Multiplicity Fluctuations

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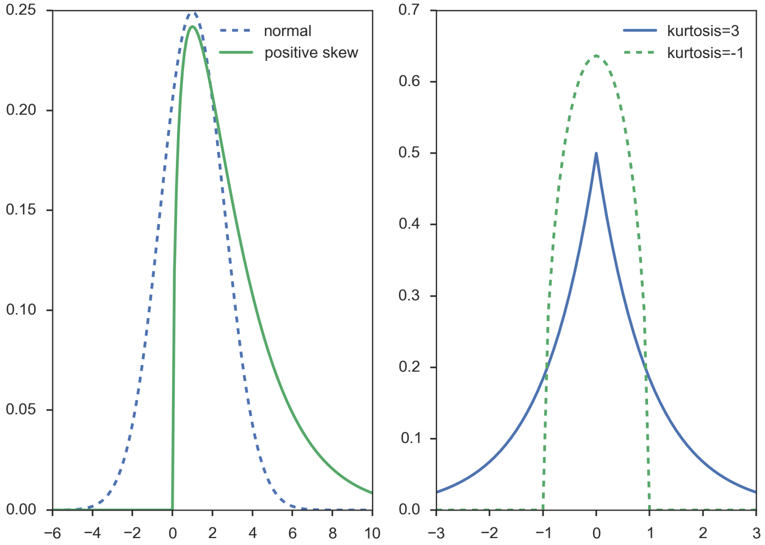
Outline

1. Statistical moments
2. Multiplicity fluctuations within a simple model
3. Multiplicity fluctuations within a Hadron Gas Model (HRG) with chemical equilibrium
4. Multiplicity fluctuations within a Hadron Gas Model (HRG) without chemical equilibrium
5. Conclusion and outlook

Statistical moments

- m-th statistical moment $\varphi_m(X)' : \varphi_m(X)' = E(X^m)$
- m-th central moment $\varphi_m(X) : \varphi_m(X) = E(X - EX)^m$
- first four central moments are of great significance
- **mean:** $M = \varphi_1$, **variance:** $\sigma^2 = \varphi_2$
- **skewness:** $S = \varphi_3/\varphi_2^{3/2}$ - measure of the assymetry of the probability distribution
- **kurtosis:** $\kappa = \varphi_4/\varphi_2^2$ - measure of the "tailedness" of the probability distribution

Skewness (left) and kurtosis (right).



Calculation of the multiplicity fluctuations within the statistical model

- grandcanonical and canonical ensemble assumed, event-by-event distributions of conserved quantities - characterized by the moments (M, σ, S, κ)
- introduction of the following products: $S\sigma = \varphi_3/\varphi_2$, $\kappa\sigma^2 = \varphi_4/\varphi_2, M/\sigma^2 = \varphi_1/\varphi_2$, $S\sigma^3/M = \varphi_3/\varphi_1$ -the volume term in the distribution gets obviously cancelled; direct comparison of experimental measurement and theoretical calculation possible
- large volume limit (V$\rightarrow \infty$) - all statistical ensembles (MCE, CE, GCE) equivalent

Partition functions in statistical ensembles - GC formalism

- HRG model - all relevant degrees of freedom contained in the partition function
- confined, strongly interacting matter - interactions that result in resonance formation included
- **GC partition function:** $Z_{GC}(\lambda_j) = \prod_j \exp\left[\sum_{n_j=1}^{+\infty} \frac{z_j(n_j)\lambda_j^{n_j}}{n_j}\right]$ where $z_j(n_j) = (\mp 1)^{n_j+1} \frac{g_j V}{2\pi^2 n_j} T m_j^2 K_2\left(\frac{n_j m_j}{T}\right)$ is the single particle partition function
- $K_2 \ldots$ modified Bessel function, $V \ldots$ volume of the hadron gas
- $\lambda_j = \exp(\frac{\mu_j}{T}) \ldots$ fugacity for each particle species j, $m_j \ldots$ hadron mass
- $\mu_j \ldots$ chemical potential of a particle species j, $g_j = 2J_j + 1 \ldots$ spin degeneracy
- $\mp \ldots$ upper sign for fermions, lower sign for bosons

Partition functions in statistical ensembles - canonical formalism

- constraint - fixed charges $\rightarrow$ partition function not factorized into one-species expressions
- let $\vec{Q} = (Q_1, Q_2, Q_3) = (B, S, Q) \cdots$ vector of charges
- let $\vec{q_j} = (q_{1,j}, q_{2,j}, q_{3,j}) = (b_j, s_j, q_j) \cdots$ vector of charges of the hadron species j
- **Wick-rotated fugacities:** $\lambda_j = \exp[i \sum_i q_{i,j}\phi_i]$
- **Canonical partition function:**
  $Z_{\vec{Q}} = \left[\prod_{i=1}^{3} \frac{1}{2\pi} \int_0^{2\pi} d\phi_i e^{-iQ_i\phi_i}\right] Z_{GC}(\lambda_j)$
- $h \ldots$ set of hadron species: $\lambda_j \rightarrow \lambda_h \lambda_j$

Results of the first four moments

$$\langle N_h \rangle = \frac{1}{Z_{\vec{Q}}} \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} |_{\lambda_h=1} = \sum_{j\in h} \sum_{n_j=1}^{\infty} z_j(n_j) \frac{Z_{\vec{Q}-n_j\vec{q}_j}}{Z_{\vec{Q}}}$$

$$\langle N_h^2 \rangle = \frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right] |_{\lambda_h=1} = \sum_{j\in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \frac{Z_{\vec{Q}-n_j\vec{q}_j}}{Z_{\vec{Q}}} + \sum_{j\in h} \sum_{n_j=1}^{+\infty} z_j(n_j) \sum_{k\in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q}-n_j\vec{q}_j-n_k\vec{q}_k}}{Z_{\vec{Q}}}$$

$$\langle N_h^3 \rangle = \frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right) \right] |_{\lambda_h=1} = \sum_{j\in h} \sum_{n_j=1}^{+\infty} n_j^2 z_j(n_j) \frac{Z_{\vec{Q}-n_j\vec{q}_j}}{Z_{\vec{Q}}} + 3 \left[\sum_{j\in h} \sum_{n_j=1}^{+\infty} n_j z_j(n_j) \sum_{k\in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q}-n_j\vec{q}_j-n_k\vec{q}_k}}{Z_{\vec{Q}}} \right] + \sum_{j\in h} \sum_{n_j=1}^{+\infty} z_j(n_j) \sum_{k\in h} \sum_{n_k=1}^{+\infty} z_k(n_k) \sum_{l\in h} \sum_{n_l=1}^{+\infty} z_l(n_l) \frac{Z_{\vec{Q}-n_j\vec{q}_j-n_k\vec{q}_k-n_l\vec{q}_l}}{Z_{\vec{Q}}}$$

$$\left\langle N_h^4 \right\rangle = \frac{1}{Z_{\vec{Q}}} \left[\frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial}{\partial \lambda_h} \left(\lambda_h \frac{\partial Z_{\vec{Q}}}{\partial \lambda_h} \right) \right) \right) \right] |_{\lambda_h = 1} =$$

$$\sum_{j \in h} \sum_{n_j = 1}^{+\infty} n_j^3 z_j(n_j) \frac{Z_{\vec{Q} - n_j \vec{q}_j}}{Z_{\vec{Q}}} + 4 \left[\sum_{j \in h} \sum_{n_j = 1}^{+\infty} n_j^2 z_j(n_j) \sum_{k \in h} \sum_{n_k = 1}^{+\infty} z_k(n_k) \frac{Z_{\vec{Q} - n_j \vec{q}_j - n_k \vec{q}_k}}{Z_{\vec{Q}}} \right]$$

$$+ 3 \left[\sum_{j \in h} \sum_{n_j = 1}^{+\infty} n_j z_j(n_j) \sum_{k \in h} \sum_{n_k = 1}^{+\infty} n_k z_k(n_k) \frac{Z_{\vec{Q} - n_j \vec{q}_j - n_k \vec{q}_k}}{Z_{\vec{Q}}} \right]$$

$$+ 6 \left[\sum_{j \in h} \sum_{n_j = 1}^{+\infty} n_j z_j(n_j) \sum_{k \in h} \sum_{n_k = 1}^{+\infty} z_k(n_k) \sum_{l \in h} \sum_{n_l = 1}^{+\infty} z_l(n_l) \frac{Z_{\vec{Q} - n_j \vec{q}_j - n_k \vec{q}_k - n_l \vec{q}_l}}{Z_{\vec{Q}}} \right]$$

$$+ [\sum_{j \in h} \sum_{n_j = 1}^{+\infty} z_j(n_j) \sum_{k \in h} \sum_{n_k = 1}^{+\infty} z_k(n_k) \sum_{l \in h} \sum_{n_l = 1}^{+\infty} z_l(n_l)$$

$$\sum_{m \in h} \sum_{n_m = 1}^{+\infty} z_m(n_m) \frac{Z_{\vec{Q} - n_j \vec{q}_j - n_k \vec{q}_k - n_l \vec{q}_l - n_m \vec{q}_m}}{Z_{\vec{Q}}}]$$

Asymptotic fluctuations in the canonical ensemble

- Poissonian distribution of fluctuations: $P_{GC} = \frac{1}{N_j!} \langle N_j \rangle^{N_j} e^{-\langle N_j \rangle}$
- **Canonical partition function:**
  $Z_{\vec{Q}} = \left[\prod_{i=1}^{3} \frac{1}{2\pi} \int_0^{2\pi} d\phi_i e^{-iQ_i\phi_i} \right] Z_{GC}(\lambda_j)$
- integration performed in the complex **w** plane: $w_i = \exp[i\phi_i]$
  $Z_{\vec{Q}} =$
  $\frac{1}{(2\pi i)^3} \oint dw_B \oint dw_S \oint dw_Q w_B^{-B-1} w_S^{-S-1} w_Q^{-Q-1} \exp \sum_j z_{j(1)} w_B^{b_i} w_S^{s_i} w_Q^{q_i}$
- obviously: $w_{B,Q,S}^{-(B,Q,S)} = \exp[-(B,Q,S) \ln w_{B,Q,S}]$
- $g(\vec{w}) = w_B^{b_j-1} w_S^{s_j-1} w_Q^{q_j-1}$; $\rho_{B,S,Q} = \frac{B,S,Q}{V}$
- $f(\vec{w}) = -\rho_B \ln w_B - \rho_S \ln w_S - \rho_Q \ln w_Q + \sum_k \frac{z_{k(1)}}{V} w_B^{b_k} w_S^{s_k} w_Q^{q_k}$
- $Z_{\vec{Q}-\vec{q_j}} = \frac{1}{(2\pi i)^3} \oint dw_B \oint dw_S \oint dw_Q g(\vec{w}) \exp[Vf(\vec{w})]$
- **method:** saddle-point expansion

Multiplicity fluctuations for a simple model I.

- **classical pion gas** - no b or s quarks $\rightarrow \vec{Q} = (0, 0, Q)$
- saddle point: $w_0 = \lambda_Q$
- only π^+ and π^- considered
- $\nu = V$, $g(w) = 1/w$, $f(w) = -\rho_Q \ln w + \frac{z_\pi}{V}(w + \frac{1}{w})$
- $Z_Q = \frac{1}{2\pi i} \oint dw_q w_q^{-Q-1} \exp\left[\sum_{j=\pm 1} z_{\pi(1)} w_Q^{q_j}\right]$
- $s_{\pi^+} = s_{\pi^-} = 0$; $m_{\pi^+} = m_{\pi^-} = 139.57$ MeV
- $z_{j(1)} = (2J_j + 1)\frac{V}{(2\pi)^3} \int d^3p \exp(-\sqrt{p^2 + m_j^2}) = \frac{V}{(2\pi)^3} \int d^3p \exp(-\sqrt{p^2 + m_j^2})$

Multiplicity fluctuations for a simple model II.

$$Z_Q^\pi = \frac{Z_{GC}}{\lambda_Q^Q}\sqrt{\frac{1}{2\pi f''(\lambda_Q)}}\left[\frac{1}{\lambda_Q} + \frac{1}{V}\left(\frac{\gamma(\lambda_Q)}{\lambda_Q} - \frac{\alpha(\lambda_Q)}{\lambda_Q^2} - \frac{1}{\lambda_Q^3 f''(\lambda_Q)}\right) + O(V^{-2})\right]$$

$$Z_{Q-Q_i}^\pi = \frac{Z_{GC}}{\lambda_Q^{Q+1}}\sqrt{\frac{1}{2\pi f''(\lambda_Q)}}[1 + \frac{1}{V}[\gamma(\lambda_Q) + (q_j - 1)\frac{\alpha(\lambda_Q)}{\lambda_Q} - \frac{1}{2}(q_j - 1)(q_j - 2)\frac{1}{\lambda_Q^2 f''(\lambda_Q)}] + O(V^{-2})]$$

- thermodynamical limit $V \to \infty$:
- $\langle \pi^\pm \rangle = z_\pi \frac{Z_{Q\mp1}^\pi}{Z_Q^\pi} = z_\pi \lambda_Q^{\pm1} + O(V^{-1}) = \langle \pi^\pm \rangle_{GC} + O(V^{-1})$

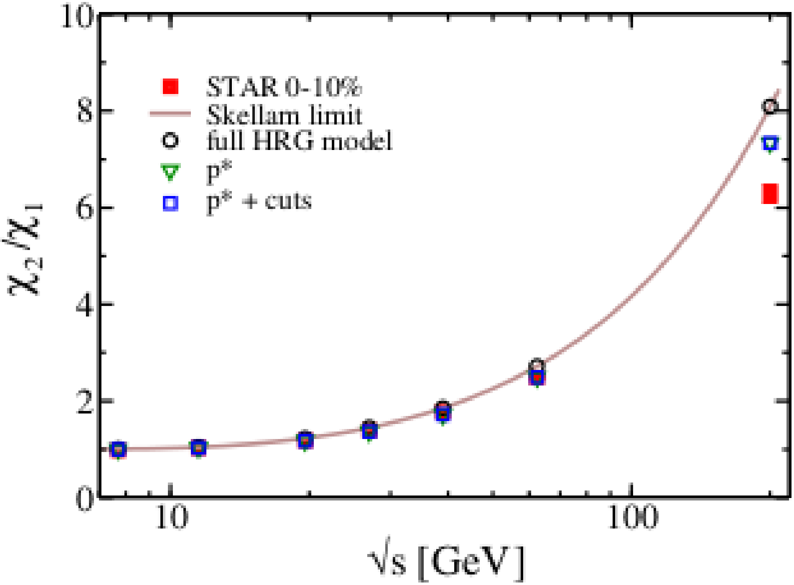
Fluctuations in a hadron resonance gas model with chemical equilibrium

Susceptibilities and cumulants:

- $\chi_l^{(i)} = \frac{\partial^l (P/T)^4}{\partial(\mu_i/T)^l} \mid_T$
- $\chi_1^{(i)} = \frac{1}{VT^3} \langle N_i \rangle_c = \frac{1}{VT^3} \langle N_i \rangle$
- $\chi_2^{(i)} = \frac{1}{VT^3} \left\langle (\Delta N_i)^2 \right\rangle_c = \frac{1}{VT^3} \left\langle (\Delta N_i)^2 \right\rangle$
- $\chi_3^{(i)} = \frac{1}{VT^3} \left\langle (\Delta N_i)^3 \right\rangle_c = \frac{1}{VT^3} \left\langle (\Delta N_i)^3 \right\rangle$
- $\chi_4^{(i)} = \frac{1}{VT^3} \left\langle (\Delta N_i)^4 \right\rangle_c = \frac{1}{VT^3} \left(\left\langle (\Delta N_i)^4 \right\rangle - 3 \left\langle (\Delta N_i)^2 \right\rangle^2 \right)$

Equilibrium pressure:

- $P/T^4 = \frac{1}{VT^3} \sum_i \ln Z_{m_i}^{M/B}(V, T, \mu_B, \mu_Q, \mu_S)$
- $\ln Z_{m_i}^{M/B} = \mp \frac{V g_i}{(2\pi)^3} \int d^3k \ln(1 \mp z_i \exp(-\epsilon_i/T))$


STAR 0-10%
Skellam limit
full HRG model
p*
p* + cuts
χ_2/χ_1
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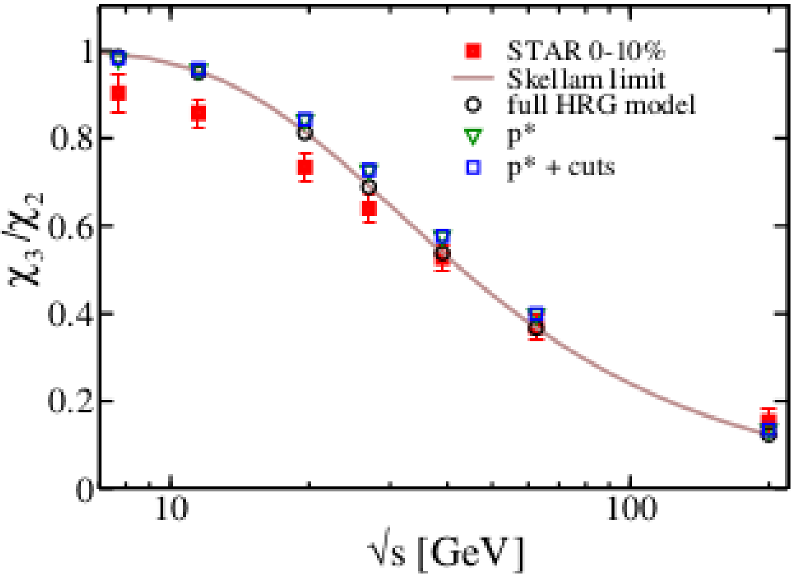

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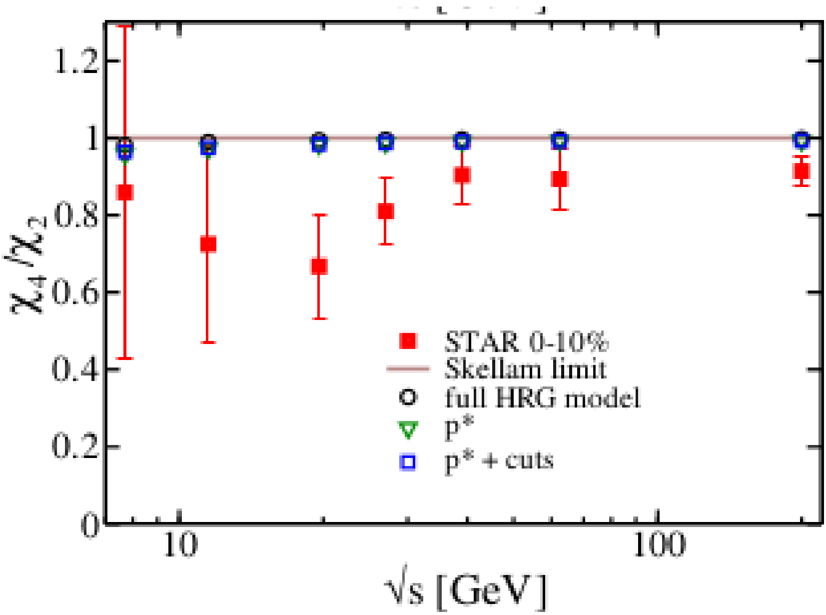

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Skellam limit
full HRG model
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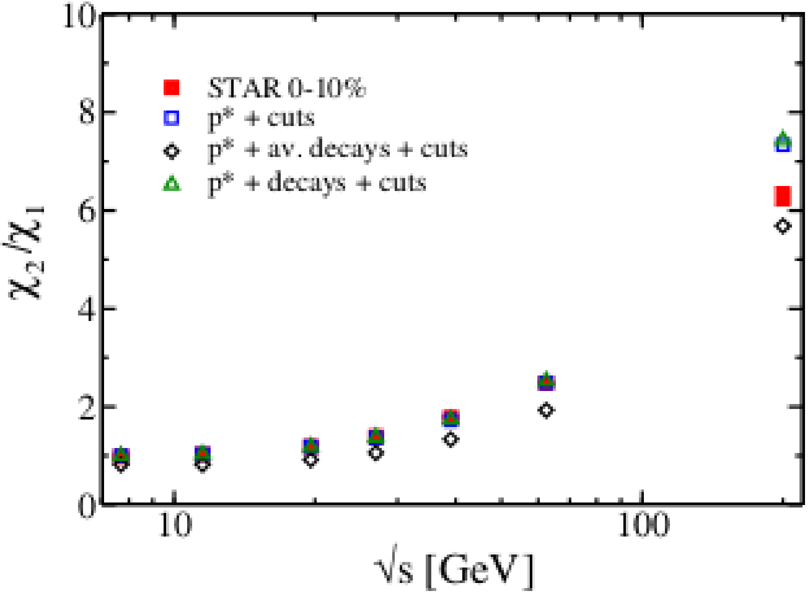
Inclusion of resonances

$$VT^3 \frac{\partial(P/T^4)}{\partial(\mu_h/T)}|_T = \langle N_h \rangle + \sum_R \langle N_R \rangle \langle n_h \rangle_R \tag{1}$$

where $\langle N_h \rangle$ and $\langle N_R \rangle$ are the means of the primordial numbers of hadrons and resonances, respectively. The sum runs over all the resonances in the model.

- 26 particle species we consider stable: $\pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \bar{K}_0, \eta$ and $p, d, \lambda^0, \sigma^+, \sigma^0, \sigma^-, \Xi^0, \Xi^-, \Omega^-$ and their respective anti-baryons
- $\langle n_h \rangle_R \equiv \sum_r b_r^R n_{h,r}^R$
- b_r^R - the branching ratio of the decay-channel and $n_{h,r}^R = 0, 1, \dots$ - number of hadrons h formed in that specific decay-channel.



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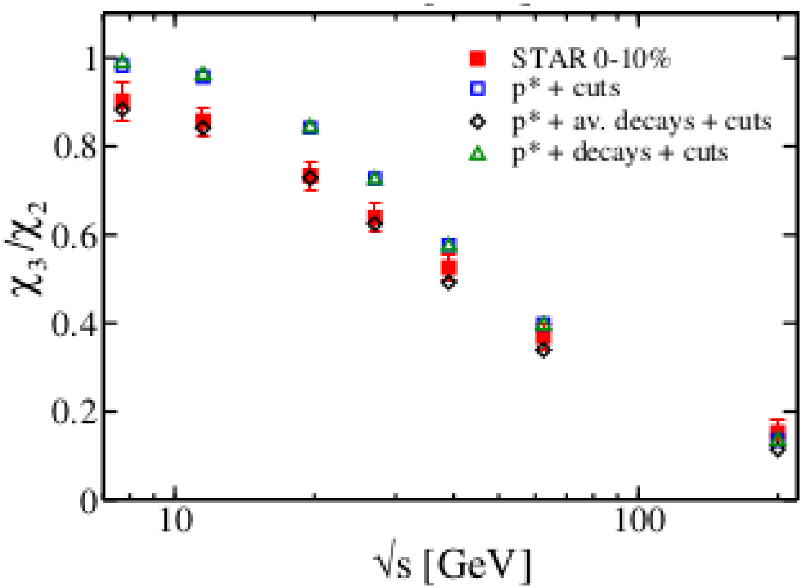
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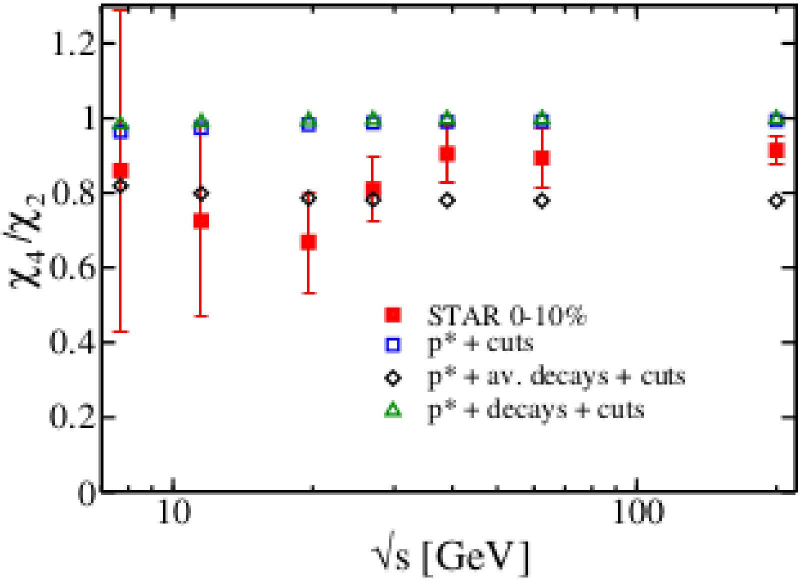

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p* + cuts
p* + av. decays + cuts
p* + decays + cuts

Fluctuations in a hadron resonance gas model with chemical non-equilibrium

- **Chemical equilibrium:** $\mu_i = B_i \mu_B + S_i \mu_S + Q_i \mu_Q$,
- **Chemical non-equilibrium:** $\mu_i = \sum_\sigma d_i^\sigma \mu_\sigma$
- d_i^σ - mean number of stable particles emerging in the decay of the level i
- assumption: chemical potential of the mother equal to the sum of the chemical potentials of the daughters
- only configurations for wich the number of particles and antiparticles is the same (e.g. $\mu_N = \mu_{\bar{N}}$.) considered
- SU(3) limit of lattice QCD taken into account - the chemical potentials of the stable **mesons** take a common value μ_π, whereas the stable **baryons** take a value of μ_N
- the equation of state involves only two independent chemical potentials and reads $P = P(T, \mu_\pi, \mu_N)$.

Conclusion and outlook

So far, the following has been introduced:

- ways to calculate multiplicity fluctuations within the statistical model
- ways to calculate multiplicity fluctuations for a classical pion gas
- multiplicity fluctuations in a hadron resonance gas model, where a particle production from resonance decays and a thermal equilibrium is assumed
- multiplicity fluctuations in a hadron resonance gas model, where a particle production from resonance decays and a thermal non-equilibrium is taken into account. The SU(3) limit of lattice QCD was taken into account.

Further research:

- Generalize the results obtained in the chapter concerning chemical non-equilibrium, where the SU(3) limit will not be taken into account and provide similar results as in the case of chemical equilibrium (see Figures above).

Backup - Saddle-point expansion I.

- $f(\vec{w}) = -\rho_B \ln w_B - \rho_S \ln w_S - \rho_Q \ln w_Q + \sum_k \frac{z_{k(1)}}{V} w_B^{b_k} w_S^{s_k} w_Q^{q_k}$
- saddle point: $\vec{w_0} = (\lambda_B, \lambda_S, \lambda_Q)$; $\frac{\partial f(\vec{w})}{\partial w_k}|_{\vec{w_0}} = 0$
- **complex d-dimensional integral:**

$$I(\nu) = \left[\prod_{k=1}^{d} \int_{\Gamma_k} dw_k \right] g(\vec{w}) e^{\nu f(\vec{w})}$$

- $\Gamma_k \ldots$ paths of integration
- ν large - dominant contribution to the integral comes from the small part of the path in the neighbourhood of the saddle point $\vec{w_0}$
- Taylor expansion: $f(\vec{w}) \simeq f(\vec{w_0}) + \frac{1}{2} \sum_{i,k} \frac{\partial^2 f}{\partial w_i w_k}|_{\vec{w_0}}$

Backup - Saddle-point expansion II.

- choice of a **real** integration variable t_k: $w_k - w_{0k} = e^{i\phi_k} t_k$; $\phi_k \dots$ phase
- "deformation" of the original path into a **line** in the complex plane
- only a small segment around the saddle point $\vec{w_0}$ contributes to the total integral value:

$$I(\nu) \simeq e^{\nu f(\vec{w_0})} \frac{1}{(2\pi)^d} \left[\prod_{k=1}^{d} \int_{-\infty}^{+\infty} dt_k \right] g(\vec{w(\vec{t})}) e^{-\frac{1}{2}\nu \vec{t}^T \mathbf{H} \vec{t}}$$

where $\mathbf{H} \dots$ Hessian matrix of $f(\vec{w})$

Backup - Saddle-point expansion III.

- expansion of $g(\vec{w})$ into a Taylor series around $\vec{w} = \vec{w_0}$
- **H** diagonalizable $\rightarrow \exists \mathbf{A} : \mathbf{H'} = \mathbf{A}\mathbf{H}\mathbf{A}^{\mathbf{T}}$
- **final solution:**

$$I(\nu) \simeq \exp(\nu f(\vec{w_0}))\sqrt{\frac{1}{(2\pi\nu)^d det\mathbf{H}}}\,[g(\vec{w_0})+$$

$$\frac{1}{\nu}\left[-\frac{1}{2}\sum_{k,m=1}^{d}\frac{\partial^2 g(\vec{w})}{\partial w_k \partial w_m}|_{\vec{w_0}}\left(\sum_{i=1}^{d}\frac{A_{im}A_{ik}}{h_i}\right)|_{\vec{w_0}} + \sum_{k=1}^{d}\alpha_i\frac{\partial g(\vec{w})}{\partial w_i}|_{\vec{w_0}} + \gamma g(\vec{w_0})\right]]$$