

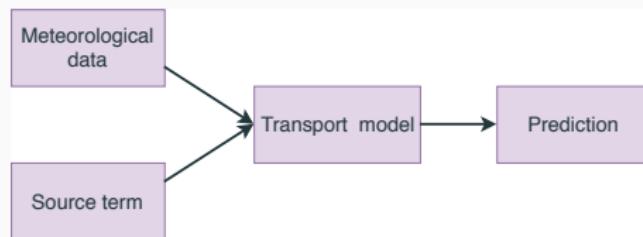
# Bayesian approach to source term estimation

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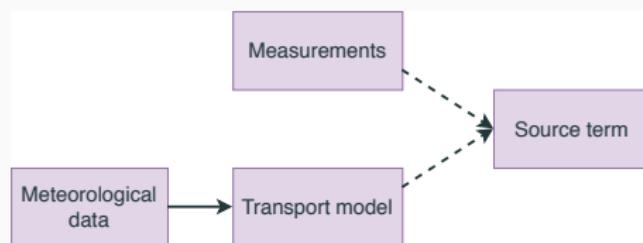
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SPMS 2018

# Atmospheric dispersion modeling



- Inverse problem



# Source term estimation

- General model

$$\mathbf{y} = \mathcal{M}(\mathbf{x})$$

- Linear model

$$\mathbf{y} = M\mathbf{x}$$

$\mathbf{y}$  – vector of measurements  $p \times 1$ ,

$M$  – SRS matrix  $p \times t$ ,

$\mathbf{x}$  – source term  $t \times 1$ .

## Regularized least squares

$$\begin{aligned}\|\mathbf{y} - M\mathbf{x}\|_2 \\ \rightarrow \hat{\mathbf{x}}^{ols} = (M^T M)^{-1} M^T \mathbf{y}\end{aligned}$$

- matrix  $M$  sparse, ill-conditioned
- + positivity of  $\hat{\mathbf{x}}$
- + constraints on norm of  $\hat{\mathbf{x}}$

$$\|\mathbf{y} - M\mathbf{x}\|_2 + \alpha \|\mathbf{x}\|$$

- Tikhonov reg., LASSO, Elastic net, Eckhardt reg., ...

## Choice of the regularization parameter

$$\|\mathbf{y} - M\mathbf{x}\|_2 + \alpha\|\mathbf{x}\|; \quad \mathbf{x} \geq \mathbf{0}$$

- Cross validation
- L-curve
- Discrepancy principle
- ...
- Bayesian estimate of  $\mathbf{x}$  along with  $\alpha$

# Bayesian estimate

1. Observation model  $f(D|\theta)$
2. Prior distribution  $f(\theta)$
3. Bayes' rule:

$$f(\theta|D) = \frac{f(D|\theta)f(\theta)}{\int_{\Theta^*} f(D|\theta)f(\theta) d\theta}.$$

4. Bayesian estimate

$$\widehat{g(\theta)} = E_{f(\theta|D)}[g(\theta)]$$

# Variational Bayes approximation

$$\|\mathbf{y} - M\mathbf{x}\|_2 + \alpha\|\mathbf{x}\|$$

$$f(\mathbf{x}, \alpha | \mathbf{y}) = \frac{f(\mathbf{y} | \mathbf{x}, \alpha) f(\mathbf{x}, \alpha)}{\int\limits_{\alpha^* \times \mathbf{x}^*} f(\mathbf{y} | \mathbf{x}, \alpha) f(\mathbf{x}, \alpha) d\mathbf{x} d\alpha}$$

- $f(\alpha | \mathbf{y}), f(\mathbf{x} | \mathbf{y})$  ?
- Approximation of marginals

$$\tilde{f}(\theta_i | D) \propto \exp \left( \text{E}_{\tilde{f}(\theta / i | D)} [\ln f(\theta, D)] \right), \quad i = 1, \dots, q$$

## Choice of the prior distribution of $\mathbf{x}$

1.

$$f(\mathbf{x}) = t\mathcal{N}(\mathbf{0}, \alpha^{-1} I_t)$$

2.

$$f(\mathbf{x}) = t\mathcal{N}(\mathbf{0}, \text{diag}([v_1, \dots, v_t])^{-1})$$

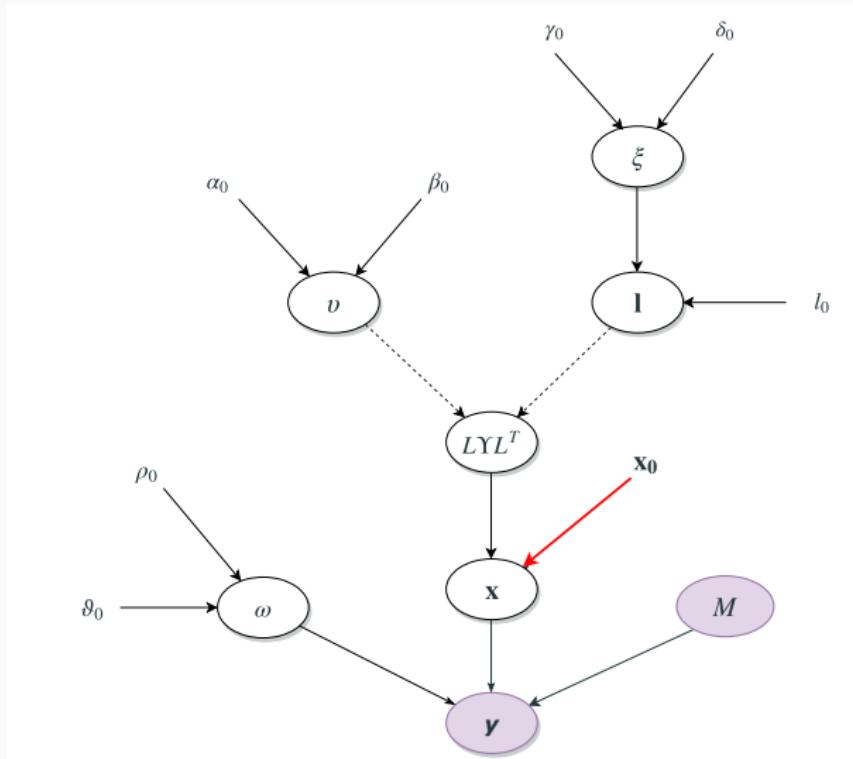
3.

$$f(\mathbf{x}) = t\mathcal{N}\left(\mathbf{0}, (L\Upsilon L^T)^{-1}\right)$$

4.

$$f(\mathbf{x}) = t\mathcal{N}\left(\mathbf{x}_0, (L\Upsilon L^T)^{-1}\right)$$

# Model LS-APC- $x_0$



## IVB algorithm

$$\tilde{f}(\theta_i|D) = \exp\left(\text{E}_{\tilde{f}(\theta/i|D)}[\ln f(\boldsymbol{\theta}, D)]\right), \quad i = 1, \dots, q,$$

$$\tilde{f}(\mathbf{x}|\mathbf{y}) \propto t\mathcal{N}(\boldsymbol{\mu}_x, \Sigma_x),$$

$$\tilde{f}(l_k|\mathbf{y}) \propto \mathcal{N}(\mu_{l,k}, \lambda_k^{-1}), \quad k = 1, \dots, t-1,$$

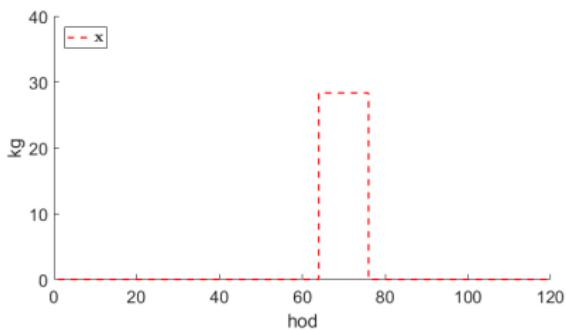
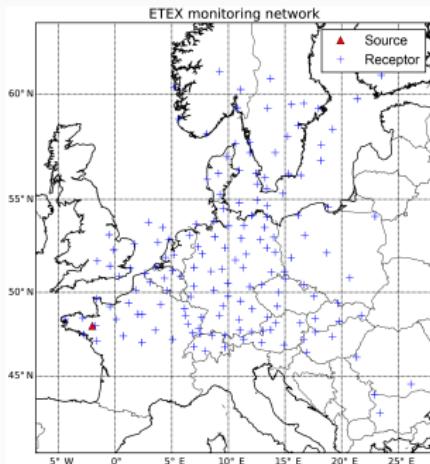
$$\tilde{f}(v_k|\mathbf{y}) \propto \mathcal{G}(\alpha_k, \beta_k), \quad k = 1, \dots, t$$

$$\tilde{f}(\xi_k|\mathbf{y}) \propto \mathcal{G}(\gamma_k, \delta_k), \quad k = 1, \dots, t-1$$

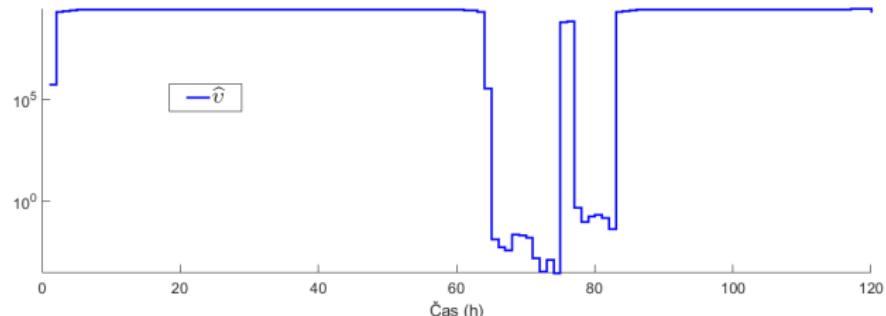
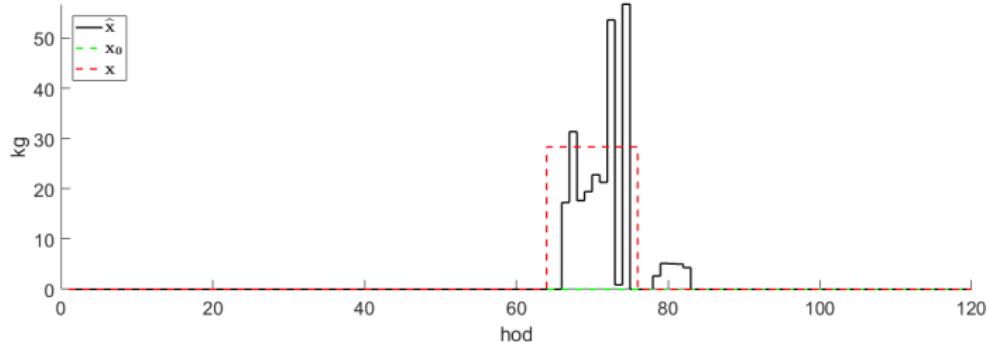
$$\tilde{f}(\omega|\mathbf{y}) \propto \mathcal{G}(\vartheta, \rho).$$

- A system of 17 equations for moments and shaping parameters:

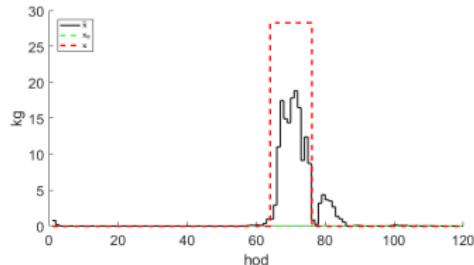
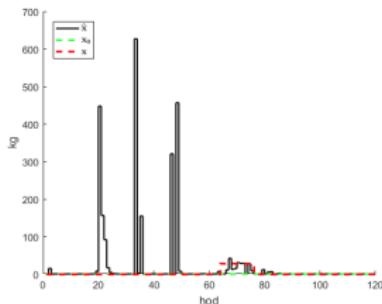
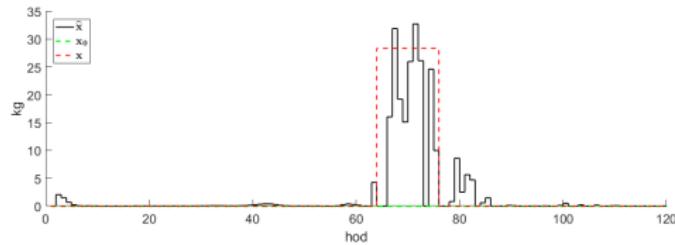
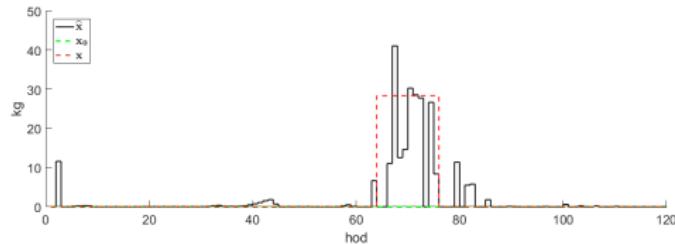
$$\begin{aligned}\Sigma_x, \quad \boldsymbol{\mu}_x, \quad \widehat{\mathbf{x}}, \quad \widehat{\mathbf{x}\mathbf{x}'}, \quad \boldsymbol{\mu}_l, \quad \lambda, \quad \widehat{\mathbf{l}}, \quad \widehat{\mathbf{ll}'} \\ \gamma, \quad \delta, \quad \widehat{\boldsymbol{\xi}}, \quad \boldsymbol{\alpha}, \quad \boldsymbol{\beta}, \quad \widehat{\Upsilon}, \quad \vartheta, \quad \rho, \quad \widehat{\omega},\end{aligned}$$



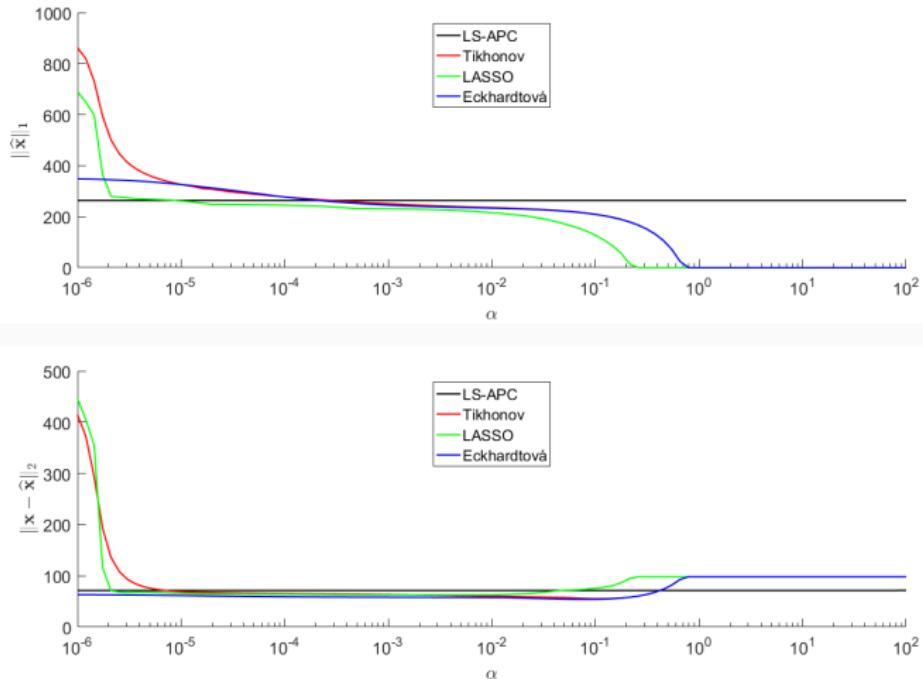
# ETEX – source term estimate



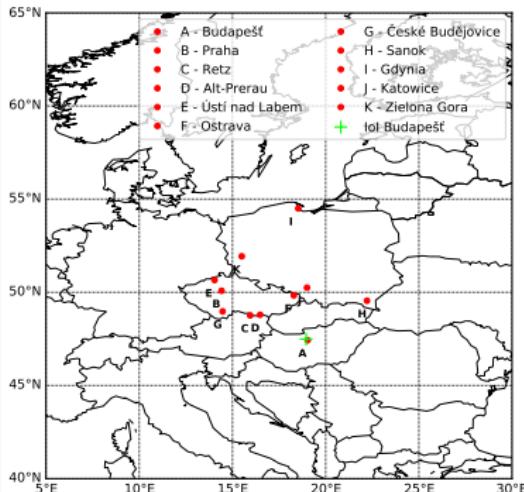
# ETEX – source term estimate



# ETEX – sensitivity to the parameter $\alpha$

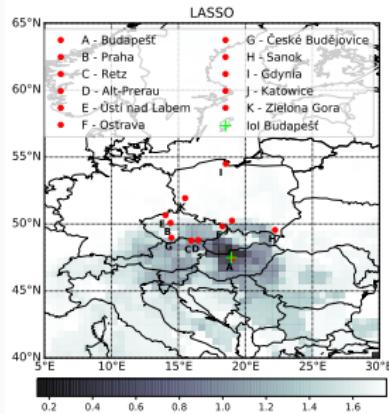
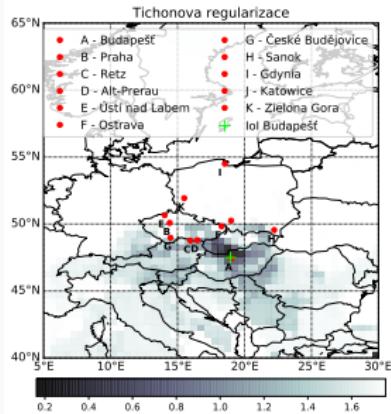
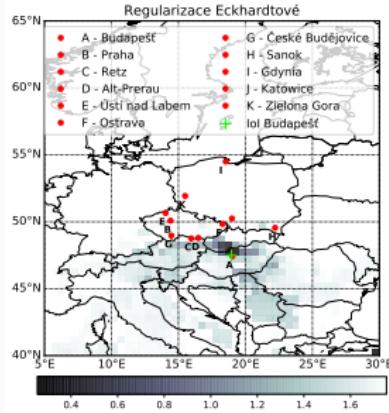
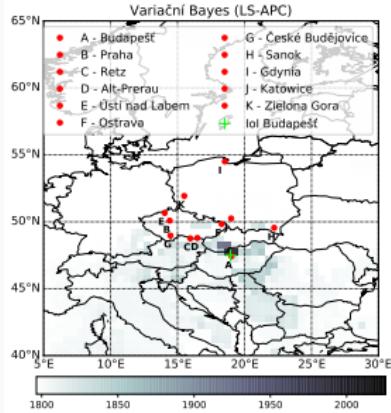


# Source localization – Hungary 2011

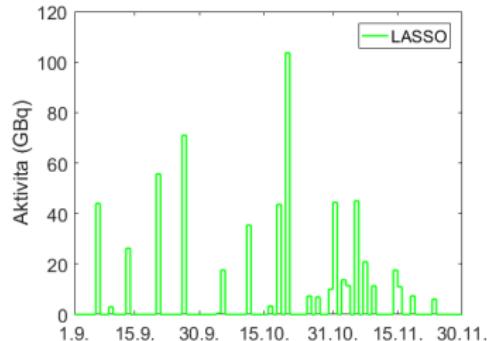
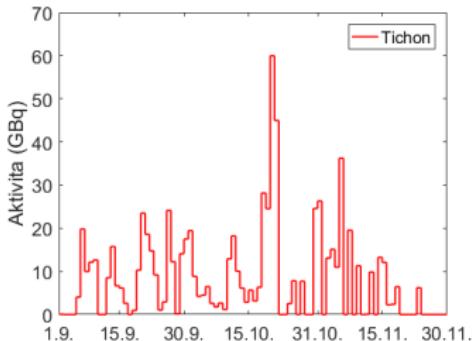
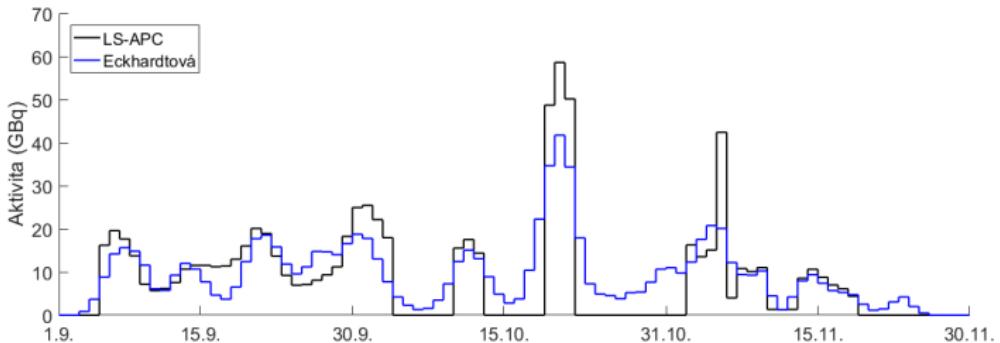


- $M^1, \dots, M^{2500}$
- $\|\mathbf{y} - M^i \hat{\mathbf{x}}^i\|_2$
- Marginal likelihood (evidence)  $f(\mathbf{y})$  for  $M^i$

# Source localization – Hungary 2011



# Source term estimate – Hungary 2011



Thank you for your attention