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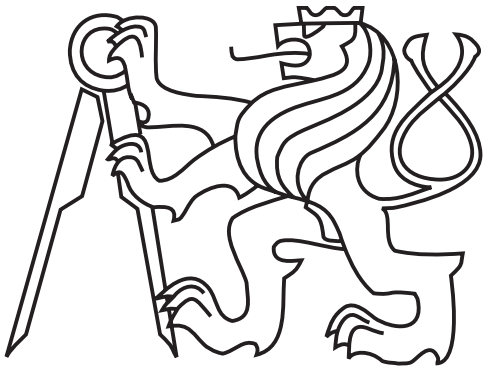

RESEARCH PROJECT

Three-body Interactions in Mean-Field Model of Nuclei and Hypernuclei

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Thesis title: **Three-body Interactions in Mean-Field Model of Nuclei and Hypernuclei**

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Abstract: In this work, we study the effects of the three-nucleon NNN interactions in the mean-field model of nuclei and hypernuclei. The mean field is constructed self-consistently using the Hartree-Fock method. The Hartree-Fock method is derived in the proton-neutron-Λ formalism for the Hamiltonian which includes the three-body NNN and ΛNN potentials. In the computer code the realistic NN and NNN interactions $\mathrm{NNLO_{sat}}$ are implemented, as well as the chiral LO ΛN potential with cut-off $\lambda = 550$ MeV. The computer code runs in the static or in the dynamic mode. The code in the static mode solves the Hartree-Fock equations for the nuclear core without the ΛN interactions. The static mode is used for calculations of the nuclear density distributions, the charge radii, and the Hartree-Fock energies of the ^{40}Ca and the ^{16}O. The NNN interactions have significant effect on the nuclear density distributions which are in a good agreement with the RMF model. The charge radii are also influenced by the NNN force and are enhanced closer to their respective experimental values. The neutron single-particle energies in the ^{40}Ca and the ^{16}O are studied as well. The NNN force considerably shrinks the gaps between the major shells and the gaps between the single-particle levels in the given major shell. The dynamic mode is used for study of the Λ single-particle spectra of the $^{41}_{\Lambda}$Ca and the $^{17}_{\Lambda}$O. We observe that the NNN force shrinks the gaps between the major shells and we obtain the opposite ordering of the p- levels in the $^{17}_{\Lambda}$O.

Keywords: NNN interactions, Hartree-Fock method, Λ hypernuclei, mean-field model

Název práce: **Tříčásticové síly v modelu středního pole jader a hyperjader**

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Abstrakt: V této práci se věnujeme studiu efektů třínukleonových NNN interakcí v modelu středního pole jader a hyperjader. Střední pole je konstruováno self-konzistentně pomocí Hartree-Fockovy metody. Hartree-Fockova metoda je odvozena v proton-neutron-Λ formalismu pro Hamiltonián obsahující tříčásticové NNN a ΛNN potenaciály. V počítačovém kódu jsou implementovány realistické NN a NNN interakce $\mathrm{NNLO_{sat}}$ a chirální LO ΛN potenciál s cut-offem $\lambda = 550$ MeV. Počítačový kód běží ve statickém nebo dynamickém módu. Kód ve statickém módu řeší Hartree-Fockovy rovnice pro jaderný core bez ΛN interakcí. Statický mód byl použit pro výpočty rozložení jaderných hustot, nábojových poloměrů a Hartree-Fock energií ^{40}Ca a ^{16}O. NNN interakce mají silný efekt na rozložení jaderných hustot, které jsou v dobré shodě s RMF modelem. Nábojové poloměry jsou také ovlivněny NNN silou a blíží se více ke svým experimentálním hodnotám. Neutronové jednočásticové energie ^{40}Ca a ^{16}O byly také studovány. NNN síla znatelně zmenšuje vzdálenosti mezi jednotlivými slupkami a také vzdálenosti mezi hladinami v dané slupce. Dynamický mód byl použit pro studium jednočásticových Λ spekter v $^{41}_{\Lambda}$Ca a $^{17}_{\Lambda}$O. Pozorujeme, že NNN síla zmenšuje vzdálenosti mezi slupkami a dostáváme opačné řazení hladin v p- slupce v $^{17}_{\Lambda}$O.

Klíčová slova: NNN interakce, Hartree-Fockova metoda, Λ hyperjádro, model středního pole

Prohlášení

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VÝZKUMNÝ ÚKOL

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Pokyny pro vypracování:

1. Provedení odvození pohybových rovnic self-konzistentního středního pole (Hartree-Fockova metoda) s použitím jaderného a hyperjaderného Hamiltoniánu s tří-částicovými interakcemi.
2. Implementace rovnic do počítačového kódu a provedení testovacích výpočtů pro případ jaderného Hamiltoniánu.
3. Ověření správnosti implementace formalizmu a určení mezí proveditelnosti výpočtů.

Výzkumný úkol bude napsán v anglickém jazyce.

Součástí zadání výzkumného úkolu je jeho uložení na webové stránky katedry fyziky.

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Contents

Chapter 1

Introduction

Theoretical description of hypernuclear structure remains a challenging task even after decades of research. The baryon potentials are described within the Chiral Perturbation Theory (ChPT), an effective field theory with a consistent hierarchy of the two-body, the three-body, and the many-body forces. In the ChPT, the three-body interactions are generated starting from the next-to-next-to leading order of perturbation. It has been shown that the three-body NNN forces have a great influence on the nuclear ground-state energies and the charge radii [1].

Hypernuclear systems with baryon number $A \gtrsim 10$ are traditionally described by the relativistic mean-field models [2, 3] or the non-relativistic mean-field models which are based on the phenomenological NN potentials [4, 5, 6]. Recently, the mean-field model based on the realistic NN interactions has been introduced. Within this model, the mean field is constructed from the two-body NN potential with the corrective density-dependent (DD) term [7, 8]. The DD term simulates the effect of the NNN interactions and it has been found important to obtain correct radial density distributions, RMS radii, and proton and neutron single-particle spectra [8, 9].

The mean-field model has been extended for the study of the hypernuclei with one bound Λ hyperon [10]. In our recent work [11], the interaction between the Λ hyperon and the nuclear medium has been described by the Nijmegen YN potential ESC08 [12]. Main drawback of this ΛN potential is its strong dependence on the Fermi momentum parameter k_{F}. The parameter k_{F} is evaluated self-consistely during each calculation by the Thomas-Fermi approximation. Moreover, the spin-orbit part of the ESC08 ΛN potential is not implemented directly. Instead, the Scheerbaum approximation [13] is used.

In this work, the realistic $\mathrm{NNLO_{sat}}$ $NN + NNN$ potential [14] is implemented into the mean-field model instead of the two-body realistic NN potential corrected

by the phenomenological DD term. The $\mathrm{NNLO_{sat}}$ potential includes both the two-body NN and the three-body NNN interactions. Instead of the Nijmegen ESC08 ΛN potential, we use the chiral LO ΛN potential with the cut-off $\lambda = 550$ MeV [15] to describe the interaction of the Λ hyperon with the nucleons in the nuclear core.

The aim of this work is to derive the Hartree-Fock method with the three-body NNN and ΛNN interactions and to document the effect of the NNN force on the description of the nuclei ^{40}Ca and ^{16}O and the hypernuclei $^{41}_{\Lambda}$Ca and $^{17}_{\Lambda}$O.

This work is organized as follows: In Chapter 2, the derivation of the Hartree-Fock method in the proton-neutron-Λ formalism with the three-body forces is shown. Chapter 3 presents numerical implementation of the Hartree-Fock method, as well as the Hartree-Fock equations of hypernuclei. The results of the numerical calculations are presented and discussed in Chapter 4. The conclusions of this work, as well as future plans are given in Chapter 5. The Hartree-Fock method in the J-scheme formalism is summarized in Appendix A.

Chapter 2

Hypernuclear mean-field model with three-body interactions

We describe single-Λ hypernucleus as a many-body system consisting of the nuclear core and one Λ hyperon. Properties of the hypernucleus are determined by the Hamiltonian

$$\widehat{H} = \widehat{T}_N + \widehat{T}_\Lambda + \widehat{V}^{NN} + \widehat{V}^{NNN} + \widehat{V}^{\Lambda N} + \widehat{V}^{\Lambda NN} - \widehat{T}_{CM}. \tag{2.1}$$

Here, $\widehat{T}_N$ and $\widehat{T}_\Lambda$ denote the kinetic operators of nucleons and the Λ particle, respectively. The terms $\widehat{V}^{NN}$ and $\widehat{V}^{\Lambda N}$ stand for sums over the two-body NN and ΛN potentials. Sums over the three-body interactions are included in the terms $\widehat{V}^{NNN}$ and $\widehat{V}^{\Lambda NN}$. The term $\widehat{T}_{CM}$ denotes the center-of-mass kinetic operator

$$\widehat{T}_{CM} = \frac{1}{2M(A+0.19)} \left(\sum_{a=1}^{A} \widehat{\vec{P}}_a^{\,2} + 2 \sum_{a<b} \widehat{\vec{P}}_a \cdot \widehat{\vec{P}}_b \right), \tag{2.2}$$

where $M \approx 938$ MeV is the mass of a nucleon, A is the baryon number, and $\widehat{\vec{P}}_a$ is the momentum operator of the a-th particle. Here we use the fact that mass of the Λ hyperon is approximately $M_\Lambda \approx 1.19M$.

The hypernuclear mean field is constructed self-consistently by the Hartree-Fock (HF) method. In Section 2.1, we derive the HF method for a system of identical fermions with the two-body interactions in the formalism of the second quantization. In Section 2.2, we show the HF method for the single-Λ hypernuclei including the three-body NNN and the ΛNN interactions in the proton-neutron-Λ formalism.

2.1 Derivation of Hartree-Fock method in the formalism of second quantization

In this section, we show the derivation of the Hartree-Fock equation in the formalism of second quantization, i.e. in terms of creation (annihilation) operators $a^\dagger(a)$.

Let us consider a system of A identical fermions interacting through the two-body potential $\widehat{V}(\vec{r}_1, \vec{r}_2)$. The Hamiltonian of this system in the second quantization is given by

$$\widehat{H} = \sum_{ij} t_{ij} a_i^\dagger a_j + \frac{1}{4} \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_l a_k, \tag{2.3}$$

where $a_i^\dagger|0\rangle = |i\rangle$ creates the single-particle state $|i\rangle$ and $a_i|i\rangle = |0\rangle$ annihilates the single-particle state $|i\rangle$. The ket $|0\rangle$ denotes the particle vacuum. In the equation (2.3), the matrix elements of the kinetic operator read $t_{ij} = \langle i|\widehat{T}|j\rangle$ and the matrix elements of the potential operator are antisymmetrized

$$V_{ijkl} = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl\rangle - \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|lk\rangle = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl - lk\rangle. \tag{2.4}$$

The antisymmetrized ground-state wave function of the studied system is a Slater determinant

$$|\Psi_0\rangle = \prod_{i=1}^{A} a_i^\dagger|0\rangle. \tag{2.5}$$

The product in Eq. (2.5) runs over the lowest single-particle states. Throughout our work, we express all physical states in the basis of spherical harmonic oscillator. The creation and annihilation operators $a_i^\dagger, a_i$ correspond to the single-particle states $|i\rangle$ as $|i\rangle = a_i^\dagger|0\rangle$. The wave functions $|i\rangle$ can be expanded into another basis represented by the operators $a_i'^\dagger, a_i'$. The bases states $|i\rangle$ and $|i'\rangle$ and their corresponding creation and annihilation operators are connected through a unitary transformation U

$$|i'\rangle = \sum_j U_{ij}|j\rangle, \tag{2.6a}$$

$$a_i'^\dagger = \sum_{ij} U_{ij} a_j^\dagger, \tag{2.6b}$$

$$a_i' = \sum_{ij} a_j U_{ij}^* = \sum_{ij} U_{ji}^\dagger a_j. \tag{2.6c}$$

The inverse transformation satisfies

$$|i\rangle = \sum_j U^*_{ji}|j'\rangle, \tag{2.7a}$$

$$a^\dagger_i = \sum_{ij} U^+_{ij} a'^\dagger_j = \sum_{ij} a'^\dagger_j U^*_{ji}, \tag{2.7b}$$

$$a_i = \sum_{ij} U_{ji} a'_j. \tag{2.7c}$$

The basis represented by the operators $a'^\dagger_i, a'_i$ is called the self-consistent basis. The ground state in the self-consistent basis is defined as the Hartree-Fock state $|\mathrm{HF}\rangle$

$$|\mathrm{HF}\rangle = \prod_{i=1}^{A} a'^\dagger_i |0\rangle. \tag{2.8}$$

It is convenient to define the density matrix of the HF state ρ^{HF}:

$$\rho^{\mathrm{HF}}_{ji} = \langle \mathrm{HF}|a^\dagger_i a_j|\mathrm{HF}\rangle. \tag{2.9}$$

Using the transformation equations (2.7b), (2.7c) leads to the following relation

$$\rho^{\mathrm{HF}}_{ji} = \sum_{kl} U^*_{ki} U^T_{jl} \langle \mathrm{HF}|a'^\dagger_k a'_l|\mathrm{HF}\rangle. \tag{2.10}$$

Let us prove the following identity

$$\begin{aligned}\langle \mathrm{HF}|a'^\dagger_k a'_l|\mathrm{HF}\rangle &= \langle \mathrm{HF}|\{a'^\dagger_k, a'_l\}|\mathrm{HF}\rangle - \langle \mathrm{HF}|a'_l a'^\dagger_k|\mathrm{HF}\rangle \\ &= \delta_{kl}\langle \mathrm{HF}|\mathrm{HF}\rangle - \langle \mathrm{HF}|a'_l a'^\dagger_k|\mathrm{HF}\rangle = \delta_{kl}, \quad \forall k : \varepsilon_k \le \varepsilon_F,\end{aligned} \tag{2.11}$$

where ε_k is the energy of the k-th level and ε_F is the energy of the highest occupied level (the Fermi level). For all occupied single-particle states k, the expression (2.10) gives

$$\rho^{\mathrm{HF}}_{ji} = \sum_{k-\mathrm{occ.}} U^*_{ki} U^T_{jk} = \sum_{k-\mathrm{occ.}} U^T_{jk} U^*_{ki} = (U^T U^*)_{ji} \tag{2.12}$$

The next step of this calculation is construction of the energy functional

$$\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \sum_{ij} t_{ij} \langle \mathrm{HF}|a^\dagger_i a_j|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V_{ijkl} \langle \mathrm{HF}|a^\dagger_i a^\dagger_j a_l a_k|\mathrm{HF}\rangle. \tag{2.13}$$

The second term in Eq. (2.13) with the transformation relations (2.7b), (2.7c) gives

$$\begin{aligned}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a_l a_k|\mathrm{HF}\rangle &= \sum_{opqr} U^*_{oi} U^*_{pj} U^T_{lr} U^T_{kq} \langle \mathrm{HF}|a'^\dagger_o a'^\dagger_p a'_r a'_q|\mathrm{HF}\rangle \\ &= \sum_{opqr-\mathrm{occ.}} U^*_{oi} U^*_{pj} U^T_{lr} U^T_{kq} (\delta_{oq}\delta_{pr} - \delta_{or}\delta_{pq}) \\ &= \sum_{op-\mathrm{occ.}} (U^*_{oi} U^T_{ko} U^*_{pj} U^T_{lp} - U^*_{oi} U^T_{lo} U^*_{pj} U^T_{kp}) \\ &= (U^T U^*)_{ki}(U^T U^*)_{lj} - (U^T U^*)_{li}(U^T U^*)_{kj}\end{aligned} \tag{2.14}$$

The proof of the identity

$$\langle \mathrm{HF}|a_o'^{\dagger}a_p'^{\dagger}a_r'a_q'|\mathrm{HF}\rangle = \delta_{oq}\delta_{pr} - \delta_{or}\delta_{pq}, \quad \forall o,p : \varepsilon_o, \varepsilon_p \leq \varepsilon_F, \tag{2.15}$$

is analogous to the one in Eq. (2.11). The energy functional (2.13) therefore reads

$$\begin{aligned}
\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle &= \sum_{ij}\sum_{o-\mathrm{occ.}} t_{ij}U^T_{jo}U^*_{oi} \\
&+ \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{ko}U^*_{oi}U^T_{lp}U^*_{pj} - \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{lo}U^*_{oi}U^T_{kp}U^*_{pj} \\
&= \sum_{ij} t_{ij}(U^TU^*)_{ji} + \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{ki}\left(U^TU^*\right)_{lj} \\
&- \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{li}(U^TU^*)_{kj}.
\end{aligned} \tag{2.16}$$

We minimize the functional (2.16) with respect to the variation of the transformation U

$$\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U}\delta U + \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*}\delta U^* = 0. \tag{2.17}$$

In general, U is an unitary matrix, hence we get two equivalent conditions

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U} = 0, \tag{2.18a}$$

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*} = 0. \tag{2.18b}$$

Variation of the functional in Eq. (2.16) with respect to U^* gives

$$\begin{aligned}
\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} = \sum_j t_{pj}U^T_{jo} &+ \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{lj}U_{ko} \\
&+ \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{ki}U_{lo} \\
&- \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{kj}U_{lo} \\
&- \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{li}U_{ko}.
\end{aligned} \tag{2.19}$$

Due to hermiticity and antisymmetry, the matrix elements V_{ijkl} satisfy the following identities

$$V_{ijkl} = -V_{jikl} = -V_{ijlk} = V_{jilk}, \tag{2.20a}$$

$$V_{ijkl} = V_{klij}. \tag{2.20b}$$

Applying (2.20a) and (2.20b) on (2.19) yields the equation

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} = \sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right\} U^T_{jo} = 0. \tag{2.21}$$

The unitarity of the matrix U presents following restriction

$$(U^T U^*)_{op} - \mathbb{I}_{op} = 0. \tag{2.22}$$

The variational problem with the restriction (2.22) is expressed as

$$\begin{aligned} &\frac{\delta}{\delta U^*_{op}} \left\{ \langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle - \varepsilon \left[U^T U^* - \mathbb{I} \right] \right\} \\ &\qquad = \sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right\} U^T_{jo} - \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo} = 0. \end{aligned} \tag{2.23}$$

By substituting the density matrix identity (2.12) we obtain the equation

$$\sum_j \left\{ t_{pj} + \sum_{kl} V_{pkjl}\rho^{\mathrm{HF}}_{lk} \right\} U^T_{jo} = \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo}, \tag{2.24}$$

which represents an eigenvalue problem of the matrix h_{mn} defined as

$$h_{mn} = t_{mn} + \sum_{kl} V_{mknl}\rho^{\mathrm{HF}}_{lk} = \varepsilon_m \delta_{mn}, \tag{2.25}$$

where we rename indices $p \to m, j \to n$. Equation (2.25) is called the Hartree-Fock equation. Through diagonalization of h_{mn} we obtain a self-consistent basis $|i'\rangle$ which is connected to the HO basis $|i\rangle$ by the unitary transformation U_{ij}, see Eq. (2.7a).

The Hamiltonian (2.3) can be rewritten into the form in which the creation and annihilation operators are normal ordered. For this procedure we use the Wick's theorem [16] and obtain the new expression of the Hamiltonian

$$\widehat{H} = E_{\mathrm{HF}} + \widehat{H}^{(1)} + \widehat{H}^{(2)}. \tag{2.26}$$

Operators $E_{\mathrm{HF}}, \widehat{H}^{(1)}$, and $\widehat{H}^{(2)}$ are defined as follows

$$E_{\mathrm{HF}} = \sum_{ij} t_{ij}\rho^{\mathrm{HF}}_{ji} + \frac{1}{2}\sum_{kl} V_{ijkl}\rho^{\mathrm{HF}}_{ki}\rho^{\mathrm{HF}}_{jl}, \tag{2.27a}$$

$$\widehat{H}^{(1)} = \sum_{ij} \left(t_{ij} + \sum_{kl} V_{kilj}\rho^{\mathrm{HF}}_{lk} \right) \ :a^\dagger_i a_j:, \tag{2.27b}$$

$$\widehat{H}^{(2)} = \frac{1}{4}\sum_{ijkl} V_{ijkl} \ :a^\dagger_i a^\dagger_j a_l a_k:, \tag{2.27c}$$

where $:a_i^\dagger a_j:$ and $:a_i^\dagger a_j^\dagger a_l a_k:$ denote the normal ordering of the operators $a_i^\dagger a_j$ and $a_i^\dagger a_j^\dagger a_l a_k$, respectively. Normal ordered operators satisfy

$$\begin{aligned}\langle \mathrm{HF}|\, :a_i^\dagger a_j:\, |\mathrm{HF}\rangle &= \langle \mathrm{HF}|\, :a_i'^\dagger a_j':\, |\mathrm{HF}\rangle = 0,\\ \langle \mathrm{HF}|\, :a_i^\dagger a_j^\dagger a_l a_k:\, |\mathrm{HF}\rangle &= \langle \mathrm{HF}|\, :a_i'^\dagger a_j'^\dagger a_l' a_k':\, |\mathrm{HF}\rangle = 0.\end{aligned} \tag{2.28}$$

Therefore, the value of the element $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$ is equal to the energy of the ground state E_{HF} which is defined by the Eq. (2.27a). The terms $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ in Eq. (2.27b) and (2.27c) represent one-body and two-body parts of the Hamiltonian (2.26). The operators in Eqs.(2.27a), (2.27b), and (2.27c) are expressed in the self-consistent basis $|i'\rangle$ as

$$E_{\mathrm{HF}} = \sum_{i-\mathrm{occ.}} \varepsilon_i - \frac{1}{2}\sum_{i,k-\mathrm{occ.}} \overline{V}_{kiki}, \tag{2.29a}$$

$$\widehat{H}^{(1)} = \sum_i \varepsilon_i\, :a_i'^\dagger a_i':, \tag{2.29b}$$

$$\widehat{H}^{(2)} = \frac{1}{4}\sum_{ijkl} \overline{V}_{ijkl}\, :a_i'^\dagger a_j'^\dagger a_l' a_k':\,. \tag{2.29c}$$

Here, $\overline{V}_{ijkl}$ denote the matrix elements V_{ijkl} transformed into the self-consistent basis $|i'\rangle$ by the equation

$$\overline{V}_{ijkl} = \sum_{opqr} V_{opqr} U^*_{oi} U^*_{pj} U^T_{lr} U^T_{kq}. \tag{2.30}$$

2.2 Hartree-Fock method in the proton-neutron-Λ formalism with 3-body interactions

We study a hypernuclear system which consists of Z protons, N neutrons and one Λ particle. The ground state of this system is described by the wave function

$$|\Psi_0\rangle = |\Psi_0\rangle_{\mathrm{p}} \otimes |\Psi_0\rangle_{\mathrm{n}} \otimes |\Psi_0\rangle_{\Lambda}, \tag{2.31}$$

where $|\Psi_0\rangle_{\mathrm{p}}$ and $|\Psi_0\rangle_{\mathrm{n}}$ are Slater determinants of protons and neutrons and $|\Psi_0\rangle_{\Lambda}$ is the single-particle wave function of the Λ hyperon. Respective wave functions in Eq. (2.31) are defined as follows

$$|\Psi_0\rangle_{\mathrm{p}} = \prod_{i=1}^{Z} a_i^\dagger |0\rangle, \tag{2.32a}$$

$$|\Psi_0\rangle_{\mathrm{n}} = \prod_{i=1}^{N} b_i^\dagger |0\rangle, \tag{2.32b}$$

$$|\Psi_0\rangle_{\Lambda} = c_1^\dagger |0\rangle. \tag{2.32c}$$

Indices i and j run over Z and N lowest occupied states in the proton and the neutron potential wells, respectively. In the proton-neutron-Λ formalism, $a_i^\dagger, a_i$ stand for the creation and the annihilation operators of protons, $b_i^\dagger, b_i$ denote the creation and the anihillation operators of neutrons, and $c_i^\dagger, c_i$ represent the creation and the anihillation operators of the Λ hyperon. Unitary transformations for each type of creation and annihilation operators are defined as

$$a_i'^\dagger = \sum_{ij} A_{ij} a_j^\dagger; \quad a_i' = \sum_{ij} a_j A_{ij}^*, \tag{2.33a}$$

$$b_i'^\dagger = \sum_{ij} B_{ij} b_j^\dagger; \quad b_i' = \sum_{ij} b_j B_{ij}^*, \tag{2.33b}$$

$$c_i'^\dagger = \sum_{ij} C_{ij} c_j^\dagger; \quad c_i' = \sum_{ij} c_j C_{ij}^*. \tag{2.33c}$$

Ground-state wave function in the self-consistent basis gives

$$|\mathrm{HF}\rangle = |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \otimes |\mathrm{HF}\rangle_\Lambda, \tag{2.34}$$

where

$$|\mathrm{HF}\rangle_\mathrm{p} = \prod_{i=1}^{Z} a_i'^\dagger |0\rangle, \tag{2.35a}$$

$$|\mathrm{HF}\rangle_\mathrm{n} = \prod_{i=1}^{N} b_i'^\dagger |0\rangle, \tag{2.35b}$$

$$|\mathrm{HF}\rangle_\Lambda = c_1'^\dagger |0\rangle. \tag{2.35c}$$

$$\tag{2.35d}$$

Respective density matrices read

$$\rho_{ij}^\mathrm{p} = {}_\mathrm{p}\langle \mathrm{HF}| a_i^\dagger a_j |\mathrm{HF}\rangle_\mathrm{p}, \tag{2.36a}$$

$$\rho_{ij}^\mathrm{n} = {}_\mathrm{n}\langle \mathrm{HF}| b_i^\dagger b_j |\mathrm{HF}\rangle_\mathrm{n}, \tag{2.36b}$$

$$\rho_{ij}^\Lambda = {}_\Lambda\langle \mathrm{HF}| c_i^\dagger c_j |\mathrm{HF}\rangle_\Lambda. \tag{2.36c}$$

The two-body NN interaction operators $\widehat{V}^{NN}$ and $\widehat{V}^{\Lambda N}$ in Eq. (2.1) are in the proton-neutron-Λ formalism defined as

$$\widehat{V}^{NN} = \widehat{V}^\mathrm{pp} + \widehat{V}^\mathrm{pn} + \widehat{V}^\mathrm{nn}, \tag{2.37a}$$

$$\widehat{V}^{\Lambda N} = \widehat{V}^{\mathrm{p}\Lambda} + \widehat{V}^{\mathrm{n}\Lambda}. \tag{2.37b}$$

The matrix elements of the operators $\widehat{V}^{NN}$ and $\widehat{V}^{\Lambda N}$ in Eq. (2.37a) and (2.37b) are expressed as

$$V^{\rm pp}_{ijkl} = \langle ij|\widehat{V}^{\rm pp}|kl - lk\rangle, \tag{2.38a}$$
$$V^{\rm nn}_{ijkl} = \langle ij|\widehat{V}^{\rm nn}|kl - lk\rangle, \tag{2.38b}$$
$$V^{\rm pn}_{ijkl} = \langle ij|\widehat{V}^{\rm pn}|kl\rangle, \tag{2.38c}$$
$$V^{\rm p\Lambda}_{ijkl} = \langle ij|\widehat{V}^{\rm p\Lambda}|kl\rangle, \tag{2.38d}$$
$$V^{\rm n\Lambda}_{ijkl} = \langle ij|\widehat{V}^{\rm n\Lambda}|kl\rangle. \tag{2.38e}$$

The three-body interaction operators $\widehat{V}^{NNN}$ and $\widehat{V}^{\Lambda NN}$ are defined in the proton-neutron-Λ formalism as

$$\widehat{V}^{NNN} = \widehat{V}^{\rm ppp} + \widehat{V}^{\rm ppn} + \widehat{V}^{\rm pnn} + \widehat{V}^{\rm nnn}, \tag{2.39a}$$
$$\widehat{V}^{\Lambda NN} = \widehat{V}^{\rm pp\Lambda} + \widehat{V}^{\rm nn\Lambda} + \widehat{V}^{\rm pn\Lambda}. \tag{2.39b}$$

The matrix elements of the operators $\widehat{V}^{NNN}$ and $\widehat{V}^{\Lambda NN}$ in Eq. (2.39a) and (2.39b) are antisymmetrized as follows:

$$V^{\rm ppp}_{ijklmn} = \langle ijk|\widehat{V}^{\rm ppp}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{2.40a}$$
$$V^{\rm nnn}_{ijklmn} = \langle ijk|\widehat{V}^{\rm nnn}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{2.40b}$$
$$V^{\rm ppn}_{ijklmn} = \langle ijk|\widehat{V}^{\rm ppn}|lmn - mln\rangle, \tag{2.40c}$$
$$V^{\rm pnn}_{ijklmn} = \langle ijk|\widehat{V}^{\rm pnn}|lmn - lnm\rangle, \tag{2.40d}$$
$$V^{\rm pp\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\rm pp\Lambda}|lmn - mln\rangle, \tag{2.40e}$$
$$V^{\rm nn\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\rm nn\Lambda}|lmn - mln\rangle, \tag{2.40f}$$
$$V^{\rm pn\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\rm pn\Lambda}|lmn\rangle. \tag{2.40g}$$

The hypernuclear Hamiltonian (2.1) in the second quantization reads

$$\begin{aligned}
\widehat{H} = &\sum_{ij} t^{\mathrm{p}}_{ij} a^\dagger_i a_j + \sum_{ij} t^{\mathrm{n}}_{ij} b^\dagger_i b_j + \sum_{ij} t^{\Lambda}_{ij} c^\dagger_i c_j \\
&+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl} a^\dagger_i a^\dagger_j a_l a_k + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl} b^\dagger_i b^\dagger_j b_l b_k + \sum_{ijkl} V^{\mathrm{pn}}_{ijkl} a^\dagger_i b^\dagger_j b_l a_k \\
&+ \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl} a^\dagger_i c^\dagger_j c_l a_k + \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl} b^\dagger_i c^\dagger_j c_l b_k \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn} a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l + \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn} b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn} a^\dagger_i a^\dagger_j b^\dagger_k b_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn} a^\dagger_i b^\dagger_j b^\dagger_k b_n b_m a_l \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn} a^\dagger_i a^\dagger_j c^\dagger_k c_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn} b^\dagger_i b^\dagger_j c^\dagger_k c_n b_m b_l \\
&+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn} a^\dagger_i b^\dagger_j c^\dagger_k c_n b_m a_l. \qquad (2.41)
\end{aligned}$$

Again, we construct the energy functional $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$

$$\begin{aligned}
\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = &\sum_{ij} t^{\mathrm{p}}_{ij} \langle \mathrm{HF}|a^\dagger_i a_j|\mathrm{HF}\rangle + \sum_{ij} t^{\mathrm{n}}_{ij} \langle \mathrm{HF}|b^\dagger_i b_j|\mathrm{HF}\rangle + \sum_{ij} t^{\Lambda}_{ij} \langle \mathrm{HF}|c^\dagger_i c_j|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl} \langle \mathrm{HF}|a^\dagger_i a^\dagger_j a_l a_k|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl} \langle \mathrm{HF}|b^\dagger_i b^\dagger_j b_l b_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{pn}}_{ijkl} \langle \mathrm{HF}|a^\dagger_i b^\dagger_j b_l a_k|\mathrm{HF}\rangle + \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl} \langle \mathrm{HF}|a^\dagger_i c^\dagger_j c_l a_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl} \langle \mathrm{HF}|b^\dagger_i c^\dagger_j c_l b_k|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn} \langle \mathrm{HF}|a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn} \langle \mathrm{HF}|b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn} \langle \mathrm{HF}|a^\dagger_i a^\dagger_j b^\dagger_k b_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn} \langle \mathrm{HF}|a^\dagger_i b^\dagger_j b^\dagger_k b_n b_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn} \langle \mathrm{HF}|a^\dagger_i a^\dagger_j c^\dagger_k c_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn} \langle \mathrm{HF}|b^\dagger_i b^\dagger_j c^\dagger_k c_n b_m b_l|\mathrm{HF}\rangle \\
&+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn} \langle \mathrm{HF}|a^\dagger_i b^\dagger_j c^\dagger_k c_n b_m a_l|\mathrm{HF}\rangle, \qquad (2.42)
\end{aligned}$$

where

$$\begin{aligned}\langle \mathrm{HF}|a_i^\dagger a_j^\dagger a_k^\dagger a_n a_m a_l|\mathrm{HF}\rangle = &(A^TA^*)_{li}(A^TA^*)_{mj}(A^TA^*)_{nk} - (A^TA^*)_{nk}(A^TA^*)_{lj}(A^TA^*)_{mi}\\ &- (A^TA^*)_{mk}(A^TA^*)_{nj}(A^TA^*)_{li} - (A^TA^*)_{mk}(A^TA^*)_{lj}(A^TA^*)_{li}\\ &+ (A^TA^*)_{nj}(A^TA^*)_{lk}(A^TA^*)_{mi} - (A^TA^*)_{lk}(A^TA^*)_{mj}(A^TA^*)_{ni},\end{aligned} \tag{2.43}$$

$$\begin{aligned}\langle \mathrm{HF}|b_i^\dagger b_j^\dagger b_k^\dagger b_n b_m b_l|\mathrm{HF}\rangle = &(B^TB^*)_{li}(B^TB^*)_{mj}(B^TB^*)_{nk} - (B^TB^*)_{nk}(B^TB^*)_{lj}(B^TB^*)_{mi}\\ &- (B^TB^*)_{mk}(B^TB^*)_{nj}(B^TB^*)_{li} - (B^TB^*)_{mk}(B^TB^*)_{lj}(B^TB^*)_{li}\\ &+ (B^TB^*)_{nj}(B^TB^*)_{lk}(B^TB^*)_{mi} - (B^TB^*)_{lk}(B^TB^*)_{mj}(B^TB^*)_{ni}.\end{aligned} \tag{2.44}$$

The terms which contain multiple types of particles satisfy

$$\langle \mathrm{HF}|a_i^\dagger a_j^\dagger b_k^\dagger b_n a_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_j^\dagger a_m a_l|\mathrm{HF}\rangle_{\mathrm{p}}\ {}_{\mathrm{n}}\langle \mathrm{HF}|b_k^\dagger b_n|\mathrm{HF}\rangle_{\mathrm{n}}, \tag{2.45a}$$

$$\langle \mathrm{HF}|a_i^\dagger b_j^\dagger c_k^\dagger c_n b_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_l|\mathrm{HF}\rangle_{\mathrm{p}}\ {}_{\mathrm{n}}\langle \mathrm{HF}|b_j^\dagger b_m|\mathrm{HF}\rangle_{\mathrm{n}}\ {}_{\Lambda}\langle \mathrm{HF}|c_k^\dagger c_n|\mathrm{HF}\rangle_{\Lambda}. \tag{2.45b}$$

By minimizing the energy functional (2.42) with respect to the unitary transformations A, B, and C we obtain three Hartree-Fock equations – one for each type of particles. The HF equation for protons:

$$\begin{aligned}t_{ij}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pp}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{p}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{ppp}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}}\\ + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda}\\ + \sum_{klmn} V_{ijklmn}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{p}}\delta_{ij}.\end{aligned} \tag{2.46}$$

The HF equation for neutrons:

$$\begin{aligned}t_{ij}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{nn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{kilj}^{\mathrm{pn}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{n}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{nnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}}\\ + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \sum_{klmn} V_{klimnj}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda}\\ + \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{n}}\delta_{ij}.\end{aligned} \tag{2.47}$$

The HF equation for the Λ hyperon:

$$\begin{aligned}t_{ij}^{\Lambda} + \sum_{kl} V_{kilj}^{\mathrm{p}\Lambda}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{kilj}^{\mathrm{n}\Lambda}\rho_{lk}^{\mathrm{n}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}}\\ + \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} = \varepsilon_i^{\Lambda}\delta_{ij}.\end{aligned} \tag{2.48}$$

After solving the Hartree-Fock equations (2.46), (2.47), and (2.48) we obtain three self-consistent bases represented by the operators $(a'^{\dagger}, a')$, $(b'^{\dagger}, b')$, $(c'^{\dagger}, c')$. Using the Wick's theorem on the hypernuclear Hamiltonian (2.41) gets us separable Hamiltonian

$$\widehat{H} = E_{\rm HF} + \widehat{H}^{(1)} + \widehat{H}^{(2)} + \widehat{H}^{(3)}, \tag{2.49}$$

where $E_{\rm HF}$ is the ground-state (HF) energy, $\widehat{H}^{(1)}$ is the contribution to the one-body operator, $\widehat{H}^{(2)}$ is the contribution to the two-body operator, and $\widehat{H}^{(3)}$ is the contribution to the three-body operator. Here, the HF energy is defined as

$$\begin{aligned} E_{\rm HF} = &\sum_{i-{\rm occ.}} \varepsilon_i^{\rm p} + \sum_{i-{\rm occ.}} \varepsilon_i^{\rm n} + \sum_{i-{\rm occ.}} \varepsilon_i^{\Lambda} - \frac{1}{2}\sum_{ij-{\rm occ.}} \overline{V}_{ijij}^{\rm pp} - \frac{1}{2}\sum_{ij-{\rm occ.}} \overline{V}_{ijij}^{\rm nn} - \sum_{ij-{\rm occ.}} \overline{V}_{ijij}^{\rm pn} \\ &- \sum_{ij-{\rm occ.}} \overline{V}_{ijij}^{\rm p\Lambda} - \sum_{ij-{\rm occ.}} \overline{V}_{ijij}^{\rm n\Lambda} - \frac{1}{3}\sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm ppp} - \frac{1}{3}\sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm nnn} - \sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm ppn} \\ &- \sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm pnn} - \sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm pp\Lambda} - \sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm nn\Lambda} - 2\sum_{ijk-{\rm occ.}} \overline{V}_{ijkijk}^{\rm pn\Lambda}. \end{aligned} \tag{2.50}$$

The one-body operator $\widehat{H}^{(1)}$ reads

$$\widehat{H}^{(1)} = \sum_i \varepsilon_i^{\rm p} :a_i'^{\dagger} a_i': + \sum_i \varepsilon_i^{\rm n} :b_i'^{\dagger} b_i': + \sum_i \varepsilon_i^{\Lambda} :c_i'^{\dagger} c_i':, \tag{2.51}$$

the two-body operator $\widehat{H}^{(2)}$ is defined as follows

$$\begin{aligned} \widehat{H}^{(2)} = &\frac{1}{4}\sum_{ijkl} \overline{V}_{ijkl}^{\rm pp} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': + \frac{1}{4}\sum_{ijkl} \overline{V}_{ijkl}^{\rm nn} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\rm pn} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\ &+ \sum_{ijkl} \overline{V}_{ijkl}^{\rm p\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\rm n\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \frac{1}{4}\sum_{ijklm} \overline{V}_{ijmklm}^{\rm ppp} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\ &+ \frac{1}{4}\sum_{ijklm} \overline{V}_{ijmklm}^{\rm nnn} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \frac{1}{4}\sum_{ijklm} \overline{V}_{ijmklm}^{\rm ppn} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\ &+ \sum_{ijklm} \overline{V}_{mijmkl}^{\rm ppn} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4}\sum_{ijklm} \overline{V}_{mijmkl}^{\rm pnn} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\ &+ \sum_{ijklm} \overline{V}_{ijmklm}^{\rm pnn} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4}\sum_{ijklm} \overline{V}_{ijmklm}^{\rm pp\Lambda} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\ &+ \sum_{ijklm} \overline{V}_{mijmkl}^{\rm pp\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \frac{1}{4}\sum_{ijklm} \overline{V}_{ijmklm}^{\rm nn\Lambda} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\ &+ \sum_{ijklm} \overline{V}_{mijmkl}^{\rm nn\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \sum_{ijklm} \overline{V}_{ijmklm}^{\rm pn\Lambda} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\ &+ \sum_{ijklm} \overline{V}_{imjkml}^{\rm pn\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijklm} \overline{V}_{mijmkl}^{\rm pn\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k':, \end{aligned} \tag{2.52}$$

the three-body part is defined as

$$\begin{aligned}\widehat{H}^{(3)} = &\frac{1}{36}\sum_{ijklmn}\overline{V}^{\rm ppp}_{ijklmn}:a_i'^{\dagger}a_j'^{\dagger}a_k'^{\dagger}a'_n a'_m a'_l: + \frac{1}{36}\sum_{ijklmn}\overline{V}^{\rm nnn}_{ijklmn}:b_i'^{\dagger}b_j'^{\dagger}b_k'^{\dagger}b'_n b'_m b'_l: \\ &+\frac{1}{4}\sum_{ijklmn}\overline{V}^{\rm ppn}_{ijklmn}:a_i'^{\dagger}a_j'^{\dagger}b_k'^{\dagger}b'_n a'_m a'_l: + \frac{1}{4}\sum_{ijklmn}\overline{V}^{\rm pnn}_{ijklmn}:a_i'^{\dagger}b_j'^{\dagger}b_k'^{\dagger}b'_n b'_m a'_l: \\ &+\frac{1}{4}\sum_{ijklmn}\overline{V}^{\rm pp\Lambda}_{ijklmn}:a_i'^{\dagger}a_j'^{\dagger}c_k'^{\dagger}c'_n a'_m a'_l: + \frac{1}{36}\sum_{ijklmn}\overline{V}^{\rm nn\Lambda}_{ijklmn}:b_i'^{\dagger}b_j'^{\dagger}c_k'^{\dagger}c'_n b'_m b'_l: \\ &+\sum_{ijklmn}\overline{V}^{\rm pn\Lambda}_{ijklmn}:a_i'^{\dagger}b_j'^{\dagger}c_k'^{\dagger}c'_n b'_m a'_l: . \end{aligned} \tag{2.53}$$

The interaction matrix elements in the Eqs. (2.50), (2.52), (2.53) are represented in the self-consistent basis. I.e. they are transformed from the interaction elements in the HO basis by the relations equivalent to the Eq. (2.30).

Chapter 3

Numerical Implementation of the Hartree-Fock Method

The Hartree-Fock method is solved numerically by the code which is an extension of the code used in the study of multipole response in neutron-rich nuclei [8]. In this project, we derive the formalism of the HF method for the hypernuclear Hamiltonian with the $NN, \Lambda N, NNN$, and the ΛNN interactions. We implement the $NN, \Lambda N$, and the NNN interactions into the current version of the HF code. All the terms which include the ΛNN interactions are not taken into account in this project. We plan to implement them in future.

All interaction elements of the $NN, \Lambda N$, and the NNN forces are represented and stored in the J-scheme formalism (see Appendix A). I.e. we work with the J-coupled two-body elements $V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, and with the JT-coupled three-body elements $V^{J'_{12},J_{12},J,T'_{12},T_{12},T}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)}$. We need to place restrictions on the indices i, j, k, l, m, n which enumerate the $(n_\bullet l_\bullet j_\bullet)$ configurations within the NNN matrix elements to reduce the demands on the computer memory. We can use the antisymmetry and store only the matrix elements for the indices

$$i \geq j \geq k, \tag{3.1a}$$

$$l \geq m \geq n. \tag{3.1b}$$

Furthermore, we can use the fact the the NNN matrix elements are hermitian. Thus we can introduce another restriction which can be defined as $(ijk) \geq (lmn)$, e.g.

$$(i \cdot 10^6 + j \cdot 10^3 + k) \geq (l \cdot 10^6 + m \cdot 10^3 + n). \tag{3.2}$$

In Eq. (3.2), we suppose that we work within a single-particle basis with dim $< 10^3$. While in the J-scheme it is a complicated task to reconstruct the three-body matrix elements which cannot be stored due to restrictions (3.1a), (3.1b), and (3.2), in the M-scheme is this task trivial. In the code, we implement the following equations for protons (neutrons), respectively,

$$
\begin{aligned}
&t^{\mathrm{p(n)}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp(nn)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p(n)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn(np)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n(p)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_j j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda(n\Lambda)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda(\Lambda)}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \frac{1}{(2j_k+1)} \\
&+ \sum_{n_k l_k j_k m_k} \sum_{n_l l_l j_j m_l} \sum_{n_m l_m j_m m_m} \sum_{n_n l_n j_n m_n} \\
&\Big\{ \frac{1}{2} V^{\mathrm{ppp(nnn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{p(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l} \\
&+ \frac{1}{2} V^{\mathrm{pnn(ppn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{n(p)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l} \\
&+ V^{\mathrm{ppn(pnn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l} \Big\} \\
&\qquad = \varepsilon^{\mathrm{p(n)}}_i \delta_{ij}. \qquad (3.3)
\end{aligned}
$$

The elements of the NNN interactions in Eq. (3.3) are decoupled into the M-scheme from the JT-coupled elements on the fly by using the transformation equations (A.11),(A.12), (A.13), and (A.14). In addition, the following HF equation for the Λ hyperon is implemented,

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l} \delta_{j_k j_l} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l} \delta_{j_k j_l} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&\qquad = \varepsilon^{\Lambda}_i \delta_{ij}. \qquad (3.4)
\end{aligned}
$$

The HF code can run either in the static or in the dynamic mode. The code in the static mode at first solves the equations for protons and neutrons (3.3) without the proton-Λ and the neutron-Λ interactions. Afterwards, it solves the equation

(3.4) for the Λ hyperon. In the static mode, the properties of protons and neutrons in the nuclear core are not affected by the presence of the Λ hyperon. We use the static mode for calculations of the bare nuclear core. The code in the dynamic mode solves the equations (3.3) and (3.4) self-consistently for the whole hypernucleus. We use the dynamic mode for calculations of the Λ single-particle spectra.

The HF method is implemented in the spherical HO basis. This basis is infinite. In practical calculations, the basis is truncated by the maximal major shell number $N_{\max}$. I.e. the single-particle configuration space is defined as a set of the single-particle states $\{|i\rangle : 2n_i + l_i = N_i \leq N_{\max}\}$. The number n_i stands for the principal quantum number and l_i is the orbital angular momentum. The two-body interaction operators are represented as the matrix elements of the products of the two single-particle states $|i\rangle|j\rangle = |ij\rangle$, $|k\rangle|l\rangle = |kl\rangle$ (see equations (2.38a)-(2.38e)). The two-particle basis is truncated consistently with the single-particle basis by the number $N_{\max}^{(12)}$. I.e. the two-particle configuration space is defined as a set of the states $\{|ij\rangle : 2n_i + l_i + 2n_j + l_j = N_i + N_j \leq 2N_{\max} = N_{\max}^{(12)}\}$. Similar logic applies to the three-body operators which are represented as matrix elements of products of the three single-particle states $|i\rangle|j\rangle|k\rangle = |ijk\rangle$, $|l\rangle|m\rangle|n\rangle = |lmn\rangle$. The three-particle basis is truncated by the number $N_{\max}^{(123)}$. The configuration space is defined as a set of the states $\{|ijk\rangle : 2n_i + l_i + 2n_j + l_j + 2n_k + l_k = N_i + N_j + N_k \leq 3N_{\max} = N_{\max}^{(123)}\}$. The current version of the code can run with configuration spaces up to $N_{\max} = 4$, $N_{\max}^{(12)} = 8$, $N_{\max}^{(123)} = 12$.

The width of the potential well of the spherical harmonic oscillator is given by the parameter $\hbar\omega$. In this work, the parameter $\hbar\omega$ is fixed to 20 MeV.

Chapter 4

Results

We study the effect of the three-body NNN interactions on the radial density distributions and the single-particle spectra of the nuclei ^{16}O and ^{40}Ca and the Λ single-particle spectra of the hypernuclei $^{17}_{\Lambda}$O and $^{41}_{\Lambda}$Ca. The studied nuclei are doubly-magic and spherically symmetric. Therefore, they are convinient systems for calculations within the spherical HO basis which we use in our model.

Throughout our work, we employ chiral NNLO$_{\rm sat}$ potential which includes both the NN and the NNN interactions [14]. The force acting between the Λ particle and the nucleons is described by the chiral LO ΛN potential [15] with the cut-off $\lambda = 550$ MeV.

The input parameters of our calculations are the numbers $N_{\rm max}, N_{\rm max}^{(12)}$, $N_{\rm max}^{(123)}$, and $\hbar\omega$ (see Chapter 3). We perform calculations only for $N_{\rm max} = 4, N_{\rm max}^{(12)} = 8, N_{\rm max}^{(123)} = 12$, and $\hbar\omega = 20$ MeV.

The plots in Fig. 4.1 show radial distributions of the nuclear densities of the ^{40}Ca and the ^{16}O. We compare the density distributions calculated with and without the presence of the NNN interactions to the Relativistic Mean-Field (RMF) model with the parametrization NL-SH [19]. The RMF model reproduces well the experimental radial nuclear density distributions [20]. The radial density distributions calculated with only the NN interactions give unrealistically compressed nuclei and as a result too small charged radii. The charged radius of a given nucleus is defined as

$$r_{\rm ch} = \sqrt{r_{\rm p}^2 + q^2}, \tag{4.1}$$

where $q^2 = 0.64$ fm^2 is the proton form-factor [20] and $r_{\rm p}$ is the proton RMS radius

$$r_{\rm p} = \sqrt{\int {\rm d}^3 r \; r^2 \rho_{\rm p}(\vec{r})}. \tag{4.2}$$

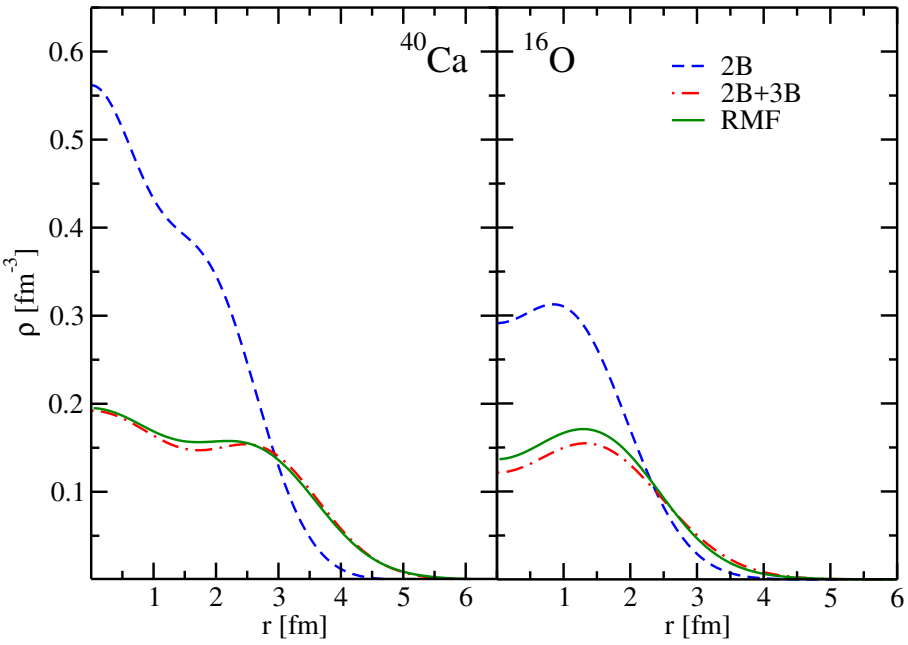

Fig. 4.1: The radial density distribution of the ^{40}Ca and the ^{16}O calculated only with the NN interactions (2B – dashed blue line), the radial density distribution of the ^{40}Ca and the ^{16}O calculated with the NN and the NNN interactions (2B + 3B – dash-dotted red line), realistic radial density distribution of the ^{40}Ca and the ^{16}O calculated with the RMF model (RMF – full green line).

The function $\rho_{\rm p}(\vec{r})$ in Eq. (4.2) is the proton density distribution. The charged radii of the ^{40}Ca and the ^{16}O calculated with the NN and with the $NN + NNN$ interactions compared to the experimental data [21] are shown in Table 4.1. The charged radii calculated with the presence of the NNN interactions give results closer to the experimental values.

In Table 4.2, there are the Hartree-Fock energies per nucleon calculated with the two-body NN and the two-body NN plus the three-body NNN interactions compared to the experimental values. The ground-state energies calculated with the

Table 4.1: The charge radii $r_{\rm ch}$ of the ^{40}Ca and the ^{16}O calculated with the NN interactions (2B) and the charged radii of the ^{40}Ca and the ^{16}O calculated with the $NN + NNN$ interactions (2B+3B) compared to the experimental data (exp) taken from [21].

	$r_{\rm ch}$ [fm]		
^{A}X	2B	2B+3B	exp
^{40}Ca	2.58	3.18	3.48
^{16}O	2.23	2.67	2.70

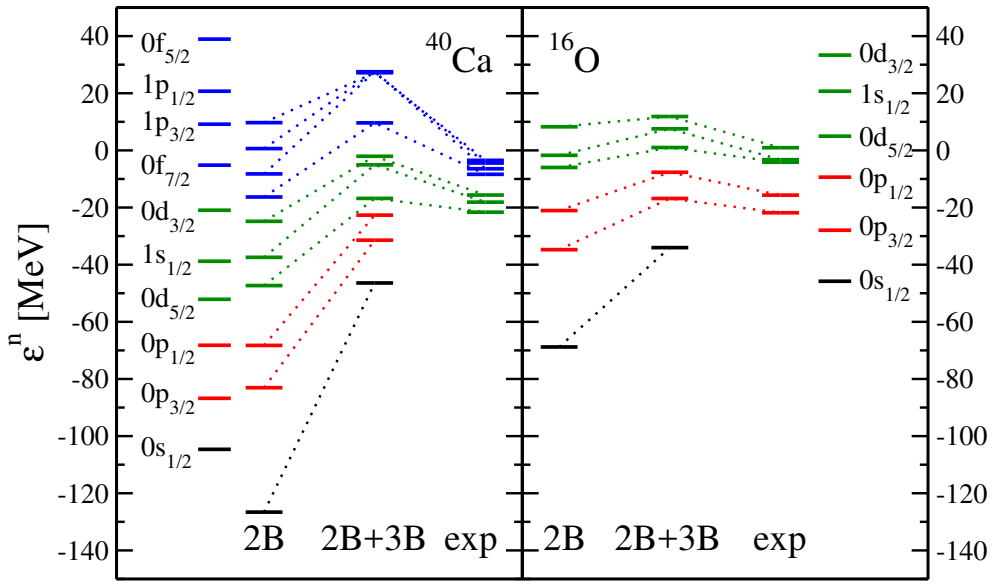

Fig. 4.2: The neutron single-particle energies $\varepsilon^{\rm n}$ in the ^{40}Ca and the ^{16}O calculated with the two-body NN interactions (2B) and with the two-body NN plus the three-body NNN interactions (2B+3B) compared to the empirical values (exp) [22].

two-body NN interactions show significant overbinding, whereas the ground-state energies calculated with the two-body NN plus the three-body NNN interactions are considerably underbinded. We did not take into account the beyond mean-field correlations.

In Fig. 4.2, we show the neutron single particle spectra of ^{40}Ca and ^{16}O calculated with the NN interactions and with the $NN + NNN$ interactions. We show the empirical values of neutron single-particle energies for comparison. Here, the empirical values are determined from the differences between binding energies of doubly-magic nuclei ^{40}Ca and ^{16}O and the corresponding neighboring odd nuclei.

Table 4.2: Values of the Hartree-Fock energies per nucleon calculated with the NN interactions (2B) and with the $NN + NNN$ interactions (2B+3B) in ^{40}Ca and ^{16}O compared to the experimental values (exp) which correspond to $E_{\rm exp} = -B(^A{\rm X})/A$.

	$E_{\rm HF}/A$ [MeV]		
^{A}X	2B	2B+3B	exp
^{40}Ca	-11.65	-0.60	-8.55
^{16}O	-7.31	-2.19	-7.98

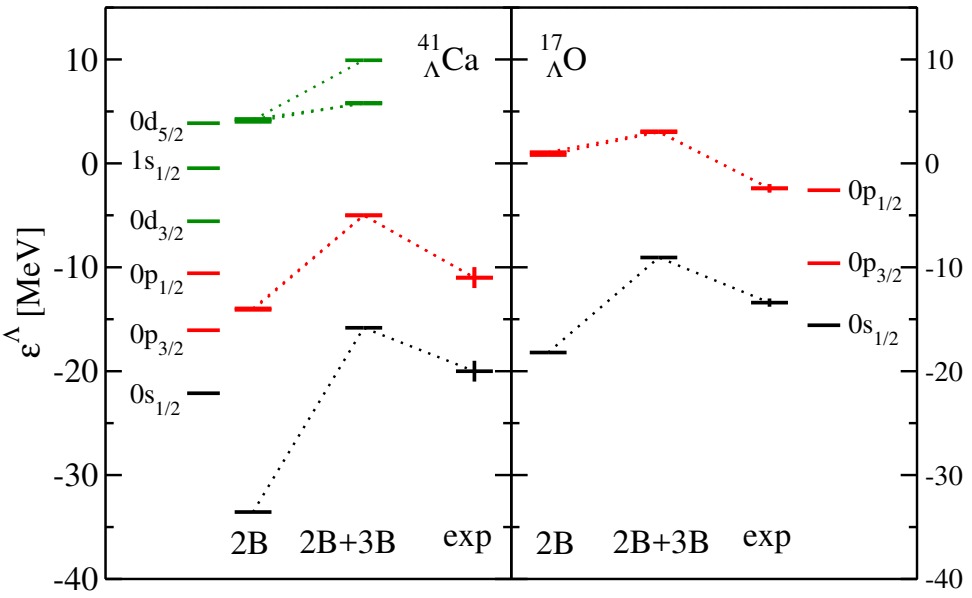

Fig. 4.3: The Λ single-particle energies ε^{Λ} in the $^{41}_{\Lambda}$Ca and the $^{17}_{\Lambda}$O calculated with the two-body NN interactions (2B) and with the two-body NN plus the three-body NNN interactions (2B+3B) compared to the experimental data (exp) [17, 18].

The unoccupied single-particle energies are calculated by the equations

$$\varepsilon^{\mathrm{n}} = B(^{40}\mathrm{Ca}) - B(^{41}\mathrm{Ca}), \tag{4.3a}$$

$$\varepsilon^{\mathrm{n}} = B(^{16}\mathrm{O}) - B(^{17}\mathrm{O}), \tag{4.3b}$$

where $B(^{A}\mathrm{X})$ is the binding energy of the given nucleus. The occupied single-particle energies are obtained by

$$\varepsilon^{\mathrm{n}} = B(^{39}\mathrm{Ca}) - B(^{40}\mathrm{Ca}), \tag{4.4a}$$

$$\varepsilon^{\mathrm{n}} = B(^{15}\mathrm{O}) - B(^{16}\mathrm{O}). \tag{4.4b}$$

The single-particle spectra calculated without the NNN interactions show unrealistically large gaps between the shells, as well as unrealistically large gaps between levels in a given shell.

In Fig. 4.3, we plot the Λ single-particle energies in the $^{41}_{\Lambda}$Ca and the $^{17}_{\Lambda}$O calculated with and without the NNN interactions compared to the experimental values. Again, the presence of the NNN interactions leads to more realistic gaps between the shells. The spectra are systematically shifted upwards in energy with respect to the experimental data. The employed ΛN interaction is strongly dependent on the cut-off parameter. Moreover, we do not take into account the $\Lambda - \Sigma$ mixing and the ΛNN interactions. These effects combined may explain the shift. The values of the energies plotted in Fig. 4.3 are written in Table 4.3. The Λ single-particle

levels in the ${}^{41}_{\Lambda}$Ca calculated with the NN interactions (2B) are ordered as follows: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{3/2}$, $1s_{1/2}$, $0d_{5/2}$. The Λ single-particle levels in the ${}^{41}_{\Lambda}$Ca calculated with the two-body NN plus the three-body NNN interactions (2B+3B) are ordered as follows: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $0d_{3/2}$, $1s_{1/2}$. However, we use too small configuration space to obtain reasonable results for the levels in the sd- shell. The Λ single-particle levels in the ${}^{17}_{\Lambda}$O calculated without and with the three-body NNN interactions (2B and 2B+3B) are ordered $0s_{1/2}$, $0p_{1/2}$, $0p_{3/2}$. Note that it does not correspond to the standard ordering of the single-particle levels $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$. This may be caused by the character of the employed ΛN interaction which has too weak tensor term.

Table 4.3: Single-particle energies of the Λ hyperon in ${}^{41}_{\Lambda}$Ca and ${}^{17}_{\Lambda}$O calculated with NN interactions (2B) and with $NN+NNN$ interactions (2B+3B) compared to the experimental data (exp) – ${}^{41}_{\Lambda}$Ca [18], ${}^{17}_{\Lambda}$O [17].

	ε^{Λ} [MeV]					
	${}^{41}_{\Lambda}$Ca			${}^{17}_{\Lambda}$O		
s.-p. level	2B	2B+3B	exp	2B	2B+3B	exp
$0s_{1/2}$	-33.561	-15.820	-20.0 ± 1.0	-18.203	-9.055	-13.5 ± 0.4
$0p_{3/2}$	-14.095	-5.016	-11.0 ± 1.0	1.076	3.090	-2.4 ± 0.4
$0p_{1/2}$	-13.958	-4.987	-11.0 ± 1.0	0.805	3.005	-2.4 ± 0.4
$0d_{5/2}$	4.275	5.762				
$1s_{1/2}$	4.180	9.928				
$0d_{3/2}$	3.998	5.820				

Chapter 5

Conclusions

In this work, we studied the effect of the three-body NNN interactions on the structure of the nuclei ^{40}Ca and ^{16}O and the hypernuclei $^{41}_{\Lambda}$Ca and $^{17}_{\Lambda}$O. We derived the formalism of the Hartree-Fock method for the Hamiltonian which includes the NNN and the ΛNN potentials. We used the proton-neutron-Λ formalism. We extended the available HF code to solve the equations which we derived. In the code, we implemented the chiral potential NNLO$_{\text{sat}}$ which includes the NN and the NNN interactions, and the chiral LO ΛN potential. We plan to implement the ΛNN interactions in future.

In this work, the HF code ran only within the configuration space up to $N_{\text{max}} = 4$, $N_{\text{max}}^{(12)} = 8$, $N_{\text{max}}^{(123)} = 12$. We intend to study another ways to truncate the configuration space, e.g. $N_{\text{max}}^{(12)} = N_{\text{max}}^{(123)} = N_{\text{max}}$. This will allow us to include larger configuration space.

We calculated the radial density distributions of the ^{40}Ca and the ^{16}O, as well as their charge radii. We concluded that the NNN interactions have a significant effect on these observables. The NNN force flattened the radial density distributions which were then in good agreement with the RMF model. In addition, it enhanced the charged radii closer to their respective experimental values.

We studied the ground-state energies E_{HF} of the ^{40}Ca and the ^{16}O. We found that the ground-state energy of the ^{40}Ca calculated purely with the two-body NN interactions overestimates the empirical value. On the other hand, the ground-state energies calculated with the two-body NN plus the three-body NNN interactions significantly underestimate empirical data in both the ^{40}Ca and the ^{16}O. Possible explanation is that we do not take into account the beyond mean-field correlations.

We explored the neutron single-particle energies of the ^{40}Ca and the ^{16}O. We discovered that the NNN force shrinks the gaps between the major shells, as well as the levels within given major shell. The neutron single-particle energies in the ^{16}O

were in better agreement with their empirical values than the ones in ^{40}Ca. However, the computations were not converged within our limited configuration space. This affected more the ^{40}Ca than the ^{16}O. We expect better agreement with the empirical values in larger configuration spaces.

We performed calculations of the Λ single-particle spectra in the $^{41}_{\Lambda}$Ca and in the $^{17}_{\Lambda}$O. The NNN force shrank the gaps between the major shells. The calculated single-particle energies were shifted systematically upwards with respect to the available experimental values. We argued that this shift was influenced by the cut-off dependence of the employed ΛN potential, as well as the effects of the $\Lambda - \Sigma$ mixing and the ΛNN interactions which were not considered in our work. In $^{17}_{\Lambda}$O we obtained the opposite spin-orbit splitting of the levels $0p_{3/2}$ and $0p_{1/2}$. We discussed that this may have been due to the weak tensor term in the ΛN potential. We plan to study the effects of the tensor term, the $\Lambda - \Sigma$ mixing and the ΛNN interaction in future.

Appendix A

J-scheme Formalism

The indices i, j, k, l, m, n which we use throughout Chapter 2 represent the eigenstates of the spherical harmonic oscillator basis

$$|i\rangle = |n_i l_i j_i m_i\rangle, \tag{A.1}$$

where n_i is the principal quantum number, l_i is the orbital angular momentum, j_i is the total angular momentum, and m_i is the projection of the total angular momentum. The numbers l_i, j_i and m_i satisfy the following relations

$$\left| l_i - \frac{1}{2} \right| \leq j_i \leq l_i + \frac{1}{2}, \tag{A.2a}$$

$$m_i = -j_i, -j_i + 1, \ldots, +j_i - 1, +j_i. \tag{A.2b}$$

The formalism which uses the eigenstates as defined in (A.1) is called the M-scheme. If the studied system exhibits spherical symmetry, we can develop a formalism which disregards the projections of the total angular momenta and represents the eigenstates as sets of three quantum numbers

$$|i\rangle \to (n_i l_i j_i). \tag{A.3}$$

This formalism is called the J-scheme.

The matrix elements of the kinetic operator of protons, neutrons, and the Λ particle, respectively, are transformed into the J-scheme formalism as follows

$$t^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{A.4a}$$

$$t^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{A.4b}$$

$$t^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{A.4c}$$

Analogically, the matrix elements of the density matrices of protons, neutrons, and the Λ particle, respectively, read

$$\rho^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \tag{A.5a}$$

$$\rho^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{A.5b}$$

$$\rho^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{A.5c}$$

The transformations of the matrix elements of the two-body NN and ΛN interaction operators into the J-scheme are expressed as:

$$V^{\mathrm{pp}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{J m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{A.6a}$$

$$V^{\mathrm{pn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{J m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{A.6b}$$

$$V^{\mathrm{nn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{J m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{A.6c}$$

$$V^{\mathrm{p}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{J m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{A.6d}$$

$$V^{\mathrm{n}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{J m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{A.6e}$$

where the symbols

$$C^{J m_i+m_j}_{j_i m_i, j_j m_j} = \langle j_i m_i, j_j m_j | J\ m_i+m_j \rangle, \quad C^{J m_k+m_l}_{j_k m_k, j_l m_l} = \langle j_k m_k, j_l m_l | J\ m_k+m_l \rangle,$$

represent the Clebsch-Gordan coefficients. The transformation of the matrix elements of the three-body interactions requires the following relations

$$|n_1 l_1 j_1, n_2 l_2 j_2, n_3 l_3 j_3; J_{12}, JM\rangle = \sum_{m_1 m_2 m_3} \sum_{M_{12}} C^{J_{12} M_{12}}_{j_1 m_1, j_2 m_2} C^{JM}_{J_{12} M_{12}, j_3 m_3} |n_1 l_1 j_1 m_1, n_2 l_2 j_2 m_2, n_3 l_3 j_3 m_3\rangle, \tag{A.7}$$

$$|n_1 l_1 j_1 m_1, n_2 l_2 j_2 m_2, n_3 l_3 j_3 m_3\rangle = \sum_{J_{12} J} C^{J_{12} m_1+m_2}_{j_1 m_1, j_2 m_2} C^{J m_1+m_2+m_3}_{J_{12} m_1+m_2, j_3 m_3} |n_1 l_1 j_1, n_2 l_2 j_2, n_3 l_3 j_3; J_{12}, J\ m_1+m_2+m_3\rangle. \tag{A.8}$$

Here, the symbol J_{12} stands for the angular momentum which couples the angular momenta j_1 and j_2. The symbol J stands for the coupling of the angular momenta J_{12} and j_3. In addition, we can introduce the isospin quantum number t and its projection m_t. Each type of particles can be expressed with distinctive values of t and m_t – protons ($t = \frac{1}{2}, m_t = +\frac{1}{2}$), neutrons ($t = \frac{1}{2}, m_t = -\frac{1}{2}$), and the Λ hyperon ($t = 0, m_t = 0$). The equations (A.7) and (A.8) can be rewritten into the JT-coupled form

$$|n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, JMTM_T\rangle = \\ = \sum_{m_1 m_2 m_3} \sum_{M_{12}} \sum_{m_{t_1} m_{t_2} m_{t_3}} \sum_{M_{T_{12}}} C^{J_{12} M_{12}}_{j_1 m_1, j_2 m_2} C^{JM}_{J_{12} M_{12}, j_3 m_3} C^{T_{12} M_{T_{12}}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{TM_T}_{T_{12} M_{T_{12}}, t_3 m_{t_3}} \\ \times |n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle, \quad \text{(A.9)}$$

and

$$|n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle \\ = \sum_{J_{12} J} \sum_{T_{12} T} C^{J_{12} m_1 + m_2}_{j_1 m_1, j_2 m_2} C^{J m_1 + m_2 + m_3}_{J_{12} m_1 + m_2, j_3 m_3} C^{T_{12} m_{t_1} + m_{t_2}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{T m_{t_1} + m_{t_2} + m_{t_3}}_{T_{12} m_{t_1} + m_{t_2}, t_3 m_{t_3}} \\ \times |n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, J m_1 + m_2 + m_3 T m_{t_1} + m_{t_2} + m_{t_3}\rangle. \quad \text{(A.10)}$$

In analogy to Eqs. (A.6a)-(A.6e), we can introduce the relations between the three-body interaction matrix elements in the JT-scheme and in the M-scheme:

$$V^{\mathrm{ppp}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \quad \text{(A.11)}$$

$$V^{\mathrm{nnn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \quad \text{(A.12)}$$

$$V^{\mathrm{ppn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\ = \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\ \left(\frac{2}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right. \\ \left. + \frac{1}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right), \quad \text{(A.13)}$$

$$
\begin{aligned}
&V^{\mathrm{pnn}}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l,n_ml_mj_mm_m,n_nl_nj_nm_n} \\
&= \sum_{J_{12}J'_{12}} \sum_{J} C^{J'_{12}m_i+m_j}_{j_im_i,j_jm_j} C^{Jm_i+m_j+m_k}_{J'_{12}m_i+m_j,j_km_k} C^{J_{12}m_l+m_m}_{j_lm_l,j_mm_m} C^{Jm_l+m_m+m_n}_{J_{12}m_l+m_m,j_nm_n} \\
&\quad \times \left(\frac{1}{2} V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \right. \\
&\qquad\qquad + \frac{1}{6} V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \\
&\qquad\qquad + \frac{1}{3} V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \\
&\qquad\qquad + \frac{1}{2\sqrt{3}} V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \\
&\qquad\qquad\qquad \left. + \frac{1}{2\sqrt{3}} V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \right), \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{pp}\Lambda}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l,n_ml_mj_mm_m,n_nl_nj_nm_n} \\
&= \sum_{J_{12}J'_{12}} \sum_{J} C^{J'_{12}m_i+m_j}_{j_im_i,j_jm_j} C^{Jm_i+m_j+m_k}_{J'_{12}m_i+m_j,j_km_k} C^{J_{12}m_l+m_m}_{j_lm_l,j_mm_m} C^{Jm_l+m_m+m_n}_{J_{12}m_l+m_m,j_nm_n} \\
&\qquad\qquad \times V^{T'_{12}=1,T_{12}=1,T=1,J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}, \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{pn}\Lambda}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l,n_ml_mj_mm_m,n_nl_nj_nm_n} \\
&= \sum_{J_{12}J'_{12}} \sum_{J} C^{J'_{12}m_i+m_j}_{j_im_i,j_jm_j} C^{Jm_i+m_j+m_k}_{J'_{12}m_i+m_j,j_km_k} C^{J_{12}m_l+m_m}_{j_lm_l,j_mm_m} C^{Jm_l+m_m+m_n}_{J_{12}m_l+m_m,j_nm_n} \\
&\quad \times \left(\frac{1}{2} V^{T'_{12}=1,T_{12}=1,T=1,J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \right. \\
&\qquad\qquad \left. + \frac{1}{2} V^{T'_{12}=0,T_{12}=0,T=0,J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)} \right), \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
&V^{\mathrm{nn}\Lambda}_{n_il_ij_im_i,n_jl_jj_jm_j,n_kl_kj_km_k,n_ll_lj_lm_l,n_ml_mj_mm_m,n_nl_nj_nm_n} \\
&= \sum_{J_{12}J'_{12}} \sum_{J} C^{J'_{12}m_i+m_j}_{j_im_i,j_jm_j} C^{Jm_i+m_j+m_k}_{J'_{12}m_i+m_j,j_km_k} C^{J_{12}m_l+m_m}_{j_lm_l,j_mm_m} C^{Jm_l+m_m+m_n}_{J_{12}m_l+m_m,j_nm_n} \\
&\qquad\qquad \times V^{T'_{12}=1,T_{12}=1,T=1,J'_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}. \qquad \text{(A.17)}
\end{aligned}
$$

A.1 Hartree-Fock Equations in the J-scheme Formalism

In this section we show the Hartree-Fock equations in the J-scheme formalism. We substitute matrix elements of the one-body, the two-body, and the three-body operators transformed into the J-scheme to respective Hartree-Fock equations in the

M-scheme (2.46), (2.47), and (2.48). We obtain the corresponding HF equations for protons, neutrons, and the Λ hyperon in the J-scheme:

$$
\begin{aligned}
&t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_j j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \frac{1}{(2j_k+1)} \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{ab} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{l_m l_k} \delta_{j_m j_k} \delta_{l_l l_n} \delta_{j_l j_n} \delta_{m_i m_j} \times \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{2} \Bigg[\frac{2}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \Bigg] \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{6} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{p}}_i\delta_{ij},
\end{aligned}
\tag{A.18}
$$

$$
\begin{aligned}
&t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_j j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{(2J+1)}{(2j_i+1)} \frac{1}{(2j_k+1)} \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{ab} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{l_m l_k} \delta_{j_m j_k} \delta_{l_l l_n} \delta_{j_l j_n} \delta_{m_i m_j} \times \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{2} \Bigg[\frac{2}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \Bigg] \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{6} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{(2J+1)}{(2j_i+1)} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\frac{1}{(2j_n+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{n}}_i\delta_{ij},
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{(2J+1)}{(2j_i+1)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=0,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{(2J+1)}{(2j_i+1)}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} = \varepsilon^{\Lambda}_i\delta_{ij}.
\end{aligned}
\tag{A.20}
$$

The HF energy corresponding to the minimized value of the energy functional (2.42) is expressed in the J-scheme as follows:

$$
\begin{aligned}
E_{\mathrm{HF}} = & \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1)\delta_{l_i l_j}\delta_{j_i j_j} \\
& \quad + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1)\delta_{l_i l_j}\delta_{j_i j_j} \\
& \quad + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\Lambda}_{(n_j l_j j_j),(n_i l_i j_i)} \delta_{l_i l_j}\delta_{j_i j_j} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{(2J+1)}{(2j_j+1)} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{(2J+1)}{(2j_j+1)} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{l_j l_l}\delta_{j_i j_l} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \times \\
& \qquad \times \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l}\delta_{j_i j_l}\delta_{l_j l_m}\delta_{j_j j_m}\delta_{l_k l_n}\delta_{j_k j_m} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \times \\
& \qquad \times \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l}\delta_{j_i j_l}\delta_{l_j l_m}\delta_{j_j j_m}\delta_{l_k l_n}\delta_{j_k j_m}
\end{aligned}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\Bigg\{\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{ab}}\Bigg\{\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\n_jl_jj_j\\n_kl_kj_k\\n_ll_lj_l\\n_ml_mj_m\\n_nl_nj_n}}\sum_{J,J_{12}}\Bigg\{\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{(2j_j+1)}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{2j_j+1}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}\times
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_lj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg\}. \qquad \text{(A.21)}
$$

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