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TASEP with generalized update and Matrix Product Ansatz

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Abstract. The steady-state distribution of Totally Asymmetric Simple Exclusion Process (TASEP) model has been studied by means of Matrix Product Ansatz (MPA) for variety of updates: random-sequential, parallel, backward-sequential. However, the newly developed generalized update has been studied very poorly. The contribution summarizes the method of MPA and defines the equations following from the MPA concept. Two different solutions of the matrix algebra are presented: by Hrabak and Krbalek (2014) from unpublished dissertation, and Aneva and Brankov (2016). Both concepts are compared and (dis)advantages and limitations discussed.

Key words: TASEP with generalized update; Matrix Product Ansatz; Stationary distribution.

1 Introduction

Hopping particle systems consisting of interacting particles hopping along a discrete lattice are often used as simple models of complex collective systems as vehicular or pedestrian traffic or motion of kinks. Their simplicity often enables to derive the exact formula for stationary distribution on the state space, i.e. the steady-state probability that the system finds himself in given configuration. These models are generally non-equilibrium systems, thus finding the stationary distribution requires other methods that the classical equilibrium.

One of possible methods to find the stationary distribution of one dimensional hopping particle system is the Matrix Product Ansatz (MPA). The idea consists in expressing the stationary solution by means of product of matrices. We recommend the reader the review [2] covering the method details and processes treatable by means of this method.

The Totally Asymmetric Simple Exclusion Process (abbr. TASEP) is the simplest and most investigated hopping particle system. The stationary distribution of continuous-time TASEP by means of MPA has been found in [5], the method has been then used for finding the stationary distribution for TASEP with variety of discrete-time updates in [7], namely for random-sequential, forward- and backward-sequential, and fully parallel update.

In [4] the new generalized update of TASEP (genTASEP) has been investigated. First attempt to find the stationary distribution by means of MPA appeared in unpublished dissertation [6]. In this dissertation, the matrix algebra is formulated and two-dimensional matrices representing the MPA found for the system on the ring (periodic boundaries). Another set of two-dimensional matrices satisfying the algebra was presented in [1]. The open boundary case has been studied in [3] using the boundary hopping rules adopted from [6]. However, finding the MPA matrices for the open-boundary case is still an opened problem.

The goal of this contribution is to summarize the unpublished results from [6] and compare them with [1].

2 Model definition

The genTASEP model is a generalization of the TASEP driven by the backward sequential update. Particles are hopping along a lattice consisting of L sites, each site can be occupied by at most one particle. The backward sequential update can be understood in the way that particles in one uninterrupted block are updated in the reverse order then the particle propagation.

In more detail, particles can hop one site to the right (from i to $i+1$). Let us consider block of n particles. The rightmost particle hops to the right with probability $p \in (0,1)$. The second rightmost particle then hops to the newly emptied site with probability $p\gamma$, where $0 \leq \gamma \leq 1/p$ reflects the repulsion/attraction forces between particles. The update proceeds analogically until the end of the cluster. The schematic description is depicted in figure 1.

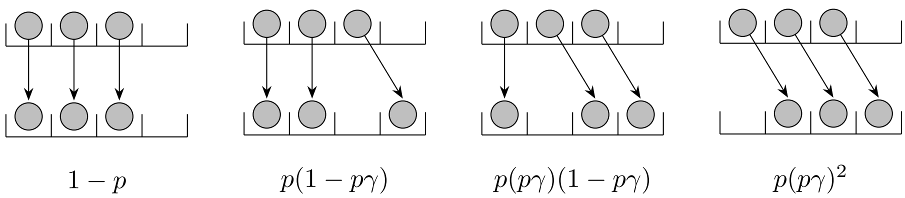

Figure 1: Illustration of genTASEP. Redrawn from [4].

The model can be considered with periodic or open boundaries. Under periodic boundaries we understand that the particles from the last site hop to the first site according to given rules, i.e. all sites are equivalent. In such a case the number of particles is preserved and the state space of such process consists of configurations $\left\{\tau = (\tau_1, \ldots, \tau_L) \in \{0,1\}^L \,\middle|\, \sum_{i=1}^{L} \tau_i = N\right\}$, where N is the number of particles.

By open boundaries we understand that particles can enter the lattice by hopping to the leftmost site and leaving the lattice by hopping from the rightmost site. The state space is then $\left\{\tau = (\tau_1, \ldots, \tau_L) \in \{0,1\}^L\right\}$. The boundary mechanism presented in [6] and used in [3] is following: Particle occupying the rightmost site L leaves the lattice with probability $\beta \in (0,1]$. If the second rightmost site is occupied, the particle hops then to the right with probability $p\gamma$ as normal. In case the leftmost site 1 is empty, new particle hops to the lattice with probability $\alpha \in (0,1]$. In the case the leftmost site gets empty during the update because the hop o a particle from 1 to 2, new particle enters the lattice with probability $\alpha\gamma$.

3 Method of Matrix Product Ansatz

The object of interest is the stationary distribution $P(\tau)$, i.e. the steady state probability that the system is in configuration $\tau = (\tau_1, \dots, \tau_L)$. The idea of Matrix Product Ansatz is to represent the probability by means of product of matrices $X_0, X_1, \dots,$ where X_j corresponds to the local state j of the corresponding site (for detailed description see [2]). Specifically, the steady state probability is expressed as

$$P_{L,N}(\tau) = \frac{1}{Z_{L,N}} \operatorname{Tr} \prod_{j=1}^{L} X_{\tau_j} \,. \tag{1}$$

for periodic boundaries or

$$P_L(\tau) = \frac{1}{Z_L} \langle W| \prod_{j=1}^{L} X_{\tau_j} |V\rangle \tag{2}$$

for open boundaries.

The stationary distribution $|P\rangle$ is the normalized eigenvector of transition matrix $\mathcal{T}$ corresponding to eigenvalue 1, i.e. satisfying

$$|P\rangle = \mathcal{T}|P\rangle \,, \tag{3}$$

where $\mathcal{T}_{\tau,\sigma}$ denotes the transition probability from state σ to state τ. The transition matrix $\mathcal{T}$ of the backward-sequential dynamics can be expressed as a product of local nearest neighbour transitions as

$$\mathcal{T} = \mathcal{T}_{1,2} \cdot \dots \cdot \mathcal{T}_{L-1,L} \cdot \mathcal{T}_{L,1} \,, \quad \text{resp.} \quad \mathcal{T} = \mathcal{T}_1 \cdot \mathcal{T}_{1,2} \cdot \dots \cdot \mathcal{T}_{L-1,L} \cdot \mathcal{T}_L \,, \tag{4}$$

where

$$\mathcal{T}_1 = T_1 \otimes I_2^{\otimes(L-1)} \,, \tag{5}$$

$$\mathcal{T}_L = I_2^{\otimes(L-1)} \otimes T_L \,, \tag{6}$$

$$\mathcal{T}_{j,j+1} = I_2^{\otimes(j-1)} \otimes T \otimes I_2^{\otimes(L-j-1)} \,. \tag{7}$$

Here T is a matrix responsible for the local bulk transition, matrices T_1 and T_L are matrices responsible for the local transition at the lattice boundaries.

As demonstrated in [2, Section 3.1], the stationary distribution can be found using auxiliary matrices $\tilde{X}_0, \tilde{X}_1, \dots$. Let us denote $X := (X_0, X_1, \dots)^T$ and $\tilde{X} := (\tilde{X}_0, \tilde{X}_1, \dots)^T$. If we find such matrices satisfying

$$T(X \otimes \tilde{X}) = \tilde{X} \otimes X \,, \quad \langle W|T_1\tilde{X} = \langle W|X \,, \quad T_L X|V\rangle = \tilde{X}|V\rangle \,, \tag{8}$$

then the formulas (1) and (2) satisfy the condition for stationary distribution.

4 MPA for genTASEP

The dynamics of backward-sequential update can be handled by means of the auxiliary state a denoting a site, which has been abandoned during the ongoing update. The local states are then: 0 = empty site, a = just abandoned by a particle, 1 = occupied site. Non-zero local transitions are summarized in table 1

Table 1: Non-zero local transitions in the MPA for TASEP with generalized update. Transition with τ applies for all $\tau \in \{0,1\}$.

site	trans.	rate
L	$1 \to a$	β
	$1 \to 1$	$1-\beta$
	$0 \to 0$	1
1	$0 \to 1$	α
	$0 \to 0$	$1-\alpha$
	$1 \to 1$	1
	$a \to 1$	$\gamma\alpha$
	$a \to 0$	$1-\gamma\alpha$

site	trans.	rate
$j, j+1$	$0\tau \to 0\tau$	1
	$10 \to a1$	p
	$10 \to 10$	$1-p$
	$1a \to a1$	$p\gamma$
	$1a \to 10$	$1-p\gamma$
	$0a \to 00$	1
	$11 \to 11$	1
	$0\tau \to 0\tau$	1

Thus, local transition matrices have the form

$$T_1 = \begin{pmatrix} 1-\alpha & 0 & 1-\gamma\alpha \\ \alpha & 1 & \gamma\alpha \\ 0 & 0 & 0 \end{pmatrix}, \quad T_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & \beta & 0 \end{pmatrix} \tag{9}$$

in basis $\{0, 1, a\}$ and

$$T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-p & 0 & 1-p\gamma & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & p\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

in basis $\{00, 01, 0a, 10, 11, 1a, a0, a1, aa\}$.

We use the common notation $X = (E, D, 0)^T$ and $\tilde{X} = (\tilde{E}, \tilde{D}, \tilde{A})^T$, i.e E for empty site, D for occupied site, and A for auxiliary state a.

The stationary-state conditions (8) are then

$$
\begin{aligned}
\tilde{E}E &= E\tilde{E} + E\tilde{A}\,, \\
\tilde{E}D &= E\tilde{D}\,, \\
\tilde{D}E &= (1-p)D\tilde{E} + (1-p\gamma)D\tilde{A}\,, \\
\tilde{D}D &= D\tilde{D}\,, \\
\tilde{A}E &= 0\,, \\
\tilde{A}D &= pD\tilde{E} + p\gamma D\tilde{A}\,,
\end{aligned}
\tag{11}
$$

with boundary conditions

$$
\begin{aligned}
\tilde{E}|V\rangle &= E|V\rangle\,, \\
\tilde{D}|V\rangle &= (1-\beta)D|V\rangle\,, \\
\tilde{A}|V\rangle &= \beta D|V\rangle\,, \\
\langle W|E &= \langle W|[(1-\alpha)\tilde{E} + (1-\alpha\gamma)\tilde{A}]\,, \\
\langle W|D &= \langle W|[\alpha\tilde{E} + \alpha\gamma\tilde{A} + \tilde{D}]\,.
\end{aligned}
\tag{12}
$$

Note that the condition $\tilde{A}E = 0$ disables to find the one-dimensional representation of matrices $E, D, \tilde{E}, \tilde{D}, \tilde{A}$, similarly to the fully-parallel update. Due to this, it is not reasonable to use this approach for backward-sequential update $\gamma = 1$. In such case the auxiliary state a is superfluous.

Here we note that according to our knowledge, the matrices satisfying the open boundary algebra (12) have not been found yet. Let us thus focus on the bluk algebra (11).

In the unpublished dissertation [6], the author presented two-dimensional matrices satisfying the bulk algebra, namely

$$
\begin{aligned}
&E = \begin{pmatrix} 1-p & 1-\gamma \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix} \\
&\tilde{E} = \begin{pmatrix} 1 & -\gamma \\ 0 & 0 \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} 0 & 1 \\ 0 & p \end{pmatrix}.
\end{aligned}
\tag{13}
$$

Properties of these matrices are

$$
D^2 = D\,, \quad E^2 = (1-p)E\,, \quad \mathrm{Tr}(DE) = 1 - p\gamma\,, \quad \det(DE) = 0\,. \tag{14}
$$

Thus the stationary distribution (1) has the form

$$
P_L(\tau) \propto (1-p)^{n_{00}}(1-p\gamma)^{n_{10}}\,, \tag{15}
$$

where n_{00} is the number of pairs 00 in the configuration τ and n_{10} is the number of pairs 10 in τ.

5 Another Solution

In [1] the authors derived the solution in more general way. Making the Ansatz

$$E = \tilde{E} + \tilde{A} - c_1\,, \quad D = \tilde{D} - c_2\,, \tag{16}$$

they found the matrices in the general form[1]

$$\begin{aligned} D = \tilde{D} &= d\begin{pmatrix} 1 & 0 \\ \frac{pe}{f(1-p)} & 0 \end{pmatrix}, \quad & E &= \begin{pmatrix} e & (1-\gamma)f \\ 0 & 0 \end{pmatrix} \\ \tilde{E} &= \begin{pmatrix} c_1 + e & -\gamma f \\ 0 & c_1 - \frac{pe}{1-p} \end{pmatrix}, \quad & \tilde{A} &= \begin{pmatrix} 0 & f \\ 0 & \frac{pe}{1-p} \end{pmatrix}, \end{aligned} \tag{17}$$

The authors claim that d, e, f, c_1 are free parameters, however, we have found out that the matrix algebra (11) is satisfied only for the choice

$$c_1 = \frac{pe}{1-p}\,. \tag{18}$$

Here we note that the authors used the notation $\tilde{p} := p\gamma$.

Choosing $d = 1$, $e = 1 - p$, $f = 1$ and thus $c_1 = p$, we obtain the matrices (**??**).

The authors of [1], however, prefer the choice $d = e = f = 1$ and thus $c_1 = p/(1 - p)$. Then

$$\begin{aligned} D = \tilde{D} &= \begin{pmatrix} 1 & 0 \\ \frac{p}{1-p} & 0 \end{pmatrix}, \quad & E &= \begin{pmatrix} 1 & 1-\gamma \\ 0 & 0 \end{pmatrix} \\ \tilde{E} &= \begin{pmatrix} \frac{1}{1-p} & -\gamma \\ 0 & 0 \end{pmatrix}, \quad & \tilde{A} &= \begin{pmatrix} 0 & 1 \\ 0 & \frac{p}{1-p} \end{pmatrix}. \end{aligned} \tag{19}$$

Properties of these matrices are

$$D^2 = D\,, \quad E^2 = E\,, \quad \mathrm{Tr}(DE) = \frac{1-p\gamma}{1-p}\,, \quad \det(DE) = 0\,. \tag{20}$$

Thus the stationary distribution (1) has the form

$$P_L(\tau) \propto \left(\frac{1-p\gamma}{1-p}\right)^{n_{10}}\,, \tag{21}$$

where n_{10} is the number of pairs 10 in τ.

Comparing equations (15) and (21) we state that both equations describe the same distribution (as necessary, since the stationary distribution is unique). Indeed, we can see, that

$$\left(\frac{1-p\gamma}{1-p}\right)^{n_{10}} = \frac{(1-p\gamma)^{n_{10}}(1-p)^{n_{00}}}{(1-p)^{n_{10}+n_{00}}} \propto (1-p\gamma)^{n_{10}}(1-p)^{n_{00}}\,, \tag{22}$$

because $n_{00} + n_{10} = L - N$ and thus the factor $(1 - p)^{-n_{10}-n00}$ is constant for all configurations having N particles.

[1] there is a misprint in the original manuscript [1], where $\tilde{E}_{1,2}$ is incorectly given as $(1 - \gamma)f$.

6 Conclusions

This paper presents the Matrix Product Ansatz method for finding the stationary distribution of TASEP with generalized update. Mainly, the original set of matrices derived in the unpublished thesis [6] is presented. Moreover, the derivation of more general class of matrices satisfying the MPA matrix algebra published in [1] is presented.

Comparing both sets of matrices we can conclude that the stationary distribution described by means of these sets are identical as expected. In [1], the authors state, that the set from [6] can be applied even in the partially deterministic case $p = 1$, but does not have all the nice properties of the set from [1]. Here it is just to say, that even in the set from [6] the partially deterministic case is problematic. Despite the set from [1], the value $p = 1$ can be substituted into the matrices, however, then the product E^2 becomes zero matrix, thus the matrix product cannot describe the stationary distribution once the number of empty sites exceeds the number of particles, i.e. $L - N > N$. Thus, the partially deterministic case $p = 1$ is to be solved using different method then the MPA.

As a final remark we note that the matrix product Ansatz for open boundary case remains, according to our knowledge, a open problem.

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