (\Delta N_p)^3 \right\rangle + \sum_R \left\langle (\Delta N_R)^3 \right\rangle \langle n_p \rangle_R^3 \tag{3.41}$$

$$\left\langle (\Delta \hat{N}_p)^4 \right\rangle_c = \left\langle (\Delta N_p)^4 \right\rangle_c + \sum_R \left\langle (\Delta N_R)^4 \right\rangle \langle n_p \rangle_R^4 . \tag{3.42}$$

The same holds for antiprotons; p is then replaced by $\bar{p}$. The related susceptibilities are given by

$$\hat{\chi}_l^{(p)} = \chi_l^{(p)} + \sum_R \chi_l^{(R)} \langle n_p \rangle_R^l . \tag{3.43}$$

In reality though, the actual numbers of decay products follow a multinomial distribution, since resonance decays are probabilistic processes. Said multinomial distribution results in fluctuations on the final particle numbers, which makes it necessary for them to be taken into account. If we assume a grandcanonical ensemble, the corresponding cumulants of the final proton distribution read as follows [8]:

$$\left\langle \hat{N}_p \right\rangle = \langle N_p \rangle + \sum_R \langle N_R \rangle \langle n_p \rangle_R \tag{3.44}$$

$$\left\langle (\Delta \hat{N}_p)^2 \right\rangle = \left\langle (\Delta N_p)^2 \right\rangle + \sum_R \left\langle (\Delta N_R)^2 \right\rangle \langle n_p \rangle_R^2 + \sum_R \langle N_R \rangle \left\langle (\Delta n_p)^2 \right\rangle_R , \tag{3.45}$$

$$\begin{aligned} \left\langle (\Delta \hat{N}_p)^3 \right\rangle &= \left\langle (\Delta N_p)^3 \right\rangle + \sum_R \left\langle (\Delta N_R)^3 \right\rangle \langle n_p \rangle_R^3 \\ &+ 3 \sum_R \left\langle (\Delta N_R)^2 \right\rangle \langle n_p \rangle_R \left\langle (\Delta n_p)^2 \right\rangle_R + \sum_R \langle N_R \rangle \left\langle (\Delta n_p)^3 \right\rangle_R \end{aligned} \tag{3.46}$$

$$\begin{aligned} \left\langle (\Delta \hat{N}_p)^4 \right\rangle_c &= \left\langle (\Delta N_p)^4 \right\rangle_c + \sum_R \left\langle (\Delta N_R)^4 \right\rangle \langle n_p \rangle_R^4 \\ &+ 6 \sum_R \left\langle (\Delta N_R)^3 \right\rangle \langle n_p \rangle_R^2 \left\langle (\Delta n_p)^2 \right\rangle_R + \sum_R \left\langle (\Delta N_R)^2 \right\rangle \Big[3 \left\langle (\Delta n_p)^2 \right\rangle_R^2 \\ &+ 4 \langle n_p \rangle_R \left\langle (\Delta n_p)^3 \right\rangle_R \Big] + \sum_R \langle N_R \rangle \left\langle (\Delta n_p)^4 \right\rangle_{R,c} . \end{aligned} \tag{3.47}$$

The factors $\langle (\Delta n_h)^2 \rangle_R$, $\langle (\Delta n_h)^2 \rangle_R$ and $\langle (\Delta n_h)^4 \rangle_{R,c}$ vanish for those resonances which have only one decay-channel or for which the number of formed hadrons $n_{h,r}^R$ of species h is the same in each decay-chanel r. As mentioned before, the subscript c denotes the value of the corresponding cumulant. The first three cumulants are equal to the corresponding central moments, which is why we can omit the subscript, whereas we cannot omit if we consider the fourth cumulant, which differs from the fourth central moment. That is why we retained the subscript c in both Eq. (3.42) and Eq. (3.47).

We may now - exactly as in the previous Section - compute the ratios of susceptibilities as defined before. We should mention that in our framework primordial protons and anti-protons are uncorrelated and no baryonic or anti-baryonic resonance decays into an anti-proton or proton, the formula given by Eq.(3.43) remains valid for the susceptibilities of the net proton distribution even when resonance decays are included.

In Fig. 3.2, we see the dependence of ratios of susceptibilities as function of the collision energy $\sqrt{s}$ and the comparison with Fig. 3.1, where the resonance decays were not taken into account. The **empty squares** show the same as in Fig. 3.1, the **empty diamonds** show the average influence of the resonance decays on the net-proton fluctuations. The **empty triangles** depict the full impact of resonance decays and include the probabilistic contribution.

3.4.1 Particle correlation after resonance decays and the Generating Function

As we have already mentioned, the resonance decay has a probabilistic character, which causes the particle number fluctuations in the final state. The main goal of this subsection is to provide information on how to determine the particle correlation. The statistical cen-

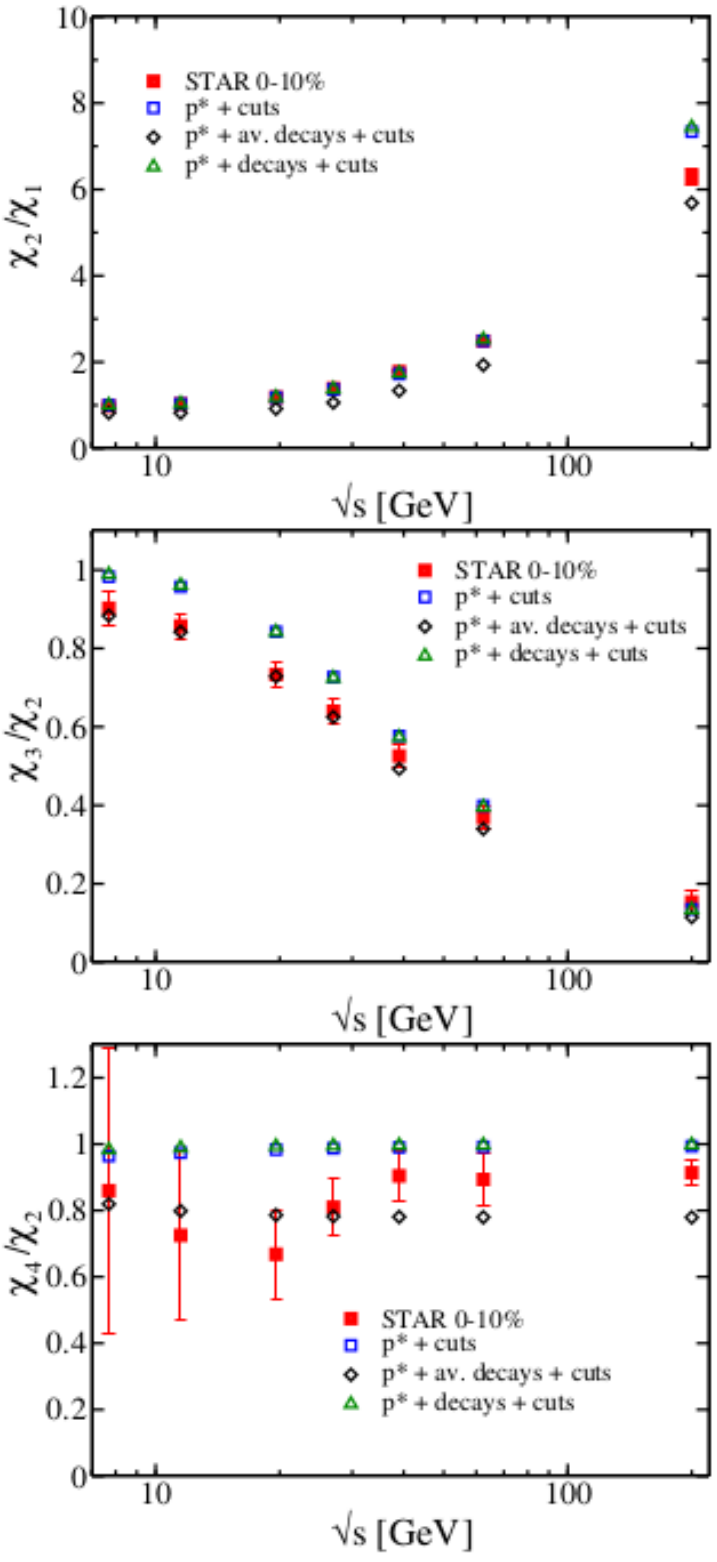

Figure 3.2: Ratios of susceptibilities as function of the collision energy with resonance decays taken into account. Taken from [8].

tral moments can be found from the following function called the **generating function**:

$$G \equiv \prod_R \left(\sum_r b_r^R \prod_i \lambda_i^{n_{i,r}^R} \right)^{N_R} \tag{3.48}$$

where b_r^R is the branching ratio of the $r-$th branch, $n_{i,r}^R$ the number of $i-$th particles produced in that decay mode and r runs over all branches with requirement $\sum_r b_r^R = 1$. The λ_i are auxiliary parameters set to one in the final formulae.

The averages from resonance decays can expressed as:

$$\bar{N_i} \equiv \sum_R \langle N_i \rangle_R = \lambda_i \frac{\partial}{\partial \lambda_i} G = \sum_R N_R \sum_r b_r^R n_{i,r}^R \equiv \sum_R N_R \langle n_i \rangle_R \,, \tag{3.49}$$

$$\begin{aligned} \overline{N_i N_j} &\equiv \sum_R \langle N_i N_j \rangle_R = \lambda_i \frac{\partial}{\partial \lambda_i} \left(\lambda_j \frac{\partial}{\partial \lambda_j} G \right) \\ &= \sum_R [N_R(N_R - 1) \langle n_i \rangle_R \langle n_j \rangle_R + N_R \langle n_i n_j \rangle_R], \end{aligned} \tag{3.50}$$

where $\langle n_i n_j \rangle \equiv \sum_r b_r^R n_{i,r}^R n_{j,r}^R$.

The origin of the formula defined by Eq. (3.48) is given by the fact that the normalized probability distribution $P(N_R^r)$ for the decay of N_R resonances is the following:

$$P(N_R^r) = N_R! \prod_r \frac{(b_r^R)^{N_R^r}}{N_R^r!} \delta \left(\sum_r N_R^r - N_R \right), \tag{3.51}$$

where N_R^r denotes the numbers of $R-$th resonances decaying via $r-$th branch.

The scaled variance ω_R^{i*} due to decays of $R-$th resonances will then read

$$\omega_R^{i*} \equiv \frac{\langle N_i^2 \rangle_R - \langle N_i \rangle_R^2}{\langle N_i \rangle_R} = \frac{\langle n_i^2 \rangle_R - \langle n_i \rangle_R^2}{\langle n_i \rangle_R} \equiv \frac{\sum_r b_r^R (n_{i,r}^R)^2 - (\sum_r b_r^R n_{i,r}^R)^2}{\sum_r b_r^R n_{i,r}^R}. \tag{3.52}$$

We can immediately see that Eq. (3.52) is equal to 0, if either $n_{i,r}^R$ are the same in all decay channels or if there is only one decay channel, which would mean $b_1^R = 1$. Also, Eq. (3.49) and (3.50) assume fixed values of N_R, while in reality, N_R fluctuates, due to which we finally arrive at

$$\omega_R^i \equiv \frac{\langle \langle N_i^2 \rangle \rangle_T - \langle \langle N_i \rangle_R \rangle_T^2}{\langle \langle N_i \rangle_R \rangle_T} = \omega_R^{i*} + \langle n_i \rangle_R \omega_R, \tag{3.53}$$

where the scaled variance

$$\omega_R = \frac{\langle N_R^2 \rangle_T - \langle N_R \rangle_T^2}{\langle N_R \rangle_T} \tag{3.54}$$

corresponds to the thermal fluctuation of the number of resonances.

Chapter 4

Multiplicity fluctuations for a resonance gas model with chemical non-equilibrium

We will now concentrate on the multiplicity fluctuations while considering a resonance gas model with chemical non-equilibrium. Before we do that, we will summarize some of the facts we have stated in the previous Chapters. The effect of resonance decays will also be taken into account. As such, we approximate the hadron gas by a collection of free particles [6], distributed according to

$$dN_i = \frac{d^3x d^3p}{(2\pi)^3} g_i \left\{ \exp\left(\frac{E-\mu_i}{T}\right) \pm 1 \right\}^{-1} \tag{4.1}$$

where μ_i is the total chemical potential and $E = \sqrt{m_i^2 + p^2}$ and $\pm$ depends on whether the particle is a fermion or a boson, $g_j = 2J_i + 1$ is a spin degeneration factor corresponding to the statistical weight ($g_\pi = 3, g_K = 4, g_\rho = 9, \dots$).

The pressure generated by the distribution (4.1) is given by

$$P = T \sum_i \pm g_i \int \frac{d^3p}{(2\pi)^3} \ln\left\{ 1 \pm \exp\left(\frac{\mu_i - E}{T}\right) \right\}. \tag{4.2}$$

where we assumed a unit volume $V = 1$.

4.1 Chemical potentials in a HRG model with chemical non-equilibrium

In order to be able to lay down the formalism describing the state of chemical non-equilibrium, we have to consider the chemical potentials first. They start building up once the chemical equilibrium is lost. We assume that the population of the excited states remains in equilibrium with the particles formed in their decay [6]. Furthermore, we set

the chemical potential of the mother equal to the sum of the chemical potentials of the daughters. If there are several decay channels (i. e. more than one) open, we multiply the various final state configurations with the corresponding branching ratio.

Let us for example consider the states $\rho(770)$, $\Delta(1231)$ and $a_2(1320)$. Considering the aforementioned assumptions, this leads to

$$\mu_\rho = 2\mu_\pi,$$

$$\mu_\Delta = \mu_\pi + \mu_N$$

and

$$\mu_{a_2} = 2.8\mu_\pi + 0.1\mu_K + 0.15\mu_\eta.$$

The condition for partial equilibrium determines the chemical potentials of the excited states as functions of the potentials corresponding to the stable particles

$$\sigma = \{\pi, K, \eta, N, \Lambda, \Sigma, \Xi, \Omega\}$$

- see Chapter 3 - which occur as end products of the decay chain

$$\mu_i = \sum_\sigma d_i^\sigma \mu_\sigma \tag{4.3}$$

where d_i^σ is the mean number of stable particles emerging in the decay of the level i.

We consider only configurations for wich the number of particles and antiparticles is the same (e.g. $\mu_N = \mu_{\bar{N}}$. Furthermore, if we take the SU(3) limit of lattice QCD into account, the situation simplifies remarkably. The chemical potentials of the stable **mesons** take a common value μ_π, whereas the stable **baryons** take a value of μ_N. In the SU(3) limit, Ω is unstable and decays into $\Xi\bar{K}$. Consequently, the equation of state (i.e. Eq. (4.2) involves only two independent chemical potentials and reads

$$P = P(T, \mu_\pi, \mu_N).$$

Due to the high level splittings generated by $m_s - m_u$, the various members of a multiplet develop somewhat different chemical potentials in the course of the expansion. The assumption is that the effect is proportional to the number of strange quarks or antiquarks contained in the particle, which would then mean

$$\mu_\Lambda - \mu_N = \mu_\Sigma - \mu_N = \mu_\Xi - \mu\Sigma = \mu_K - \mu_\pi, \tag{4.4}$$

$$\mu_\Omega - \mu_\Lambda = 3(\mu_K - \mu_\pi). \tag{4.5}$$

If we use this approximation, reactions such as $\pi\Sigma \to \bar{K}N$ are in equilibrium, because they conserve the number of strange quarks and antiquarks. For η we use

$$\mu_\eta = (4\mu_K - \mu_\pi)/3.$$

This value is suggested by ideal mixing where the probability that this particle contains an $s\bar{s}$ pair. Therefore, the chemical potentials of the various levels can be rewritten in terms of μ_π, μ_K and μ_N:

$$\mu_i = \bar{d}_i^{\,\pi}\mu_\pi + \bar{d}_i^{\,K}\mu_K + \bar{d}_i^{\,N}\mu_N \tag{4.6}$$

where the coefficients are determined by the conditions of the partial chemical equilibrium mentioned earlier:

$$\bar{d}_i^{\,\pi} = d_i^\pi - \frac{1}{3}d_i^\eta - d_i^\Lambda - d_i^\Sigma - 2d_i^\Xi - 2d_i^\Omega, \tag{4.7}$$

$$\bar{d}_i^{\,K} = d_i^K + \frac{4}{3}d_i^\eta + d_i^\Lambda + d_i^\Sigma + 2d_i^\Xi + 3d_i^\Omega, \tag{4.8}$$

$$\bar{d}_i^{\,N} = d_i^N + d_i^\Lambda + d_i^\Sigma + d_i^\Xi + d_i^\Omega. \tag{4.9}$$

We may also write down the number densities conjugate to μ_π, μ_K, μ_N. The densities will be denoted $\bar{n}_\pi$, $\bar{n}_K$ and $\bar{n}_N$, respectively:

$$\bar{n}_\pi = \partial P/\partial\mu_\pi = \sum_i \bar{d}_i^{\,\pi} n_i, \tag{4.10}$$

$$\bar{n}_K = \partial P/\partial\mu_K = \sum_i \bar{d}_i^{\,K} n_i, \tag{4.11}$$

$$\bar{n}_N = \partial P/\partial\mu_N = \sum_i \bar{d}_i^{\,N} n_i. \tag{4.12}$$

where n_i is the occupation number of the level i per unit volume.

Obviously, the coefficient $\bar{d}_i^{\,N}$ is equal to one for baryonic levels and equal to zero for mesonic levels. Thus, the number density $\bar{n}_K$ counts the number of baryons and antibaryons per unit volume.

The coefficient $\bar{n}_K$ enumerates strange quarks and antiquarks. This is to be understood the following way: the excitations decay according to the branching ratios given in the particle data tables and count the strange valence quarks contained in the stable particles emerged in these decays. For low T, $\bar{n}_K$ counts the kaons occuring after disintegration of the excited states.

If strange baryons are rare, $\bar{n}_\pi$ counts the pions emerging if all resonances decayed.

4.2 Cumulants in a HRG model with chemical non-equilibrium

We will now use a similar formalism as in Chapter 3, but will now implement the chemical non-equilibrium. This is represented by the different expression for the total chemical potential μ_i. Whereas for the chemical equilibrium, the potential reads

$$\mu_i = B_i\mu_B + S_i\mu_S + Q_i\mu_Q,$$

in case of chemical non-equilibrium, it is given by Eq. (4.6). The expressions for the first four cumulants - the first one C_1 being the Mean M and the second one C_2 being the variance σ - are the following:

$$C_1 = M = \langle N_i \rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right) \ln Z_i \right]_{T,V} = \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i, \tag{4.13}$$

$$\begin{aligned} C_2 &= \sigma^2 = \left\langle (\Delta N_i)^2 \right\rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^2 \ln Z_i \right]_{T,V} && (4.14) \\ &= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp n_i), && (4.15) \end{aligned}$$

$$\begin{aligned} C_3 &= \left\langle (\Delta N_i)^3 \right\rangle = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^3 \ln Z_i \right]_{T,V} && (4.16) \\ &= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp 3n_i + 2n_i^2), && (4.17) \end{aligned}$$

$$\begin{aligned} C_4 &= \left\langle (\Delta N_i)^4 \right\rangle - 3 \left\langle (\Delta N_i)^2 \right\rangle^2 = \left[\left(T \frac{\partial}{\partial \mu_i} \right)^4 \ln Z_i \right]_{T,V} && (4.18) \\ &= \frac{V g_i}{2\pi^2} \int_0^{+\infty} p^2 dp \; n_i (1 \mp 7n_i + 12n_i^2 \mp 6n_i^3). && (4.19) \end{aligned}$$

where

$$n_i = \frac{1}{\exp[(E_i - \mu_i)/T] \pm 1},$$
$$E_i = \sqrt{p^2 + m_i^2}$$

and

$$\begin{aligned} \mu_i &= (d_i^\pi - \frac{1}{3} d_i^\eta - d_i^\Lambda - d_i^\Sigma - 2d_i^\Xi - 2d_i^\Omega)\mu_\pi \\ &+ (d_i^K + \frac{4}{3} d_i^\eta + d_i^\Lambda + d_i^\Sigma + 2d_i^\Xi + 3d_i^\Omega)\mu_K \\ &+ (d_i^N + d_i^\Lambda + d_i^\Sigma + d_i^\Xi + d_i^\Omega)\mu_N. \end{aligned}$$

with the partition function

$$\ln Z_i(T, V, \mu_i) = \frac{V g_i}{2\pi^2} \int_0^{+\infty} \pm p^2 dp \ln[1 \pm \exp(-(E_i - \mu_i)/T)].$$

4.3 Effects of resonance decays

The equations in the previous section apply for all the particles in the system, as is the case in the previous Chapter. If we want to include the resonance decays and write down the cumulants for stable particles h, we receive the following equations from the previous Chapter:

$$\left\langle \hat{N}_h \right\rangle = \langle N_h \rangle + \sum_R \langle N_R \rangle \langle n_h \rangle_R \tag{4.20}$$

$$\left\langle (\Delta\hat{N}_h)^2 \right\rangle = \left\langle (\Delta N_h)^2 \right\rangle + \sum_R \left\langle (\Delta N_R)^2 \right\rangle \langle n_h \rangle_R^2 + \sum_R \langle N_R \rangle \left\langle (\Delta n_h)^2 \right\rangle_R , \tag{4.21}$$

$$\begin{aligned} \left\langle (\Delta\hat{N}_h)^3 \right\rangle &= \left\langle (\Delta N_h)^3 \right\rangle + \sum_R \left\langle (\Delta N_R)^3 \right\rangle \langle n_h \rangle_R^3 \\ &+ 3 \sum_R \left\langle (\Delta N_R)^2 \right\rangle \langle n_h \rangle_R \left\langle (\Delta n_h)^2 \right\rangle_R + \sum_R \langle N_R \rangle \left\langle (\Delta n_h)^3 \right\rangle_R \end{aligned} \tag{4.22}$$

$$\begin{aligned} \left\langle (\Delta\hat{N}_h)^4 \right\rangle_c &= \left\langle (\Delta N_h)^4 \right\rangle_c + \sum_R \left\langle (\Delta N_R)^4 \right\rangle \langle n_h \rangle_R^4 \\ &+ 6 \sum_R \left\langle (\Delta N_R)^3 \right\rangle \langle n_h \rangle_R^2 \left\langle (\Delta n_h)^2 \right\rangle_R + \sum_R \left\langle (\Delta N_R)^2 \right\rangle \Big[3 \left\langle (\Delta n_h)^2 \right\rangle_R^2 \\ &+ 4 \langle n_h \rangle_R \left\langle (\Delta n_h)^3 \right\rangle_R \Big] + \sum_R \langle N_R \rangle \left\langle (\Delta n_h)^4 \right\rangle_{R,c} . \end{aligned} \tag{4.23}$$

Here, $\langle n_h \rangle_R = d_i^\sigma$ introduced in the previous Section, $\hat{N}_h$ is the final distribution of the stable particle h, $\Delta N_R = N_R - \langle N_R \rangle$, $(\Delta n_h)_R = n_h - \langle n_h \rangle_R$.

Conclusion and Outlook

In **Chapter 1**, the mathematical apparatus necessary to perform all the calculations was provided. Statistical moments in form of central moments were introduced along with close elaboration of the first four moments (mean M, variance σ, skewness S, kurtosis κ), which are of great importance for describing multiplicity fluctuations in the statistical model. Moreover, the basics of canonical and grandcanonical formalism along with the Wick-rotated fugacities which are useful for expressing the canonical partition function in terms of the grandcanonical one were introduced and the first four statistical moments were explicitly written down. Furthermore, the methods of computing the asymptotic fluctuations in the canonical ensemble were provided and the corresponding method called "saddle-point expansion" along with a quantity called "scaled variance", which effectively describes the fluctuations, was introduced.

In **Chapter 2**, a simple model in form of a classical pion gas and a full hadron gas was considered and the corresponding calculations using the formalism introduced in Chapter 1 were performed. In the former case, only one charge (the electric charge) was considered, while the other two (strangeness and baryon number) were neglected. In the latter case, all three charges were considered. The scaled variances for both net charge and total particle number distributions were provided for both cases.

Chapter 3 provided formalism for multiplicity fluctuations in a hadron resonance gas model with the assumption of chemical equilibrium. At first, the exact charge conservation in statistical systems was elaborated and the exact conservation laws were implemented. Afterwards, the hadron resonance gas model was introduced and the corresponding susceptibilities were defined. Using said susceptibilities, the first four cumulants in the ideal hadron gas were derived. Subsequently, the loss of chemical equilibrium and the chemical freeze-out parametrization were elaborated, which enabled us to finally lay down the formalism necessary for the resonance decays to be accounted for.

In **Chapter 4**, the multiplicity fluctuations for a resonance gas model with chemical non-equilibrium were introduced. The corresponding adjusted chemical potentials were introduced and the cumulants in a hadron resonance gas where chemical non-equilibrium is taken into account were introduced and the resonance decays were also accounted for. So far, only the comparison to the chemical equilibrium case was performed, as we have used a simplified approximation in form of the SU(3) limit.

Contrary to the original goal, the chemical non-equilibrium case was not fully elaborated. However, we have laid down the formalism we could base upon in further research.

Appendix

In this Chapter, some brief trivia on Bessel functions and their practical use will be presented. A brief theoretical overview concerning the mathematical apparatus used in this research project will be provided. Specifically, the Bessel functions will be addressed.

Bessel functions

The Bessel functions were first defined by Daniel Bernoulli and later on generalized by Friedrich Bessel. They are defined as canonical solutions $y = y(x)$ of a differential equation better known as the "Bessel differential equation":

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0 \tag{4.24}$$

where the arbitrary complex number α is called the **order** of the Bessel function.

The Bessel functions can be distinguished according to the parameter α. If α is an integer, we talk about **cylinder functions** or the **cylindrical harmonics** because they appear in the solution to Laplace's equation in cylindrical coordinates. Once α is a half-integer (i.e. for each α there is an $n \in \mathbb{N}$ such that $\alpha = n + \frac{1}{2}$), then we call the functions $y = y(x)$ the **spherical Bessel functions** and they are obtained when the **Helmholtz equation** is solved in spherical coordinates.

Modified Bessel functions

The Bessel functions are well defined, even though their argument x is complex. However, if a special case occurs - when this argument is purely complex - we talk about the **modified Bessel function** (also called the **hyperbolic Bessel function**) of the **first kind** (denoted as $I_\alpha(x)$) and of the **second kind** (denoted as $K_\alpha(x)$). Those are defined by the following equations:

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{4.25}$$

$$K_\alpha(x) = \frac{\pi}{2}\frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)} \tag{4.26}$$

Those solutions are two independent solutions of the modified Bessel equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \alpha^2)y = 0. \tag{4.27}$$

Unlike the classical Bessel functions, which oscillate as functions of a real argument in both $I_\alpha(x)$ and $K_\alpha(x)$, the Modified Bessel functions grow exponentially. We will now present the integral form of the Modified Bessel functions (assuming that $Re(x) > 0$):

$$I_\alpha(x) = \frac{1}{\pi}\int_0^\pi \exp(x\cos(\theta))\cos(\alpha\theta)d\theta - \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \exp(-x\cosh t - \alpha t)dt \tag{4.28}$$

$$K_\alpha(x) = \int_0^\infty \exp(-x\cosh t)\cosh(\alpha t)dt \tag{4.29}$$

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